ON THE DAMPING OSCILLATORY INTEGRALS

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Abstract

We suppose that $S$ is a smooth hypersurface in $\mathbb{R}^{n+1}$ with Gaussian curvature $K$ and surface measure $d\sigma$, $\psi$ is a compactly supported cut-off function, and we let $\mu_q$ be the surface measure with $d\mu_q = \psi|K|^q d\sigma$. In this paper we consider the cases where $S$ is a smooth hypersurface and $S$ is a real analytic hypersurface and estimate the Fourier transform $\hat{\mu}_q$. We show that if $S$ is a real analytic hypersurface, then $\hat{\mu}_q(\xi)$ decays as $|\xi|^{-\frac{n}{2}}$ provided $q > \frac{n}{2}$. We also prove that if $S$ is a smooth hypersurface and $n \geq 3$, then $\hat{\mu}_q(\xi)$ decays optimally whenever $q > \frac{n}{2}$. The damping oscillatory integrals related to Lagrangian manifold are considered. It is obtained the local optimal estimation for such kind of integrals. The well-known examples show that the results are sharp for the case of real analytic hypersurfaces in $\mathbb{R}^3$. 

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§1. Introduction

In this paper we obtain estimates for the decay at infinity of certain oscillatory integrals related to the Fourier transform of surface carried measures. Let $D \subset \mathbb{R}^{n+1}$ be a compact domain with smooth boundary and let $u \in C^\infty(\mathbb{R}^{n+1})$. Consider an integral of the following form

$$\hat{u}_D(r\xi) = \int_D u(x)e^{ir(x,\xi)}dx, \quad r \in \mathbb{R}_+, \xi \in S^n,$$

where $S^n$ is a unit sphere in $\mathbb{R}^{n+1}$ centered at the origin and $(x,\xi)$ is the inner product of the vectors $x$ and $\xi$.

Let $S$ denote a smooth hypersurface in $\mathbb{R}^{n+1}$ with Gaussian curvature $K$ and an element of surface measure $d\sigma$. We fix a smooth function $\psi$ with compact support in $S$ and consider an integral of the following form:

$$\hat{\psi}_S(r\xi) = \int_S \psi(x)e^{ir(x,\xi)}d\sigma, \quad r \in \mathbb{R}_+, \xi \in S^n,$$

where $\psi \in C^\infty_c(S)$.

A critical point is such a point $\omega \in \partial D(S)$ so that $n_\omega = \pm \xi$. It should be noted that by the stationary phase method a character of both functions $\hat{u}_D(r\xi)$ and $\hat{\psi}_S(r\xi)$ when $r$ gets large is defined by small a neighbourhood of the critical points. So, the character of both functions $\hat{u}_D(r\xi)$ and $\hat{\psi}_S(r\xi)$ (when $r$ gets large) depends on the geometric properties of $\partial D$ and $S$ (see [St]). If $D$ is a strictly convex domain (i.e. $\partial D$ is a smooth hypersurface with positive principal curvatures) then $\hat{u}_D(r\xi) = O(r^{-(n+2)/2})$ (as $r \to \infty$). Moreover, the last asymptotic equality holds uniformly with respect to $\xi \in S^n$. By the analogy if the Gaussian curvature of the surface $S$ does not vanish, then $\hat{\psi}_S(r\xi) = O(r^{-n/2})$ (as $r \to \infty$) (see [He]).

The behavior of the integrals (1.1) and (1.2) was studied for the cases:

(a) when the domain $D$ is strictly convex by Hlawka, Herz. Note that in this case for any fixed $\xi \in S^n$ there exist only two critical points $\omega_1(\xi), \omega_2(\xi) \in \partial D$ and for the integral (1.1) the following estimation

$$|\hat{u}_D(r\xi)| \leq \text{const}(|K(\omega_1(\xi))|^{-1/2} + |K(\omega_2(\xi))|^{-1/2})r^{-\frac{n+1}{2}}$$

holds;

(b) when $D$ is convex possesses an analytic boundary by Randol [R1, R2]. Randol proved the estimation (1.3) for this case. Moreover, he proves that $|K(\omega_1(\xi))|^{-1/2} + |K(\omega_2(\xi))|^{-1/2} \in L^{2+\varepsilon}(S^n)$ for some $\varepsilon > 0$;

(c) when $D$ is a convex domain and has sufficiently smooth boundary and the boundary has no tangent lines of infinite order by Svensson [S]. Svensson proved the analogue of Randol results, and also Bruna Nagel and Wainger [BNW] proved the following estimation:

$$|\hat{\psi}_D(r\xi)| \leq \text{const}(\sigma(B(\omega_1(\xi), r^{-1})) + \sigma(B(\omega_2(\xi), r^{-1}))),$$

where

$$\sigma(B(\omega(\xi), t)) = \sup_{(x,\xi) \in B(\omega(\xi), t), \theta \in [0, 2\pi]} |\psi(x)e^{i\theta r(x,\xi)}|.$$
where $B(\omega, \delta)$ is a ball on $\partial D$ and $\sigma(B(\omega, r^{-1}))$ is a measure of the ball $B(\omega, r^{-1})$. Note that if $\xi$ is a fixed unit vector and $K(\omega(\xi)) \neq 0$ then

$$\sigma(B(\omega(\xi), r^{-1})) \asymp |K(\omega(\xi))|^{-1/2} r^{-n/2} \quad \text{as} \quad r \to \infty.$$  

(d) Varchenko [V] showed that, although for compact domains with smooth boundary the dependence of the Fourier transform of the indicator function on the direction can be complicated but, "on the average" it behaves as the Fourier transform of the indicator function of a ball;

(e) Podkorytov [Po] considered the estimation of $\hat{\psi}_S(r\xi)$ for the case of plane convex curves. L. Brandolini, L. Colzani and G. Travaglini [BCT] considered the estimation of $\hat{u}_D(r\xi)$ by some measure in the case of when $D$ is a polygon;

(f) Sogge and Stein ([SS1, SS2] considered oscillatory integrals with mitigating factor and proved that the following, uniformly with respect to $\xi$, asymptotic estimation

$$\int_S e^{irx\xi} |K(X)|^q \psi(x) d\sigma = O(r^{-n/2}) \quad \text{as} \quad r \to +\infty$$

holds, whenever $q \geq 2n$;

(g) Cowling, Disney, Maukeri and Mueller (see [CDMM], [CM]) showed that if $S$ is a convex hypersurface of finite type then

$$\int_S e^{irx\xi} a(X) \psi(x) d\sigma = O(r^{-n/2}) \quad \text{as} \quad r \to +\infty$$

holds, whenever $a$ is smooth and it satisfies the following condition: $|a(x)| \leq c|K(X)|^{1/2}$. In particular, if $a(X) = K(X)$ then we have the last asymptotic estimation.

Notice that the behavior of both $\hat{u}_D(r\xi)$ and $\hat{\psi}_S(r\xi)$ is different from these estimations for the non-convex domains and when Gaussian curvature of the surface is vanishing.

First, we assume that for any $\xi \in S^{n-1}$ there exists only a finite number of critical points: $\omega_1(\xi), \omega_2(\xi), \ldots, \omega_k(\xi)$. The number of such points i.e. (the number $k$) depends on $\xi$. According to [BNW] and [CDMM] consider a "ball"

$$B(\omega(\xi), \delta) = \{y \in S : \text{dist}(y, T_{\omega(\xi)}) < \delta\},$$

where $T_{\omega}$ is a tangent plane of $S$ at the point $\omega$.

**Proposition 1.1.** There exists a surface $S \subset \mathbb{R}^3$ which is in generic in $\mathbb{R}^3$ and $\phi \in C_0^\infty(\mathbb{R}^3)$ such that the following relation:

$$\sup_{r > 0} \sum_l \sigma(B(\omega_l(\xi), r^{-1})) \notin L^\infty(S^2)$$

holds.

The proof of Proposition 1.1 follows from [I2, Theorem 2.1] (see also [Popi]).

Our main results are the following Theorems.
Theorem 1.2. Let $S \subset R^{n+1}$ be a real analytic hypersurface and $\psi \in C_0^\infty(S)$. Then the following inequality
\[
\left| \int_S e^{ir \varphi_\xi} |K(X)|^q \psi(X) d\sigma \right| \leq c r^{-n/2}
\]
holds, whenever $q > \frac{n}{2}$.

Note that [CDMM] considered an example which indicated sharpness of Theorem 1.2 in the case of $n = 2$.

Theorem 1.3. Let $n \geq 3$, $S \subset R^{n+1}$ be a smooth hypersurface and $\psi \in C_0^\infty(S)$. Then the following inequality
\[
\left| \int_S e^{ir \varphi_\xi} |K(X)|^q \psi(X) d\sigma \right| \leq c r^{-n/2}
\]
holds, whenever $q > \frac{n}{2}$.

This paper is organized as follows.

In §2 we consider Vitali type covering of the surface $S$ by using principal curvatures of the surface. In §3, we prove local estimations for damping oscillatory integrals related to Lagrangian manifolds. In §4, we obtain estimations of some one-dimensional oscillatory integrals. Finally, in §5, we prove our main results.

§2. Decomposition of the surface $S$ into balls.

Let $S \subset R^{n+1}$ be a smooth hypersurface and let the system of coordinates be fixed in $R^{n+1}$. Then we have one-to-one correspondence between the points of $U \subset R^n$ and the points $X \in S$. Let $X_0 \in S$ be a fixed point of the surface. We consider the covering of $S \cap \{X : K(X) \neq 0\}$ in a small neighbourhood of the point $X_0$. Without loss of generality we may suppose that the surface $S$ is a graph of some smooth function $f$, and $X_0 = (0,0)$. So, we assume:
\[
S = \{X = (x, f(x)) \in R^{n+1} : x \in U\},
\]
where $U$ is a neighbourhood of zero in $R^n$. In fact, we consider the covering of the set $U_0 = U \cap \pi(\{X : K(X) \neq 0\})$, where $\pi : R^{n+1} \mapsto R^n$ is a projection along the $x_{n+1}$ axis. For any $x \in S$ denote by $\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)$ the principal curvatures of the surface. Note that $\lambda_1(x)\lambda_2(x)\ldots\lambda_n(x) = K(X)$.

Let $\lambda_1(x)$ be maximal by module principal curvature, i.e. $|\lambda_1(x)| \geq |\lambda_k(x)|$ for any $k = 2, \ldots, n$.

Note that $c$ will denote a constant, which may not be the same in each occurrence. The constants $c_1, c_2$ etc. will denote other bounds, which are small and do not depend on parameters $\delta(0 < \delta \leq 1)$ (where $\delta \simeq |\lambda|$ is a radius of the ball) and $t(0 < t < \infty)$ ($t$ is an oscillation parameter). Denote by $B_x$ a ball defined by $B_x = B(x, c|\lambda_1(x)|)$, i.e. $B_x$ is a ball centered at $x$ and has a radius $c|\lambda_1(x)|$, where $c$ is a small positive. The number $c$ depends only on $C^3$ norm of the function $f$. 


The following Lemma is needed for the sequel.

**Lemma 2.1.** There exist positive numbers $c, c_1, c_2$ such that for any $x \in U_0$ and for any $y \in B_x$ holds the following inequality holds

$$ c_1|\lambda_1(y)| \leq |\lambda_1(x)| \leq c_2|\lambda_1(y)|. $$  \hspace{1cm} (2.1)

**Proof of Lemma 2.1.** Without loss of generality we can assume that $x = 0$, $f(0) = 0$, $\nabla f(0) = 0$. We represent the function $f$ (after a possible rotation of the axes) by the following expression:

$$ f(x) = \sum_{j=1}^{n} \lambda_j x_j^2 + \sum_{k,l,m=1}^{n} x_k x_l x_m H_{klm}(x), $$

where $\lambda_k = \lambda_k(0)$, $H_{klm} \in C^\infty(\mathbb{R}^n)$. Note that $|\lambda_1| = \max_l |\lambda_l|$, and $C^k$ norm of $H_{klm}$ is bounded by $C^{k+3}$ norm of $f$. It is easy to see that if $|x| < c|\lambda_1|$ and $c$ is sufficiently small positive then there exists a positive number $c_3$ such that for any $x \in B(0, c|\lambda_1|)$ the following inequalities

$$ \left| \frac{\partial^2}{\partial x_i \partial x_j} \sum_{k,l,m=1}^{n} x_k x_l x_m H_{klm}(x) \right| \leq c_3|\lambda_1|, \quad i, j = 1, n $$

hold. Moreover, we may suppose that $c_3 < 1$. If $\xi \in S^n$ then we have

$$ |\lambda_1(x)| = \sup_{\xi \in S^n} |(\text{Hess} f(x)\xi, \xi)| = $$

$$ \sup_{\xi \in S^n} \left| \sum_{k=1}^{n} \lambda_k \xi_k^2 + (\text{Hess} \sum_{k,l,m=1}^{n} x_k x_l x_m H_{klm}(x)\xi, \xi) \right| \leq (1 + c_3)|\lambda_1(0)|. $$

On the other hand

$$ |\lambda_1(x)| \geq \left| \frac{\partial^2 f}{\partial x_i^2}(x) \right| \geq (1 - c_3)|\lambda_1| = c_2|\lambda_1|. $$

Note that the positive constants $c, c_1, c_2$ depend only on $C^3$ norm of the function $f$, therefore we can choose the constants such that the inequality (2.1) holds uniformly with respect to $x \in U_0$. This completes the proof of Lemma 2.1.

Note that the $\lambda_1(x)$ is a continuous function. Let $c_3 < 1$ be a fixed positive close to one number. We use the following Vitaly procedure. Denote by $B_1$ a ball $B(x_1, c|\lambda_1(x_1)|)$, where the point $x_1$ is chosen from the condition $|\lambda_1(x_1)| \geq c_3 \sup_{x \in U_0} |\lambda_1(x)|$. Let $B_1, B_2, \ldots, B_k (k \geq 1)$ be balls to be chosen. We will choose the ball $B_{k+1} :$ If for any $x \in U_0 \setminus \bigcup_{l=1}^{k} B_l$ we have $\bigcup_{l=1}^{k} B_l \cap B(x, c|\lambda_1(x)|) \neq \emptyset$ then we shall finish our procedure. Otherwise, we can choose a ball $B_{k+1} = B(x_{k+1}, c|\lambda_1(x_{k+1})|)$, so that

$$ |\lambda_1(x_{k+1})| \geq c_3 \sup_{x \in U_0} |\lambda_1(x)| : B(x, c|\lambda_1(x)|) \bigcap \bigcup_{l=1}^{k} B_l = \emptyset. $$
It should be noted that the sequence of balls \( \{B_k\} \) may be both finite or infinite number. If there exists an infinite number of such balls then we have

\[ \lambda_1(x_k) \to 0 \quad \text{as} \quad k \to \infty. \]

**Lemma 2.2.** Let \( c = \frac{c_3}{2} \) (where \( c \) is a positive number chosen from Lemma 2.1) be a fixed positive then the collection of balls \( \{B_k\} = \{B(x_k, c|\lambda_1(x_k)|)\} \) possesses the following properties:

(i) \( U_0 \subset \bigcup_k B'_k \), where \( B'_k = B(x_k, c|\lambda_1(x_k)|) \);

(ii) the collection \( \{B'_k\} \) has a bounded overlap property: there exists an \( N \) so that no point is contained in more than \( N \) balls among \( \{B'_k\} \);

(iii) \( \sum_k (\text{radius } B_k)^n < \infty \).

**Proof.** Let \( \{B_k\} \) be collection of balls chosen as above i.e. centered at the point \( x_k \) with radius \( c|\lambda_1(x_k)| \) and the ball \( B'_k \) has the same center as \( B_k \) with radius \( \frac{2c}{c_3}|\lambda_1(x_k)| \). We show that the collection of balls \( \{B'_k\} \) covers \( U_0 \). Let \( x \in U_0 \) be a fixed point. If \( x \in B_k \) for some \( k \) then there is nothing to prove. We may assume that \( x \notin B_k \) for any \( k \). Consider a ball \( B_x \) centered at the point \( x \) with radius \( c|\lambda_1(x)| \) i.e. \( B_x = B(x, c|\lambda_1(x)|) \). Note that \( \lambda_1(x) \neq 0 \), since \( x \in U_0 \).

It is clear that there exists a ball \( B_k \) such that \( B_k \cap B_x \neq \emptyset \). Let \( B_k \) be such a ball with minimal number of \( k \). So, we have \( \bigcup_{k=1}^{N-1} B_k \cap B_x = \emptyset \). It is easy to see that \( |\lambda_1(x)| \leq \frac{1}{c_3}|\lambda_1(x_k)| \). Hence

\[ B \left( x_k, \frac{c}{c_3}|\lambda_1(x_k)| \right) \cap B \left( x, \frac{c}{c_3}|\lambda_1(x)| \right) \neq \emptyset. \]

Thus, we obtain \( x \in B'_k \equiv B(x_k, \frac{2c}{c_3}|\lambda_1(x_k)|) \).

Finally, we show the bounded overlap property: there exists an \( N \) so that no point is contained in more than \( N \) of the balls among \( \{B'_k\} \). One can see this by a packing argument [SS1]. Suppose that \( B'_1, B'_2, \ldots, B'_N \) contain \( \bar{x} \), by Lemma 2.1 they all have a radius comparable to \( |\lambda_1(\bar{x})| \), and so their union is contained in a ball \( \bar{B} \) centered at \( \bar{x} \) of radius comparable to \( |\lambda_1(\bar{x})| \). Moreover, the balls \( B_1, B_2, \ldots, B_N \) are disjoint, they have a radius of comparable sizes, and their union is also contained in \( \bar{B} \). Comparing the volume of \( \bigcup_{k=1}^{N} B_k \) with \( \bar{B} \) gives us a bound for \( N \). As a result (iii) also holds, since the sum then represents essentially the \( \sigma \)-measure of a bounded set. In fact, it is bounded by \( \sigma(U_0) + \text{const.} \). This proves Lemma 2.2.

§3. On the estimation of damping oscillatory integrals with elliptic singularities.

Let \( S \) be a smooth \( n \)-dimensional manifold and \( T^*S \) be a cotangent bundle of \( S \). Denote by \( T_0^*S \) the conic subset \( T_0^*S = T^*S \setminus \{0\} \), where

\[ T^*S \setminus \{0\} = \{(x, s) \in T^*S : x \neq 0\}. \]

Following [D], [Pa] denote by \( C^*S \) the quotient manifold by multiplication group \( \mathbb{R}_+ \) i.e. \( C^*S = T_0^*S/\mathbb{R}_+ \).

Consider the non-degenerate phase function \( \Phi(x, s) \) which is homogeneous with respect to \( x \) i.e. \( \Phi(tx, s) = t\Phi(x, s)(t > 0) \). Let \( C_\Phi \) be a set of critical points of the phase function: \( C_\Phi = \)}
\( \{ (x, s) : \frac{\partial \Phi}{\partial x} = 0 \} \). We have a natural imbedding of \( C_\Phi \) to \( T^*_0 S \) by map: \( \Phi : (x, s) \mapsto (x, d_s \Phi(x, s)) \). Notes that \( \Lambda = \hat{\Phi}(C_\Phi) \) is a Lagrangian manifold with respect to canonical symplectic structure of \( T^*_0 S \). Denote by \( \Lambda_c \) a quotient manifold, i.e. \( \Lambda_c = \Lambda / \mathbb{R}_+ \). Following [D] and [Pa] we call it a contact Lagrangian manifold.

Let \( L : \Lambda_c \mapsto S \) be associated to the Lagrangian map. Denote by \( J(\Phi) \) Jacobian of the map.

Now, consider the oscillatory integral:

\[
I(t, s) = \int_S e^{i t \Phi(x, s)} a(x, s) |J(\Phi)|^q dx
\]

(3.1)

The following proposition is needed for the sequel.

**Proposition 3.1.** Let \( (x^0, s^0) \in \Lambda_c \) be a fixed point and the phase function \( \Phi(x, s) \) be \( \mathbb{R}_+ \) equivalent (Lagrangian equivalent) to \( \mathbb{R}_+ \) versal deformation of elliptic singularity in some neighbourhood of the point. Then there exists a neighbourhood \( X^0 \times S^0 \) of the point \( (x^0, s^0) \) such that for any amplitude function \( a \in C^\infty_0 (X^0 \times S^0) \) the following estimation holds, whenever \( q > 1 \).

It is well known that the elliptic singularities up to diffeomorphism have a type \( A_k, (k \geq 1), \) \( D_k (k \geq 4), \) \( E_k (6 \leq k \leq 8) \) [AGV].

First, we consider the estimation of damping oscillatory integrals with phase function having singularities of type \( A_k \). These singularities have a codimension one. For this reason, we consider one dimensional oscillatory integrals.

Let \( p(x, s) \) be \( \mathbb{R}_+ \) versal deformation of \( A_n \) type singularity:

\[
p(x, s) = \frac{x^{n+1}}{n+1} + s_1 \frac{x^n}{n-1} + \cdots + s_{n-1} x
\]

In this case the oscillatory integral (3.1) is reduced to the following:

\[
I(t, s) = \int_\mathbb{R} e^{i t p(x, s)} a(x, s) |p''(x, s)|^{\frac{1}{2}} dx,
\]

(3.2)

where \( a \in C^\infty_0 (\mathbb{R} \times \mathbb{R}^{n-1}) \).

First we prove the following Lemma.

**Lemma 3.2.** There exists a constant \( c \) such that for the integral (3.2) the following estimation holds.

\[
|I(x, t)| \leq \frac{c}{|t|^\frac{1}{2}}
\]

(3.3)

**Proof.** Lemma 3.2 is proved by the induction method over \( n \). For the case of \( n = 1 \) the required estimation easily follows from the Van der Corpute Lemma (see [AKCh], [Va], [Il] ). Let \( n = 2 \). In this case the phase function has the form: \( p(x, s) = \frac{x^3}{3} + s_1 x \), Note that if \( s_1 = 0 \)
then the estimation (3.3) follows from Van der Corpute type estimation (see [E]). Let $s_1 \neq 0$. We use a change of variables $x \mapsto |s_1|^{1/2}x$ and have

$$I(t, s) = |s_1|^{3/4} \int_{S} e^{it|s_1|^{3/2}p_1(x, s)} a(|s_1|^{1/2}x, s)|x|^{1/2}dx,$$

where $p_1(x, s) = \frac{x^2}{3} + \text{sgn}(s_1)x$.

Note that if $\text{sgn}(s_1) = 1$ then the phase function $p_1(x, s)$ has no critical points. For the sake of being definite we consider the case $\text{sgn}(s_1) = -1$. In this case the phase function has only two critical points $x = \pm 1$. Let us consider a partition of unity $\{\varphi_1, \varphi_2, \varphi_3\}$ such that $\text{supp}\varphi_1 \subset (-2, 2), \text{ supp}\varphi_2 \subset (1, \infty)$ and $\text{supp}\varphi_3 \subset (-\infty, -1)$.

By using the partition of unity the oscillatory integral $I(t, s)$ is represented as the sum of three integrals:

$$I_k(t, s) = |s_1|^{3/4} \int_{S} e^{it|s_1|^{3/2}p_1(x, s)} a(|s_1|^{1/2}x, s)|x|^{1/2}\varphi_k(x)dx,$$

where $k = 1, 2, 3$.

Let us consider the estimation of $I_3(t, s)$. In the integral we use a change of variables $x \mapsto x^{2/3}$ and obtain:

$$I_3(t, s) = |s_1|^{3/4} \int_{1}^{\infty} e^{it|s_1|^{3/2}p_2(x, s)} a(|s_1|^{1/2}x^{2/3}, s)|x^{2/3}\varphi_3(x^{2/3})dx,$$

where $p_2(x, s) = x^2 - x^{4/3}$.

It is easy to see that $|p''_2(x, s)| \geq \frac{2}{3}$ for any $x \geq 1$. By using the generalized Van der Corput Lemma we have (see [I1]):

$$|I_3(t, s)| \leq \frac{c||a(., s)||_V}{|t|^{1/2}}.$$

The oscillatory integral $I_2(t, s)$ is estimated by the analogy and one has the same estimation.

Finally, consider the estimation of the integral $I_1(t, s)$. Note that the phase function of the oscillatory integral has only non-degenerate critical points. Therefore we can use the Generalized Van der Corpute estimation and get the required inequality (3.3) for the case of $n = 2$ (see [AKCh], [I1]).

Now, assume that $n > 2$ and the assertion of Lemma 2.2 have proved for all cases $k \leq n - 1$. We follow the method of [I3]. Let $\rho = |s_1|^{n+1} + \cdots + |s_{n-1}|^{n+1}$ be a quasidistance and $K_S = \{s \in \mathbb{R}^{n-1} : \rho(s) = 1\}$ be a quasisphere. First, we consider the estimation of integral (3.2) for some neighbourhood of a fixed point $s^0 \in K_S$. If $s = s^0$ is a fixed point then the phase function $p(x, s^0)$ has critical points $x_0^0$ of the type $A_{\nu}$, where $\nu \leq n - 1$ and $\nu = 1, \ldots, q$. Moreover, if $S(s^0)$ is a bounded neighbourhood of the point $s^0$ then the set of critical points of the phase function $p(x, s)$ is contained in some compact set $[-M, M]$, whenever $s \in S(s^0)$.

Consider a partition of unity $\{\varphi_1, \varphi_2\}$ associated to the covering $(-M - 1, M + 1) \cap (\mathbb{R} \setminus [-M, M])$ of $\mathbb{R}$. With the help of this partition of unity the oscillatory integral (3.2) can be represented as the sum of two integrals:

$$I_k(t, s) = \int_{\mathbb{R}} e^{itp(x, s)} a(x, s)|p''(x, s)|^{1/2}\varphi_k(x)dx; \quad k = 1, 2.$$
First, consider the estimation of the integral $I_2(t, s)$. We assume that $M$ is sufficiently large so that the function $\left(1 + \frac{\frac{n-2}{2}s_1}{x^2} + \cdots + \frac{s_{n-1}}{x^{n-1}}\right)$ and its derivatives are uniformly bounded. Let us write the integral $I_2(t, s)$ as the sum of two integrals:

$$I_2(t, s) = I_{21}(t, s) + I_{22}(t, s),$$

where

$$I_{21}(t, s) = \int_{-\infty}^{M} e^{tp(x, s)} a(x, s) x^{\frac{n-2}{2}} \left(1 + \frac{\frac{n-2}{2}s_1}{x^2} + \cdots + \frac{s_{n-1}}{x^{n-1}}\right) \varphi_2(x) dx$$

and

$$I_{22}(t, s) = \int_{M}^{\infty} e^{tp(x, s)} a(x, s) x^{\frac{n-2}{2}} \left(1 + \frac{\frac{n-2}{2}s_1}{x^2} + \cdots + \frac{s_{n-1}}{x^{n-1}}\right) \varphi_2(x) dx.$$

Now, consider the estimation of $I_{22}(t, s)$. In this integral we use a change of variables $x \mapsto x^{\frac{n-2}{2(n+1)}}$ and obtain

$$I_{22}(t, s) = \int_{M^{(n+1)/2}} e^{tp(x^{\frac{n-2}{2(n+1)}})} a_1(x, s) dx,$$

where

$$a_1(x, s) = a(x^{\frac{n-2}{4(n+1)}}, s) \left(1 + \frac{\frac{n-2}{4}s_1}{x^{(n-1)/2}} + \cdots + \frac{s_{n-1}}{x^{(n-1)/(n+1)}}\right) \varphi_2(x^{\frac{n-2}{2(n+1)}}).$$

It is readily seen that $|p''_n(x^{(2/(n+1))}, s)| \geq c > 0$ for any $x \in (M^{(n+1)/2}, \infty)$ and $s \in S^0$. In addition, we obviously have $\text{Var}[a_1(x, s)] \leq c\|a(., s)\|_V$. Thus, by using the generalized Van der Corput Lemma we obtain:

$$|I_{22}(t, s)| \leq c\|a(., s)\|_V |t|^{\frac{1}{2}}.$$

In the same way it follows the required estimation of the integral $I_{21}(t, s)$.

Furthermore, we consider the estimation of the oscillatory integral $I_1(t, s)$. It is well known that the phase function $p(x, s)$ is a versal deformation of a singularity $A_{l_\nu}$ in some neighbourhood of the point $(x_{\nu,0}, s^0)$. Hence, there exists a neighbourhood $(X(x_{\nu,0}), S(s^0))$ of the point $(x_{\nu,0}, s^0)$ and a deformation of the diffeomorphism

$$F : X(x_{\nu,0}) \mapsto X(0),$$

such that the phase function $p(x, s)$ can be reduced to the form

$$F^*p(x, s) = x^{l_{\nu}+1} + \sigma_1 x^{l_{\nu}-1} + \cdots + \sigma_{l_{\nu}-1} x + \sigma_{l_{\nu}},$$

(3.4)

where $\sigma_1, \ldots, \sigma_{l_{\nu}}$ are analytic in some neighbourhood of the point $s^0$, $\sigma_j(s^0) = 0$, $j = 1, \ldots, l_{\nu} - 1$, and the map

$$F_1 : (\mathbb{R}^k, s^0) \mapsto (\mathbb{R}^{l_{\nu}-1}, 0),$$

has the maximal rank at the point $s = s^0$, where $F_{ij}(s) = \sigma_j(s)$, $j = 1, 2, \ldots, l_{\nu} - 1$. 

9
Let $X_0 \cup \bigcup_{\nu=0}^q X(x_0^\nu)$ be a covering of $[-1 - M, M + 1]$ and let $\{h_\nu\}_{\nu=0}^q$ be a partition of unity subordinate to this covering. By using this partition of unity the oscillatory integral $I_1(t, s)$ can be represented as the sum of the following $q + 1$ integrals:

$$I_{1\nu}(t, s) = \int_\mathbb{R} e^{itp(x, s)} a(x, s) |p''(x, s)|^{\frac{1}{2}} \varphi_\nu(x) dx, \quad \nu = 0, 1, \ldots, q.$$ 

Note that the support of the amplitude function of the integral $I_{10}(t, s)$ contains no critical points of the phase function; hence for the integral we have:

$$|I_{10}(t, s)| \leq c\|a(., s)\|_V |t|^{-\frac{1}{2}}.$$

Now, consider the estimation of the integral $I_{1\nu}(t, s)$ for the case of $\nu \geq 1$. We already know that the phase function $p(x, s)$ of the integral $I_{1\nu}(t, s)$ is reduced to the form (3.4) on the support of $h_\nu$. By using covariant property of the Jacobian of a Lagrangian map we obtain the same integral as (3.2) with $l_\nu \leq n - 1$. Hence, by the induction hypothesis we obtain:

$$|I_{1\nu}(t, s)| = \left| \int_\mathbb{R} F^* (a h_\nu)(x, s) (F^* p)'(x, s) \frac{1}{2} e^{itF^* p(x, s)} dx \right| \leq c\|a(., s)\|_V |t|^{-\frac{1}{2}},$$

where $\nu = 1, 2, \ldots, q$.

We can repeat this argument for each point of the quasisphere $K_S$. Since $K_S$ is a compact set, there exists a neighborhood

$$U_\varepsilon = \{1 - \varepsilon < \rho(s) < 1 + \varepsilon\}$$

of the quasisphere (where $\varepsilon$ is a positive number) such that for the integral (3.2) we have the estimation (3.3). It should be remarked that the estimation (3.3) can be obtained as above (as the integrals $I_{21}(t, s)$ and $I_{22}(t, s)$ were estimated) for the case of $s = 0$ (see [E]). For the same reason, we consider the bound of the integral (3.2) for $s \in \mathbb{R}^{n-1} \setminus \{0\}$. We make a change of variables $x \mapsto \rho^{\frac{1}{n+1}} x$ in the oscillatory integral (3.2) and obtain:

$$I(t, s) = \rho^{\frac{1}{2}} \int_\mathbb{R} e^{itp(x, \xi)} a(\rho^{\frac{1}{n+1}} x, s) |p''(x, \xi)|^{\frac{1}{2}} dx,$$

where $\xi_k = \frac{x_k}{\rho^{\frac{1}{n+1}}}$, therefore $\xi \in K_S$.

As was proved the integral $I(t, s)$ satisfies the inequality

$$|I(t, s)| \leq c\|a(\rho^{\frac{1}{n+1}} ., s)\|_V |t|^{-\frac{1}{2}}.$$ 

Indeed, we have $\|a(\rho^{\frac{1}{n+1}} ., s)\|_V = \|a(., s)\|_V$. This completes the proof of Lemma 3.2.

It is well known that the elliptic singularities $D_k (k \geq 4)$ and $E_k (6 \leq k \leq 8)$ have codimension two (see [AGV]). In this case we consider two-dimensional oscillatory integrals. Let $p(x, s)$ be a deformation of the elliptic singularities of the form:

$$p(x, s) = f(x) + \sum_{j=1}^k s_j e_j(x),$$

(3.5)
where \( f \) is a quasihomogeneous polynomial with singularity at the origin, \( \{ e_j \}^k_{j=1} \) are basis monomials of the local algebra \( Q_f = M/\mathfrak{m} \) of the singularity, \( \mathfrak{m} \) is the maximal ideal of the algebra of germs of smooth functions at zero (see [AGV]). In this case, we consider the following oscillatory integral:

\[
I(t,s) = \int_{\mathbb{R}^2} e^{itp(x,s)}a(x,s)|\text{Hess}p(x,s)|^{1/2}dx.
\] (3.6)

The next Lemma is an analogue of Lemma 3.2.

**Lemma 3.3.** If \( p(x,s) \) is the phase function defined by (3.5), then for the oscillatory integral (3.6) the following inequality:

\[
|I(t,s)| \leq c\|a(.,s)\|_{C^2} / |t|
\]

holds, whenever \( a \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^k) \).

Lemma 3.3 can be proved by the analogy of Lemma 3.2 (see [I3]).

**Proof of Proposition 3.1.** If the phase function \( \Phi(x,s) \) is equivalent to \( \mathbb{R}_+ \) versal deformation of the elliptic singularities. Then up to diffeomorphism we can write

\[
\Phi(x,s) = p(x_1,x_2,s) + q(x''),
\]

where \( p(x_1,x_2,s) \) is a polynomial defined by (3.5) and \( q(x'') \) is a non-degenerate quadratic form of variables \( (x_2, \ldots, x_n) \).

Note that the following identity

\[
J(\Phi) = \varphi(x,s)\text{Hess}p(x_1,x_2,s)
\]

holds, where \( \varphi(x,s) \) is some nonvanishing function. Therefore the integral (3.1) is reduced to the following

\[
I(t,s) = \int_{\mathbb{R}^{n-2}} e^{itq(x'')}dx'' \int_{\mathbb{R}^2} e^{itp(x,s)}a(x,s)|\text{Hess}p(x,s)|^{1/2}dx.
\]

Finally, by using the stationary phase method (see [Ho]) or the Van der Corput type estimation and by using Lemma 3.3 we get the proof of Proposition 3.1.

**§4. Estimations of one-dimensional oscillatory integrals.**

Let \( f(x,s') \in C^\infty(\mathbb{R} \times \mathbb{R}^{n-1}) \) be a smooth function and \( a(x,s) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n) \) be an amplitude function. Consider an oscillatory integral of the form:

\[
I(t,s) = \int_{\mathbb{R}} a(x,s)|f''(x,s')|^{1/2}e^{it(f(x,s') + s_1x)}dx.
\] (4.1)

The next Lemma is needed for the sequel.

**Lemma 4.1.** There exists a constant \( c \) such that for the integral \( I(t,s) \) the following inequality

\[
|I(t,s)| \leq c\text{Var}a(.,s)|t|^{1/2}
\] (4.2)
holds, where $\text{Var}[a(., s)]$ is a total variation of the amplitude function $a$ on $\mathbb{R}$.

**Proof of Lemma 4.1.** Let $s' \in \mathbb{R}^{n-1}$ be a fixed point. We prove that the constant $c$ in the estimation (4.2) depends only on $C^3$ norm of the function $f$. We use the method of Sogge-Stein [SS1] based on Vitaly type covering Lemma. Let $\text{supp}(a(., s)) \subset (a, b)$ and let $V = (a, b) \cap \{x : f''(x, s') \neq 0\}$. We represent the set $V$ as a disjoint union of intervals $V = \bigcup_k (\alpha_k, \beta_k)$. Let $(\alpha_k, \beta_k) \equiv (\alpha, \beta)$ be one of the intervals. Note that $f''(x, s') \neq 0$ for any $x \in (\alpha, \beta)$. For the sake of being definite suppose that $f''(x, s') > 0$ whenever $x \in (\alpha, \beta)$. We consider the following integral:

$$I_{\alpha\beta}(t, s) = \int_\alpha^\beta a(x, s)|f''(x, s')|^{3/2} e^{it(f(x, s') + s_1 x)} dx.$$ 

Arguing as in [SS1] we can find a collection of intervals $[\alpha^k, \beta^k]$ with the following properties:

(I) $(\alpha^k, \beta^k) \cap (\alpha^l, \beta^l) = \emptyset$ for any $k \neq l$;

(II) $(\alpha, \beta) \subset \bigcup_k [\alpha^k, \beta^k]$;

(III) $\beta - \alpha = \sum_k (\beta^k - \alpha^k)$;

(IV) $\beta^k - \alpha^k < c f''(\frac{\alpha^k + \beta^k}{2}, s')$, and for any $x \in (\alpha^k, \beta^k)$ the following inequality holds.

$$c_1 f''(x, s') \leq f''\left(\frac{\beta^k + \alpha^k}{2}, s'\right) \leq c_2 f''(x, s')$$

Now, we consider the estimation of the oscillatory integral defined by:

$$I_k(t, s) = \int_{\alpha^k}^{\beta^k} a(x, s)|f''(x, s')|^{3/2} e^{it(f(x, s') + s_1 x)} dx.$$ 

By using translation we reduce the last integral to the form:

$$I_k(t, s) = \int_{|x| < c|\lambda|} a(x + x_k, s)(f''(x, s'))^{3/2} e^{it(f(x, s') + s_1 x)} dx,$$

where $x_k = \frac{\alpha^k + \beta^k}{2}$, and $\lambda = f''(\frac{\alpha^k + \beta^k}{2}, s')$.

Now, we use the following variant of the Van der Corpute type estimation:

$$\left| \int_a^b \phi(x)e^{iF(x)} dx \right| \leq \frac{c\|\phi\|_V}{(\min_{x \in [a, b]} |F''(x)|)^{1/2}},$$

where $c$ is an absolute constant, $\|\phi\|_V = |\phi(a)| + \text{Var}_a^b[\phi]$ is a variation norm (see [Va] [AKCh], I1]). Note that the inequality $\|\phi \psi\|_V \leq 2\|\phi\|_V \|\psi\|_V$ holds. By using these inequalities we obtain:

$$|I_k(t, s)| \leq \frac{c\|a(., s)\|_V(\beta^k - \alpha^k)}{|t|^{1/2}},$$

where $c$ is a constant depending only on the $C^3$ norm of the function $f(., s')$ which is uniformly bounded with respect to $s'$. Finally, by summing these inequalities over all $k$ we get:

$$|I_{\alpha\beta}(t, s)| \leq \frac{c\|a(., s)\|_V(\beta - \alpha)}{|t|^{1/2}}.$$
Note that the inequality \( \|a(.,s)\| \leq Var_0^b[a(.,s)] \) is fulfilled, since \( a(.,s) \) is a smooth function with compact support. Arguing as above we arrive to proof of Lemma 4.1.

Now, we will formulate the Lemma on the Weierstrass-Malgrange preparation theorem (see [Hi], [M], [I4]). Let \( f : (\mathbb{R} \times \mathbb{R}^n, 0) \mapsto (\mathbb{R}, 0) \) be a real analytic function.

**Lemma 4.2.** There exists a real analytic manifold \( Y \) and a mapping \( \pi : Y \mapsto \mathbb{R}^n \) which is the composite of a finite sequence of blowing-up with smooth analytic centers such that for every point \( y^0 \in Y \) there exists a chart \( (y_1, y_2, \ldots, y_n) \) in which we have

\[
(\pi^*)(f)(x, y) = y_1^{k_1}y_2^{k_2} \ldots y_n^{k_n}g(x, y)p(x, y),
\]

where \( \pi^*(x, y) = (x, \pi(y)) \), \( g(0, y^0) \neq 0 \) is a real analytic function and \( p(x, y) \) is a unitary pseudopolynomial.

Let \( f(x, s) \) be a real analytic function defined as above and \( a(x, s) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n) \) be an amplitude function with compact support. Consider a one-dimensional oscillatory integral defined by:

\[
I(t, s) = \int_{\mathbb{R}} e^{it(f(x,s') + s_1x)}|f^\alpha(x, s')|^\frac{1}{2}a(x, s)dx
\]

**Lemma 4.3.** There exists a neighbourhood \( U = U_1 \times U_2 \subset \mathbb{R} \times \mathbb{R}^n \) of zero such that for the integral (4.3) the following estimation

\[
|I(t, s)| \leq \frac{c}{|t|^{1/2}}
\]

holds, whenever \( a \in C_0^\infty(U) \).

**Proof of Lemma 4.3.** By using Lemma 4.2 we may assume that

\[
f(x, s') = s_2^{m_2} \ldots s_n^{m_n}b(x, s')p(x, s'),
\]

where \( b(x, s') \) is some non-vanishing real analytic function and

\[
p(x, s') = x^n + g_1(s')x^{n-2} + \cdots + g_n(s')
\]

is a pseudopolynomial.

For the parameters \( s \) consider two-cases:

1-case: \( |s_2^{m_2} \ldots s_n^{m_n}| < M|s_1| \), where \( M \) is a fixed positive number. If \( U \) is a sufficiently small neighbourhood of zero then by using integration by part we obtain:

\[
|I(t, s)| \leq \frac{c|s_2^{m_2} \ldots s_n^{m_n}|^{1/2}\|a(., s)\|_V}{|ts_1|^{1/2}} \leq \frac{c\|a(., s)\|_V}{|t|^{1/2}}.
\]

2-case: \( |s_1| < \varepsilon|s_2^{m_2} \ldots s_n^{m_n}| \), where \( \varepsilon \) is a sufficiently small positive. In this case, we use the following change of variables:

\[
\sigma_1 = s_1s_2^{-m_2} \ldots s_n^{-m_n}, \quad \sigma' = s'
\]
and have
\[ I(t, s) = s_2^{m_2} \cdots s_n^{m_n} \int_{\mathbb{R}} e^{i t x_2^{m_2} \cdots x_n^{m_n} F_t(x, \sigma)} |f''(x, \sigma')|^\frac{1}{2} a(x, s) \, dx, \]
where \( F(x, s) = b(x, \sigma') p(x, \sigma') + \sigma_1 x \), \( f(x, s') = b(x, s') p(x, s') \). Now, by using Proposition 3.1 we get (4.4). This completes the proof of Lemma 4.3.

§5. Proof of main results.

Let \( f(x) \) be a smooth function in some neighbourhood of zero and \( f(0) = 0, \quad \nabla f(0) = 0 \). The function \( f \) (after a possible rotation of the axes) is represented by the following expression:
\[ f(x) = \sum_{j=1}^{n} \lambda_j x_j^2 + \sum_{k,l,m=1}^{n} x_k x_l x_m H_{klm}(x), \]
Moreover, we suppose that \( |\lambda_1| \geq |\lambda_k| > 0, \) for \( k = 2, \ldots, n \). Consider the phase function defined by \( \Phi_1(x, s) = f(x) + (s, x) \). Let us use a change of variables \( x \mapsto \lambda_1 x \) and obtain
\[ \Phi_1(\lambda_1 x, s) = \lambda_1^3 \Phi(x, \sigma), \]
where
\[ \Phi(x, \sigma) = \sum_{j=1}^{n} \frac{\lambda_j}{\lambda_1} x_j^2 + \sum_{k,l,m=1}^{n} x_k x_l x_m H_{klm}(\lambda_1) + (\sigma, x), \quad \sigma = \frac{s}{\lambda_1^2}. \]

Now, we consider the proof of some auxiliary Lemma.

**Lemma 5.1.** There exist positive numbers \( c \) and \( \varepsilon \) such that the following statements hold:
(i) if \( |\sigma_1| > \varepsilon \) then the following inequality
\[ \left| \frac{\partial \Phi(x)}{\partial x_1} \right| \geq \frac{1}{2} |\sigma_1| \]
holds for any \( |x| < c \);
(ii) if \( |\sigma_1| \leq \varepsilon \) then there exists a deformation of the diffeomorphism
\[ H : \{|x_1| < 2c\} \times \{|x'| < c\} \mapsto U_1 \times \{|x'| < c\} \]
of the form \( H_1 = H_1(x, \sigma_1), \quad H_t = x_t, \quad l = \frac{2}{n}, \quad H(x, 0) = x \) such that
\[ (H^* \Phi)(x) = x_1^2 + \Phi_1(x', \sigma_1) + \sigma' x', \]
where \( \Phi_1 \in \mathfrak{m}^2, \quad \mathfrak{m} \) is a maximal ideal of the local ring \( C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}) \) at the origin and \( U_1 \) is a neighbourhood of zero in \( \mathbb{R} \).

**Proof of Lemma 5.1.** The proof of (i) is trivial. We shall prove the assertion (ii). Following [D] and [Pa] we consider an equation \( \frac{\partial \Phi(x, \sigma)}{\partial x_1} = 0 \). If \( c \) and \( \varepsilon \) are sufficiently small positive numbers and \( |x| < c, \quad |\sigma_1| < \varepsilon \) then the last equation has a unique solution with respect to \( x_1 \) of the form
\[ x_1(x', \sigma_1) = -\frac{\sigma_1}{2} + \sum_{l,m=2}^{n} x_l x_m H_{lm}(x, \lambda_1). \]
Note that the $C^k$ norm of $H_{lm}(x, \lambda_1)$ is uniformly bounded with respect to $\lambda_1$ by $C^{k+4}$ norm of the phase function $\Phi(x, \sigma)$.

Now, we use a change of variables

$$y_1 = x_1 + \frac{\sigma_1}{2} - \sum_{l,m=2}^{n} x_l x_m H_{lm}(x, \lambda_1), \quad y' = x'.$$

Then the phase function $\Phi(x, \sigma)$ is reduced to the form

$$\Phi(x, \sigma) = y_1^2 + \frac{1}{2} \sigma_1^2 + \sum_{l=2}^{n} \frac{\lambda_l}{\lambda_1} y_l^2 + \phi_2(y', \sigma_1) + (\sigma', y'),$$

where $\phi_1$ is a smooth function $\phi_1(0, 0) = 1$ and $\phi_2 \in \mathcal{M}^2$ is a uniformly bounded function.

Finally, we apply a change of variables $z_1 = y_1(\phi_1(y, \sigma_1))^{1/2}$, $z' = y'$ and have

$$\Phi_1(x, \sigma) = z_1^2 + \frac{1}{2} \sigma_1^2 + \sum_{l=2}^{n} \frac{\lambda_l}{\lambda_1} z_l^2 + \phi_2(y', \sigma_1) + \sigma'y',$$

The last equality proves Lemma 5.1.

**Remark.** Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth hypersurface. Then there exists a natural imbedding $i : \Sigma \hookrightarrow T^* S^n$, where $T^* S^n$ is a cotangent bundle of the unit sphere. The $i(\Sigma)$ is a Lagrangian manifold. The Gaussian curvature is just a Jacobian of the associated Lagrangian map (see [Pa]). If $k$ of $n$ principal curvatures are non-vanished then a generating function $F$ locally exists and has the form

$$F(x^{(k)}, s) = f(x^{(k)}, s_{n-k+1}, \ldots, s_n) + s^{(k)} x^{(k)},$$

where $x^{(k)} = (x_1, \ldots, x_{n-k})$, $s^{(k)} = (s_1, \ldots, s_{n-k})$ (see [AGV]). So, it is easy to see that the Gaussian curvature $K(X)$ is expressed by the form $K(X) = \psi(x) det \text{Hess} f(x^{(k)}, s^{(k)})$ where $\psi$ is some non-vanishing function.

Let $S \subset \mathbb{R}^3$ be a smooth hypersurface and for each $x \in S$ we let $K(x)$ denote its Gaussian curvature. We write $d\sigma$ for the induced Lebesgue measure on $S$. Consider the following integral:

$$I(r, \xi) = \int_S e^{ix\xi} |K(x)|^{3/2} \psi(X) d\sigma,$$

where $\xi \in S^2$, $S^2$ is a unit sphere centered at zero, $r \in \mathbb{R}^+$ and $\psi$ is a smooth function with little support.

**Proposition 5.2.** The following inequality holds:

$$|I(r, \xi)| \leq \frac{C}{r}, \text{ whenever } \psi \in C_0^\infty(S).$$

**Proof of Proposition 5.2.** Without loss of generality we can assume that $S$ contains the point $0 \in \mathbb{R}^3$ and the function $\psi$ is supported in a small neighbourhood of zero. Moreover, we may suppose that the surface $S$ is represented as a graph of some function $f(x)$ so that
\( f(0) = 0, \ \nabla f(0) = 0. \) Thus \( S = \{(x, f(x)) : x \in U\}. \) In this case the Gaussian curvature is represented by [DFN]:

\[
K(X) = \frac{\det \text{Hess} f(x)}{(1 + |\nabla f(x)|^2)^2}
\]

We can assume that \( \frac{|\xi'|_{\xi}}{|\xi|} \) is sufficiently small, otherwise if \( U \) is sufficiently small and \( \frac{|\xi'|_{\xi}}{|\xi|} > \varepsilon > 0 \) then we can apply integration by part and have an estimation : \( |I(r, \xi)| \leq \frac{\varepsilon}{r} \). So it is sufficient to estimate the following integral:

\[
I(t, s) = \int_{\mathbb{R}^2} e^{itF(x, s)}|\det \text{Hess} f(x)|^{3/2}\psi_1(x)dx,
\]

where \( F(x, s) = f(x_1, x_2) + s_1x_1 + s_2x_2. \)

Now, we consider covering of the set

\[
U_0 = \{x \in U : \det \text{Hess} f(x) \neq 0\}.
\]

Let \( \{B_k\} \) be collection of balls chosen from Lemma 2.2, so that \( U_0 \subset \bigcup B'_k \), where \( B'_k \) has the same center as \( B_k \) with radius \( \frac{3c}{c_3^2} \). With this family of balls now fixed we can also construct in the usual way a partition of unity [SS1]:

\[
1 = \sum_j \psi_j(x) \quad \text{for any} \quad x \in U_0
\]

with \( \psi_j \) supported in \( B'_j \) and

\[
\left| \left( \frac{\partial}{\partial x} \right)^\alpha \psi_j \right| \leq c(\text{radius}B_j)^{-|\alpha|}.
\]

By the partition of unity the integral \( I(t, s) \) represents the sum of integrals:

\[
I_j(t, s) = \int_{\mathbb{R}^2} e^{itF(x, s)}|\det \text{Hess} h(x)|^{3/2}\psi_1(x)\psi_j(x)dx.
\]

Let us consider the estimation of one of the integrals \( I_j(\lambda, s) \). Following [SS1] for each ball \( B'_j \equiv B \) choose a coordinate system in \( \mathbb{R}^3 \) so that the center \( x^j \) of \( B \equiv B_j \) is the origin, and the tangent plane of \( S \) at \( x^j \) is given by the hyperplane \( x_3 = 0 \). Then if we choose \( c \) sufficiently small, the portion of \( S \) in \( B \) is realized as a graph \( x_3 = f(x_1, x_2) \), with \( f(0) = 0, \nabla f(0) = 0 \). Also the support of \( \psi_j \) corresponds to a portion of the graph where \( (x_1, x_2) \in B \). The function \( f(x) \) (after a possible rotation of the axes) represents:

\[
f(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \sum_{k, l, m=1}^{2} x_kx_lx_mH_{klm}(x)
\]

in the neighbourhood \( B \) of zero.

Note that \( |\lambda_1| \geq |\lambda_2| \) and \( B \) has a radius \( c'|\lambda_1| \), where \( c' \) is a sufficiently small positive. Let us use a change of variables \( x \mapsto \lambda_1 x \) then we have:

\[
I_j(t, s) = \lambda_1^3 \int_{|x|<c'} e^{it\lambda_1^2F_1(x, \sigma)}|\det \text{Hess} f(\lambda_1 x)|^{3/2}\psi_1(\lambda_1 x)\psi_j(\lambda_1 x)dx,
\]
where
\[ F_1(x, \sigma) = x_1^2 + \frac{\lambda_2}{\lambda_1} x_2^2 + \sum_{k,l,m=1}^{2} x_k x_l x_m H_{klm}(\lambda_1 x) + \sigma_1 x_1 + \sigma_2 x_2 \]
and \( \sigma_1 = \frac{s_1}{\lambda_1^2} \) \( \sigma_1 = \frac{s_1}{\lambda_1} \). Note that \( \text{det Hess } f(\lambda_1 x) = \lambda_1^2 h(\lambda_1, x) \) and \( h(\lambda_1, x) \) is some uniformly bounded function in \( C^n \) for any \( n \).

Now, for the parameter \( \sigma_1 \) consider two cases:

1-case: \( |\sigma_1| \geq \varepsilon \), where \( \varepsilon \) and \( \varepsilon \) are defined by Lemma 5.1. Then by using integration by part we obtain:

\[ |I_j(t, s)| \leq \frac{C\lambda_1^2}{|t|} \] (5.1)

2-case: \( |\sigma_1| < \varepsilon \). Then by using Lemma 5.1 we reduce the integral \( I_j(t, s) \) to the form:

\[ I_j(t, s) = \lambda_1^5 \int_{\mathbb{R}} e^{it\lambda_1^2 x_1^2} dx_1 \int_{\mathbb{R}} e^{it\lambda_1^2 (f(x_2, \sigma_1) + \sigma_2 x_2)} |f''(x)(x_2, \sigma_1)|^{3/2} a(x, \lambda_1, \sigma) dx_2. \]

Note that the amplitude function \( a \) and the phase function \( f(x_2, \sigma_1) \) are uniformly bounded functions depending only on \( c \) and the \( C^3 \) norm of the function \( f \). Also we have used the covariant property of Gaussian curvature. Now, by using Lemma 4.1 we obtain the estimation of the form (5.1). Finally, by summing this estimation over all numbers \( j \) we arrive to the proof of Proposition 5.2.

Let \( S \) be a hypersurface which smoothly depends on parameters \( \sigma \) and let \( I(\sigma, \xi, r) \) be an associated oscillatory integral.

**Corollary 5.3.** For the integral \( I(\sigma, \xi, r) \) the following estimation
\[ |I(\sigma, \xi, r)| \leq \frac{C}{r} \]
holds. Moreover, the constant \( C \) does not depend on the additional parameters \( \sigma \).

**Proposition 5.4.** Let \( S \subset \mathbb{R}^{n+1} \) be a smooth hypersurface. If \( n \geq 3 \) and \( q > \frac{n}{2} \), then the following inequality:
\[ | \int_{S} e^{i(x, \xi)} |K(X)|^q \psi(X) d\sigma(X)| \leq \frac{c}{|\xi|^{\frac{n}{2}}} \] (5.2)
holds, whenever \( \psi \in C_0^\infty(S) \) and \( \psi \) has sufficiently small support.

**Proof of Proposition 5.4.** Arguing as above we assume that the surface \( S \) contains the point \( 0 \in \mathbb{R}^{n+1} \) and the function \( \psi \) is supported in a small neighbourhood of zero. Moreover, we may suppose that the surface \( S \) is represented as a graph of some function \( f(x) \) so that \( f(0) = 0, \ \nabla f(0) = 0 \). In this case the Gaussian curvature is represented by (see [DFN]):
\[ K(X) = \frac{\text{det Hess } f(x)}{(1 + |\nabla f(x)|^2)^2}. \]

Without loss of generality we may assume that the point \( \xi \in S^n \) ranges in a small neighbourhood of the points \((0, \ldots, \pm 1)\) otherwise if the neighbourhood of zero sufficiently small and \( |\xi|_{\mathbb{R}^{n+1}} > \varepsilon > 0 \) then we can apply \( \frac{n}{2} \) times integration by part and have the estimation:
\[ | \int_{S} e^{i(x, \xi)} |K(X)|^q \psi(X) d\sigma(X)| \leq \frac{c}{|\xi|^{\frac{n}{2}}} \].
whenever $q > \frac{n}{2}$, for the case of when $n$ is an even. If $n$ is an odd; then arguing as [CM] we use $[\frac{n}{2}]$ times integration by part. Then we will use the fact that $|K(X)|^{\frac{1}{2}+1}$ belongs to $B_{1/2}^{\infty}$ Besov space [Pe] and obtain the required estimation. So, it is enough to estimate the following integral:

$$I(t, s) = \int_{\mathbb{R}^n} e^{itF[x,s]}|\det \text{Hess } f(x)|^q \psi_1(x)dx,$$

where $F(x,s) = f(x) + (s,x)$.

Proposition 5.4 is proved by the induction method over $n$. Let $n \geq 3$. We consider the covering of the set $U_0 = \{x \in U : \det \text{Hess } f(x) \neq 0\}$.

Let $\{B_k\}$ be a collection of balls chosen from Lemma 2.2, so that $U_0 \subset \bigcup B'_k$, where $B'_k$ has the same center as $B_k$ with radius $c' = \frac{3c}{c_0}$ and $U_0 \subset \bigcup U B'_k$. Arguing as above denote by $\{\psi_k\}$ a partition of unity. By this partition of unity the integral $I(t,s)$ represents the sum of integrals:

$$I_j(t,s) = \int_{\mathbb{R}^n} e^{itF(x,s)}|\det \text{Hess } f(x)|^q \psi_1(x)\psi_j(x)dx.$$

Let us consider the estimation of one of the integrals $I_j(t,s)$. Following [SS1] for each ball $B \equiv B'_j$ choose a coordinate system in $\mathbb{R}^{n+1}$ so that the center $x^j$ of $B$ is the origin, and the tangent plane of $S$ at $x^j$ is given by the equation $x_{n+1} = 0$. Then if we choose $c$ sufficiently small, the portion of $S$ in $B$ is realized as a graph $x_{n+1} = f(x_1,x_2,\ldots,x_n)$, with $f(0) = 0, \nabla f(0) = 0$. Also the support of $\psi_j$ corresponds to a portion of the graph where $(x_1,x_2,\ldots,x_n) \in B$. The function $f(x)$ (after a possible rotation of the axes) is represented by:

$$f(x) = \sum_{k=1}^{n} \lambda_k x_k^2 + \sum_{k,l,m=1}^{n} x_k x_l x_m K_{klm}(x)$$

in the neighbourhood $B$ of zero.

Note that $|\lambda_1| \geq |\lambda_k|, \; k = 2,\ldots,n$ and $B$ has a radius $c'|\mu_1|$, where $c'$ is a sufficiently small positive. Let us use a change of variables $x \mapsto \lambda_1 x$ then we have:

$$I_j(\lambda,s) = \lambda_1^n \int_{|x|<c'} e^{it\lambda_1^2 F_1(x,\sigma)}|\det \text{Hess } f(\lambda_1 x)|^q \psi_1(\lambda_1 x)\psi_j(\lambda_1 x)dx,$$

where

$$F_1(x,\sigma) = \sum_{k=1}^{n} \frac{\lambda_k x_k^2}{\lambda_1} + \sum_{k,l,m=1}^{n} x_k x_l x_m H_{klm}(\lambda_1 x) + (\sigma, x), \; \text{ and } \; \sigma = \frac{s}{\lambda_1^2}.$$ 

Note that $\det \text{Hess } f(\lambda_1 x) = \lambda_1^n h(\lambda_1, x)$ and $h(\lambda_1, x)$ is some uniformly bounded function on $C^k$ for any $k$.

Now, for the parameter $\sigma_1$ consider two cases:

1-case: $|\sigma_1| \geq \varepsilon$, where $c$ and $\varepsilon$ are defined from Lemma 5.1. Then by using integration by part as above we obtain:

$$|I_j(t,s)| \leq \frac{C\lambda_1^n}{|t|^\frac{n}{2}}$$

$$\text{(5.3)}$$
2-case: $|\sigma_1| < \varepsilon$. Then by using Lemma 5.1 we reduce the integral $I_j(\lambda, s)$ to the form:

$$I_j = \lambda_1^{n+q} \int_{\mathbb{R}} e^{i\mu_1^q x_1^q} dx_1 \int_{\mathbb{R}^{n-1}} e^{i\mu_1^q \left| f_1'(x', \sigma_1) + (\sigma', x') \right|} |f_1''(x', \lambda_1, \sigma_1)| q a(x, \lambda_1, \sigma) dx',$$

where $f_1'(x', \lambda_1, \sigma_1) = \text{det Hess } f_1(x', \lambda_1, \sigma_1)$, $x' = (x_2, \ldots, x_n)$. Note that the amplitude function $a$ and the phase function $f_1(x', \sigma_1)$ are uniformly bounded functions depending only on $c$ and the $C^3$ norm of the function $f$. It should be noted that $\text{supp } a \subset U \times \pi_n(B(0, c'))$, where $\pi_n : \mathbb{R}^n \mapsto \mathbb{R}^{n-1}$ along the axes $x_n$. Also we have used the covariant property of the Gaussian curvature.

Now we use the Van der Corpute Lemma and obtain:

$$|I_j| \leq \frac{c|\lambda_1|^{n+qn-3/2}}{|t|^{1/2}} \int dx_1 \int_{\mathbb{R}^{n-1}} e^{i\mu_1^q F_1(x', \sigma)} f_1''(x', \lambda_1, \sigma_1)|q a(x, \lambda_1, \sigma) \frac{\partial a(x, \lambda_1, \sigma)}{\partial x_1} dx',$$

where $F_1(x', \sigma) = f(x', \sigma_1) + (\sigma', x')$.

Note that for the case of $n=3$, we can apply Corollary 5.3. So, we obtain

$$|I_j(\lambda, s)| \leq \frac{c|\mu_1|^n}{|\lambda|^{n/2}} = \frac{c \sigma(B(0, c|\mu_1|))}{|\lambda|^{n/2}}.$$

Finally, we can use induction hypothesis for the case of $n > 3$. By summing these estimations over all numbers $j$ we arrive to the proof of Proposition 5.4.

The proof of Theorem 1.3 follows from Proposition 5.4 by covering amplitude support and by using standard methods.

Let $S \subset \mathbb{R}^{n+1}$ be a real analytic hypersurface.

The following result holds.

**Proposition 5.5.** Let $S \subset \mathbb{R}^{n+1}$ be a real analytic hypersurface and $\psi \in C_0^\infty(S)$. Then the following inequality

$$\left| \int_S e^{i(x, \xi)} |K(X)| q \psi(X) d\sigma(X) \right| \leq \frac{c}{|\xi|^{n/2}},$$

holds whenever $q > n/2$ and $\psi$ have a little support.

Proposition 5.5 is proved by the analogy of the proof Proposition 5.4 by using Lemma 4.3 instead of Lemma 4.1. In fact, it is enough to prove Proposition 5.5 for the case of $n = 1$.

Finally, the proof of Theorem 1.2 follows from Proposition 1.2 as above.

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