ON THE DOLBEAULT COHOMOLOGY
OF COMPACT LIE GROUPS

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Abstract

An algebraic way to compute p’th Dolbeault cohomology groups on even dimensional compact Lie groups considered with a left invariant complex structure is given. In particular, a description of the p’th Dolbeault cohomology groups of some left invariant complex structures on SU(2n+1) is presented.

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1 Introduction

We consider even dimensional compact Lie groups with a left-invariant complex structure. R.Bott showed in [?] that all Dolbeault cohomology of a compact Lie group (more general, on a compact homogeneous space) with respect to an invariant complex structure can be expressed in terms of Lie algebra cohomology. In [?] the Dolbeault cohomology ring of compact even dimensional Lie groups of rank 2 as well as of some groups of type $B_n, C_n, D_n$ considered with a left invariant complex structure are described relying on Bott’s formula [?].

In this note we apply the vanishing theorem of [?] to even dimensional compact Lie group considered with left invariant complex structure. This approach gives a purely algebraic way to compute p’th Dolbeault cohomology groups and is completely different from that given by Bott in [?]. We apply this result to the groups of type $A_n$ and determine the p’th Dolbeault cohomology groups of some left invariant complex structures on $SU(2n + 1)$. We show that these groups are generated by the chosen (0,1)-space in the Cartan subalgebra (see Theorem ??) which agrees with the results in [?].

Bott’s work [?] leads to the independence of the Hodge numbers of a left-invariant complex structure on a compact Lie group on the chosen left-invariant complex structure. In view of this fact and Theorem ?? we propose

**Theorem 1.1** Hodge numbers of a left-invariant complex structure on $SU(2n+1)$ are given by

$$h^{0,p} = \binom{n}{p}, \quad p = 1, 2, \ldots, n,$$

$$h^{0,p} = 0, \quad p > n.$$

2 Dolbeault cohomology

Let $(M, g, J)$ be a 2n-dimensional $(n > 1)$ Hermitian manifold with complex structure $J$ and compatible metric $g$. Let $\Omega$ be the Kähler form of $(M, g, J)$, defined by $\Omega(X, Y) = g(X, JY)$. Denote by $\theta$ the Lee form of $(M, g, J)$, $\theta = \frac{1}{n-1} d^c \Omega o J$. Any Hermitian manifold $(M, g, J)$ carries a unique Hermitian connection with completely skew-symmetric torsion, the Bismut connection (cf. [?] and [?]). The Bismut connection is given by

$$g(\nabla^B_X Y, Z) = g(\nabla^L_X Y, Z) + \frac{1}{2} d^c \Omega(X, Y, Z),$$

where $\nabla^L$ is the Levi-Civita connection of $g$.

Recall that $d^c = i(\bar{\partial} - \partial)$. In particular, $d^c \Omega(X, Y, Z) = -d\Omega(JX, JY, JZ)$. This connection has been used by Bismut in [?] to prove a local index theorem for the Dolbeault operator on Hermitian manifold.

A Hermitian manifold equipped with the Bismut connection is also called Kähler with torsion or KT manifold, see e.g. [?]. KT manifolds arise in a natural way in physics as target spaces.
of (2,0)-supersymmetric sigma models with Wess-Zumino term (torsion) [?, ?] (see also [?] and the references there). If the torsion is closed then the manifold is called strong KT manifold. We use the following result from [?]

**Theorem 2.1** [?] Let $(M,g,J)$ be a compact $2n$-dimensional $(n > 1)$ Hermitian manifold with Kähler form $\Omega$. Suppose that $\Omega$ is $d\bar{d}$-harmonic, i.e. $d\bar{d}\Omega = 0$ and $(d\bar{d})^*\Omega = 0$. Suppose also that the $(1,1)$-part of the Ricci form of the Bismut connection is non-negative everywhere on $M$.

a) Then every $\bar{\partial}$-harmonic $(0,p)$-form, $p = 1,2,\ldots,n$, must be parallel with respect to the Bismut connection.

b) If moreover the $(1,1)$-part of the Ricci form of the Bismut connection is strictly positive at some point, then the cohomology groups $H^p(M,\mathcal{O})$ vanish for $p = 1,2,\ldots,n$.

We note that $d\bar{d}\Omega = \partial \bar{\partial} \Omega$ and the condition $(d\bar{d})^*\Omega = 0$ is equivalent to $d^*\theta = 0$.

### 3 Compact Lie groups

Any compact Lie group with bi-invariant metric and compatible left-invariant complex structure has $d\bar{d}\Omega = 0$, $(d\bar{d})^*\Omega = 0$. On a compact even dimensional Lie group there exists a flat strong KT structure [?, ?]. Indeed, on every compact Lie group $G$ there exists at least one left invariant complex structure $J$ [?]. The definition corresponds to a Cartan decomposition of the Lie algebra. The generators of the Cartan subalgebra, including $U(1)$ generators, define an invariant subspace under the action of $J$. The (complex) generators associated to positive roots are eigenvectors of $J$ with eigenvalue $\sqrt{-1}$, while the generators associated to negative roots are eigenvectors with eigenvalue $-\sqrt{-1}$ [?]. Taking any biinvariant metric $g$ on $G$ and a left-invariant complex structure $J$ the Bismut connection $\nabla^B$ of the hermitian structure $(J,g)$ is defined by the requirement that an orthonormal left invariant basis is parallel. (In fact the definition of $\nabla^B$ depends only on the biinvariant metric chosen and does not depend on the complex structure). The connection $\nabla^B$ is flat and the torsion is just the commutator. The corresponding torsion 3-form $T^B$ is $\nabla^B$-parallel and hence closed because of the Jacobi identity. We know from (??) that $T^B = d\varphi^B$. The last equality implies that the Lee form $\theta$ is $\nabla^B$-parallel. Then, (??) shows that the Lee form is coclosed. Thus, the manifold $(G,J,g)$ is a compact Hermitian manifold which satisfies $d\bar{d}\Omega = d^*\theta = \rho^B = b = 0$. Now, Theorem ?? yields

**Proposition 3.1** Let $(G,J,g)$ be a compact $2n$-dimensional Lie group with a left-invariant complex structure $J$ and a biinvariant metric $g$. Then every $\bar{\partial}$-harmonic $(0,p)$-form, $p = 1,\ldots,n$ is parallel with respect to the Bismut connection.

The above proposition allows us to reduce computations of the $p$'th, $p = 1,2,\ldots,n$ Dolbeault cohomology groups of a compact $2n$ dimensional Lie group $G$ endowed with left-invariant complex structure $J$ to purely algebraic calculations on the Lie algebra.
We recall that the action of the operators $\bar{\partial}$ and $\bar{\partial}^*$ on a $(0,p)$-form $\psi$ are expressed in terms of any hermitian connection $\nabla$ with torsion $T$ by ([?], Lemma 5)

\begin{align}
(\bar{\partial}\psi)((Z_0, \ldots, Z_p)) &= \sum_{j=1}^{p} (-1)^j (\nabla Z_j \psi)(Z_0, \ldots, \hat{Z}_j, \ldots, Z_p) \\
&+ \sum_{j<k} (-1)^{j+k} \psi \left( T^{2,0}(Z_j, Z_k), Z_0, \ldots, \hat{Z}_j, \ldots, \hat{Z}_k, \ldots Z_p \right), \\
(\bar{\partial}^*\psi)((Z_1, \ldots, Z_{p-1})) &= - \sum_{j=1}^{p-1} (\nabla_{e_j}\psi)(e_i, Z_1, \ldots, Z_{p-1}) - \sum_{j=1}^{p-1} (-1)^j < \left( Z_j, T^{2,0} \right), \psi(., Z_1, ..., \hat{Z}_j, ..., Z_{p-1}) >,
\end{align}

for any vector fields $Z_i$ of type $(0,1)$, where $T^{2,0}$ is the $(2,0)$ part of $T$ characterized by the property $T^{2,0}(Y, Z) = JT^{2,0}(Y, Z)$; $trT^*$ is the dual vector field to the 1-form defined by:

$Y \rightarrow \sum(e_i, T(e_i, X)); (Z_j, T^{2,0})$ denotes the complex 2-form defined by $Y, Z \rightarrow (Z_j, T^{2,0}(Y, Z)); \psi(., Z_1, ..., \hat{Z}_j, ..., Z_{p-1})$ denotes the complex 2-form defined by $Y, Z \rightarrow \psi(Y, Z, Z_1, ..., \hat{Z}_j, ..., Z_{p-1})$ (as usual, the symbol $\hat{Z}_j$ denotes the missing argument and $\{e_i\}$ is a $J$-adapted orthonormal basis).

**Theorem 3.2** Let $(G, J, K)$ be a $2n$-dimensional compact Lie group with a left invariant complex structure $J$ and the biinvariant Killing metric $K$. Then every $\bar{\partial}$-harmonic $(0,p)$-form $\psi$, $p = 1, \ldots, n$ is left invariant and satisfies the following algebraic conditions

\begin{align}
&\sum_{j<k} (-1)^{j+k} \psi \left( [Z_j, Z_k]^{0,1}, Z_0, \ldots, \hat{Z}_j, \ldots, \hat{Z}_k, \ldots Z_p \right) = 0, \\
&\sum_{j=1}^{p-1} (-1)^j < \left( Z_j, [., .]^{1,0} \right), \psi(., Z_1, ..., \hat{Z}_j, ..., Z_{p-1}) > = 0,
\end{align}

for any unitary basis of left invariant vector fields of type $(0,1)$, where $[,]^{1,0}$ (resp. $[,]^{0,1}$ denotes the $(1,0)$-part (resp. $(0,1)$-part) of the commutator.

Conversely, every left invariant $(0,p)$-form $\psi$ satisfying (??) and (??) belongs to the Kernel of the Dolbeault operator.

**Proof.** If $\psi \in \text{Ker} \Box$ then Proposition ?? implies that $\psi$ is parallel with respect to the Bismut connection $\nabla^B$. This means that $\psi$ is left invariant by the very definition of $\nabla^B$. Considering (??) and (??) with respect to the Bismut connection we deduce that $\psi$ fulfills (??) and (??).

For the converse, every left invariant $(0,p)$-form is $\nabla^B$-parallel. Substituting (??) into (??) and (??) into (??) we get $\bar{\partial}\psi = \bar{\partial}^*\psi = 0$. **Q.E.D.**

We shall give more precise explanations. Let $G$ be a semi-simple, compact Lie group with compact Lie algebra $g$ and $g^c$ be its complexification. Let $h$ be the Cartan subalgebra of $g$ and $\Delta$ be the root system of $g^c$ with respect to the complexification $h^c$ of $h$. Then (see for example [?], p.165) $g^c = h^c \oplus_{\alpha \in \Delta} g^c_{\alpha}$, where the root subspaces are $g^c_{\alpha} = \{g_{\alpha} \in g^c : [h, g_{\alpha}] = \alpha(h)g_{\alpha}, \quad h \in h^c\}$.
We denote by $\Delta^+(\Delta^-)$ the space of positive (negative) roots of $g'$. Any root of $g'$ can be written as the sum of simple roots, $\Delta^+ := \{\alpha_i, i = 1, ..., l = \text{rank}(g) = \text{dim}(h)\}$, with positive integers coefficients ([?], p.177). For each linear function $\alpha$ on $h^e$ there exists a unique element $h_\alpha \in h^e$ defined by the Killing metric $K$ in the usual way ([?], p.166). The Killing metric induces an inner product on the root space by $\langle \alpha, \beta \rangle = \alpha(h_\beta) = K(h_\alpha, h_\beta)$. The commutation relations of $g'$ are

\begin{equation}
[h_\alpha, e_\beta] = 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} e_\beta, \quad [e_\alpha, e_\beta] = N_{\alpha, \beta} e_{\alpha+\beta} + \delta_{\alpha, -\beta} h_\alpha,
\end{equation}

where $\{e_\alpha\}$ are the step operators, $\{h_\alpha\}$ are the generators of the Cartan subalgebra and the structure constants $N_{\alpha, \beta}$ are integers ($N_{\alpha, \beta} = 0$ if $\alpha + \beta \notin \Delta$). A left invariant complex structure can be defined with $+\sqrt{-1}$-eigenspace to be the positive root space and the Cartan subalgebra has to be preserved by the structure. Further, we refer to this complex structure as a root-space complex structure.

We consider the groups of $A_n$-types i.e. $SU(n+1)$. The Lie algebra $su(n+1)$ is the space of matrices of trace zero. Cartan subalgebra $h$ of $su(n+1)$ is a group of all diagonal matrices and the root spaces $g_{\alpha, i} = \text{const}.e_i$, $1 \leq i, j \leq n, i \neq j$ are consisted of matrices with nonzero entry only in the $ij$ position. The rank of $SU(n + 1)$ is $n$. The root associated to the root space is defined by $\alpha_{ij} = e_i - e_j$, where $e_i$ is the linear map which is the projection onto the $i$th coordinate. The corresponding generator of the Cartan subalgebra is denoted by $h_{ij}$. The usual choice of simple roots are $\alpha_{12}, \alpha_{23}, ..., \alpha_{n+1}$. With this choice the positive roots are $\alpha_{ij}$ with $i < j$. A root-space complex structure on $SU(2n + 1)$ can be defined by the following $(0,1)$-space $\{A_1 = h_{12} + \sqrt{-1} h_{23}, ..., A_n = h_{2n-2n-1} + \sqrt{-1} h_{2n-12n}\}$ in the Cartan subalgebra $h$. Let $\Lambda^{0,p}$ denote the space of $(0,p)$-forms on the Cartan subalgebra $h$. We have

**Theorem 3.3** Let $J$ be a root-space complex structure on $SU(2n + 1)$. Then the $p$'th Dolbeault cohomology group $H^p(SU(2n + 1), \mathcal{O})$ is generated by $\Lambda^{0,p}$ for $p = 1, ..., n$ and is zero for $p > n$. The Hodge numbers are given by

\begin{align*}
\hat{h}^{0,p} &= \binom{n}{p}, \quad p = 1, 2, ..., n, \\
\hat{h}^{0,p} &= 0, \quad p > n.
\end{align*}

**Proof.** Follows from Theorem ?? and commutation relations (??) for $SU(2n + 1)$ by long but straightforward computations. Q.E.D.

**Example 1.** We obtain from Theorem ?? that the Hodge numbers of $SU(3)$ with respect to a root-space complex structure $J$ are

$\hat{h}^{0,1} = 1$, \quad $\hat{h}^{0,2} = \hat{h}^{0,3} = \hat{h}^{0,4} = 0$.

This is incompatible with Bott’s example ([?], p.205) where it is claimed $\hat{h}^{0,1} = 0$ which seems to be a typographical missprint. However, our result agrees with those in [?]
**Example 2.** We consider the non-Kähler space $S^3 \times S^3 = SU(2) \times SU(2) \in SU(4)$. The Cartan subalgebra is 2-dimensional. The positive roots are $\alpha_{12}, \alpha_{34}$. Cartan subalgebra is spanned by $h_{12}, h_{34}$. The root-space complex structure $J$ is defined by the following $(0,1)$-space \{e_{12}, e_{34}, A = h_{12} - ih_{34}\}. Applying Theorem ?? and using (??) we get

$$h^{0,1} = 1, \quad h^{0,2} = h^{0,3} = 0.$$ 

The $\bar{\partial}$-harmonic representative of $H^1(S^3 \times S^3, \mathcal{O})$ is the $(0,1)$-form $A^\ast$ dual to $A$. This agrees with the results in [?].

**Example 3.** We consider the exceptional 14-dimensional group $G_2$. Cartan subalgebra of $G_2$ is two-dimensional. The roots are (see for example [?], p.472):

$$\pm(e_2 - e_3), \pm(e_3 - e_1), \pm(e_1 - e_2); \pm(2e_1 - e_2 - e_3), \pm(2e_3 - e_1 - e_2).$$

The roots $\alpha_1 = e_1 - e_2, \quad \alpha_2 = -2e_1 + e_2 + e_3$ form a basis. The positive roots are: $\beta_1 = \alpha_1 + \alpha_2, \quad \beta_2 = 2\alpha_1 + \alpha_2, \quad \beta_3 = 3\alpha_1 + \alpha_2, \quad \beta_4 = 3\alpha_1 + 2\alpha_2$. Applying Theorem ?? and using (??) we obtain for the first Dolbeault cohomology groups of the root-space complex structure $J$ on $G_2$ that $h^{0,1}(G_2, J) = 1, \quad h^{0,2}(G_2, J) = 0$ which is compatible with the results in [?].

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**References**


