HARMONIC AND HOLOMORPHIC 1-FORMS
ON COMPACT BALANCED HERMITIAN MANIFOLDS

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Abstract

On compact balanced Hermitian manifolds we obtain obstructions to the existence of harmonic 1-forms, $\partial$-harmonic $(1,0)$-forms and holomorphic $(1,0)$-forms in terms of the Ricci tensors with respect to the Riemannian curvature and the Hermitian curvature. Vanishing of the first Dolbeault cohomology groups of the twistor space of a compact irreducible hyper Kähler manifold is shown. A necessary and sufficient condition the $(1,0)$-part of a harmonic 1-form to be holomorphic and vice versa, a real 1-form with a holomorphic $(1,0)$-part to be harmonic are found.

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1 Introduction

The well-known vanishing theorem of Bochner says that if the Ricci tensor of a compact Riemannian manifold is nonnegative, then every harmonic 1-form is parallel; moreover, if the Ricci tensor is nonnegative and positive at least at one point, then there are no nonzero harmonic 1-forms and the first Betti number $b_1 = 0$.

It is also well known that on a compact Kähler manifold the (1,0)-part of a harmonic 1-form is holomorphic, i.e. it is $\bar{\partial}$-closed; conversely, every holomorphic (1,0)-form is $\partial$-closed or equivalently, the corresponding real 1-form is harmonic [?]. Certainly, the Bochner theorem is true for compact Kähler manifolds and could be expressed also in terms of holomorphic forms.

However, the (1,0)-part of a harmonic 1-form may not be holomorphic on a compact Hermitian manifold. Nevertheless, there exists a Bochner type theorem for holomorphic (1,0)-forms on a compact Hermitian manifold. This theorem is formulated in terms of the Chern connection and its mean curvature. In fact, on a compact Hermitian manifold with nonnegative mean curvature every holomorphic (1,0)-form is parallel with respect to the Chern connection; if in addition the mean curvature is positive at least at one point, then there are no nonzero holomorphic (1,0)-forms [?, ?, ?, ?] and the Hodge number $h_{1,0} = 0$. We note that this is a part of the general result for the nonexistence of holomorphic sections of a holomorphic vector bundle over a compact Hermitian manifold [?, ?] (see also [?, ?]).

In the present paper we consider questions of existence of harmonic 1-forms, holomorphic (1,0)-forms and find relations between them on compact balanced Hermitian manifolds.

Balanced Hermitian manifolds are Hermitian manifolds with a co-closed fundamental form or equivalently with a zero Lee form. They have been studied intensively in [?, ?, ?, ?]; in [?] they are called semi-Kähler of special type. This class of manifolds includes the class of Kähler manifolds but also many important classes of non-Kähler manifolds, such as: complex solv-manifolds, twistor spaces of oriented Riemannian 4-manifolds, 1-dimensional families of Kähler manifolds (see [?]), some compact Hermitian manifolds with a flat Chern connection (see [?]), twistor spaces of oriented distinguished Weyl structure on compact self-dual 4-manifolds [?], twistor spaces of quaternionic Kähler manifolds [?, ?], manifolds obtained as modification of compact Kähler manifolds [?] and of compact balanced manifolds [?] (see also [?]).

On a balanced Hermitian manifold $(M, g, J)$ there are two Ricci tensors $\rho$ and $\rho^c$ associated with the Levi-Civita connection $\nabla$ of the metric $g$ and two Ricci tensors $k$ and $k^*$ associated with the canonical Chern connection $D$ generated by the metric $g$ and the complex structure $J$. We note that the (1,1)-form corresponding to the tensor $k$ represents the first Chern class of $M$ and the (1,1)-form corresponding to the tensor $k^*$ is the mean curvature. All these Ricci tensors coincide on a Kähler manifold.

Let $(M, g, J)$ be a Hermitian manifold. If $X$ is an arbitrary $C^\infty$ vector field on $M$, we denote by $\omega_X$ its corresponding 1-form with respect to the metric $g$ and use the decomposition
\( \omega_X = \omega_X^{1,0} + \omega_X^{0,1} \) with respect to the complex structure \( J \). We find obstructions to the existence of harmonic and holomorphic 1-forms in terms of the Ricci tensors of the Levi-Civita and Chern connection. The aim of the paper is to prove the following

**Theorem 1.1** Let \((M, g, J)\) be a compact balanced Hermitian manifold.

i) If the \( \ast \)-Ricci tensor \( \rho^* \) is nonnegative on \( M \), then:
   a) every holomorphic \((1,0)\)-form \( \omega_X^{1,0} \) is \( \partial \)-harmonic (\( \omega_X \) is harmonic);
   b) every \( \partial \)-harmonic \((1,0)\)-form \( \omega_X^{1,0} \) satisfies the conditions
   \[
   \rho^*(X, X) = 0, \quad \nabla'' \omega_X = 0,
   \]
   where \( \nabla'' \omega_X \) is the \((2,0)\)-part of \( \nabla \omega_X \).

ii) If the tensor \( \rho^* \) is nonnegative on \( M \) and positive at least at one point in \( M \), then there are neither holomorphic \((1,0)\)-forms, nor \( \partial \)-harmonic \((1,0)\)-forms other than zero. Consequently, the Hodge numbers \( h^{1,0}(M) = h^{0,1}(M) = 0 \) and the first Betti number \( b_1(M) = 0 \).

iii) If the tensor \( c \rho^* + (1 - c) \rho^* \) is nonnegative on \( M \) for some constant \( c \geq 0 \), then any harmonic \((1,0)\)-form \( \omega_X \) is \( \nabla \)-parallel and satisfies the conditions
   \[
   \rho(X, X) = \rho^*(X, X) = 0.
   \]

iv) If the tensor \( c \rho^* + (1 - c) \rho^* \) is nonnegative on \( M \) and positive at least at one point in \( M \), then there are no harmonic \((1,0)\)-forms other than zero and \( b_1 = 0 \).

Note that these conditions agree with the classical Bochner conditions on Kähler manifolds.

In Example 1 we apply Theorem ?? to the complex twistor space \((Z, J)\) of a compact hyper Kähler manifold which holonomy group is exactly \( \text{Sp}(n) \) to show the vanishing of the cohomology group \( H^1(Z, \mathcal{O}_Z) \) (see Theorem 5.1 in the last section).

On a compact balanced Hermitian manifold we find necessary and sufficient conditions for a \( \partial \)-harmonic \((1,0)\)-form to be \( \bar{\partial} \)-harmonic (holomorphic) in terms of the Ricci tensors of the Levi-Civita and Chern connections and show that it is also necessary and sufficient condition for a \( \bar{\partial} \)-harmonic \((1,0)\)-form to be \( \partial \)-harmonic. Constructing the tensor \( H = 2\rho^* - k - k^* \) we prove

**Theorem 1.2** On a compact balanced Hermitian manifold the following conditions are equivalent:

(i) The \((1,0)\)-part of a harmonic 1-form \( \omega_X \) is holomorphic;
(ii) A real 1-form \( \omega_X \) with a holomorphic \((1,0)\)-part is harmonic;
(iii) \( \int_M H(X, X) \, dv = 0 \).

We note that the tensor \( H \) vanishes identically on a Kähler manifold and measures the deviation of a balanced Hermitian manifold from a Kähler one (see section 3 below).

Finally, in Example 2 we show that the third condition of Theorem ?? is essential.
2 Preliminaries

Let \((M, g, J)\) be a \(2n\)-dimensional Hermitian manifold with metric \(g\) and complex structure \(J\). The algebra of all \(C^\infty\) vector fields on \(M\) will be denoted by \(\mathcal{X}M\). The Kähler form \(\Omega\) of the Hermitian structure \((g, J)\) is defined by \(\Omega(X, Y) = g(JX, Y); \ X, Y \in \mathcal{X}M\). The associated Lee form \(\theta\) is given by \(\theta = -\delta \Omega \circ J\).

We denote by \(\nabla\) and \(R = [\nabla, \nabla] - \nabla_{[\cdot, \cdot]}\) the Levi-Civita connection of the metric \(g\) and the Riemannian curvature tensor, respectively. The corresponding curvature tensor of type \((0, 4)\) is given by the equality \(R(X, Y, Z, V) = g(R(X, Y)Z, V), \ X, Y, Z, V \in \mathcal{X}M\).

Further \(\rho\) and \(\rho^*\) will stand for the Ricci tensor and \(*\)-Ricci tensor, respectively. We have

\[ \rho^*(X, Y) = \sum_{j=1}^{2n} R(e_j, X, JY, Je_j), \ X, Y \in \mathcal{X}M. \]

Henceforth \(\{e_1, ..., e_{2n}\}\) will denote an orthonormal frame.

We denote by \(D, T, K\) the canonical Chern (Hermitian) connection of the Hermitian structure, its torsion tensor and its curvature tensor (Hermitian curvature tensor), respectively. We recall that the Chern connection \(D\) is the unique linear connection preserving the metric \(g\) and the complex structure \(J\), so that the torsion tensor \(T\) of \(D\) has the property \(T(JX, Y) = T(X, JY), \ X, Y \in \mathcal{X}M\). This implies (e.g. [?]):

\[ T(JX, Y) = JT(X, Y), \ X, Y \in \mathcal{X}M. \]

The corresponding torsion tensor of type \((0, 3)\) is defined by the equality

\[ T(X, Y, Z) = g(T(X, Y), Z), \ X, Y, Z \in \mathcal{X}M. \]

The curvature tensor \(K\) of \(D\) has the following properties:

\[ K(JX, JY)Z = K(X, Y)Z, \ K(X, Y)JZ = JK(X, Y)Z, \ X, Y, Z \in \mathcal{X}M. \]

The Ricci identity for the Chern connection is expressed in the following form:

\[ D_X D_Y Z - D_Y D_X Z = K(X, Y)Z - D_{T(X, Y)} Z, \ X, Y, Z \in \mathcal{X}M. \]

The two connections \(\nabla\) and \(D\) are related by the following identity

\[ g(\nabla_X Y, Z) = g(D_X Y, Z) + \frac{1}{2} d\Omega(JX, Y, Z), \ X, Y, Z \in \mathcal{X}M. \]

This equality implies that

\[ T(X, Y, Z) = -\frac{1}{2} d\Omega(JX, Y, Z) - \frac{1}{2} d\Omega(X, JY, Z), \ X, Y, Z \in \mathcal{X}M. \]

There are three Ricci-type tensors \(k, k^*\) and \(s\) associated with the curvature tensor \(K\) defined by

\[ k(X, Y) = \frac{1}{2} \sum_{j=1}^{2n} g(K(X, JY)e_j, Je_j); \ k^*(X, Y) = \frac{1}{2} \sum_{j=1}^{2n} g(K(e_j, Je_j)X, JY); \ s(X, Y) = \frac{1}{2} \sum_{j=1}^{2n} g(K(e_j, Je_j)X, JY); \]
\[ s(X,Y) = \sum_{j=1}^{2n} g(K(e_j, X)Y, e_j), \quad X,Y \in \mathcal{X}M. \]

The corresponding scalar curvatures are defined by \( \tau = tr\rho, \tau^* = tr\rho^*, u = trk = trk^* \), \( v = trs \).

The \((1,1)\)-form \( \kappa \) corresponding to the tensor \( k \) represents the first Chern class of \( M \) (further we shall call it the Chern form) and the \((1,1)\)-form \( \kappa^* \) corresponding to the tensor \( k^* \) is the mean curvature of the holomorphic tangent bundle \( T^{1,0}M \) with the hermitian metric induced by \( g \).

For an arbitrary vector field \( X \) in \( \mathcal{X}M \) we denote by \( \omega_X \) its dual 1-form defined by \( \omega_X(Y) = g(X, Y), \quad Y \in \mathcal{X}M \). From (2.6) it follows that

\[ \delta \omega_X = -\sum_{i=1}^{2n} (D_{e_i} \omega_X) e_i - \theta(X). \]

3 Balanced Hermitian manifolds

We recall the definition of a balanced Hermitian manifold and some equivalent conditions given in [?, ?] for completeness:

**Definition:** A Hermitian manifold \( (M, g, J) \) is said to be balanced if it satisfies one of the following equivalent conditions:

i) \( \delta \Omega = 0 \) \( (\theta = 0) \);

ii) \( d\Omega^{n-1} = 0 \);

iii) \( \Delta_\theta f = \Delta_\tilde{\theta} f = \frac{1}{2} \Delta_d f \) for every smooth function \( f \) on \( M \), where \( \Delta_\theta, \Delta_\tilde{\theta} \) and \( \Delta_d \) denote the Laplacians with respect to the operators \( \theta, \tilde{\theta} \) and \( d \), respectively.

We shall use local holomorphic coordinates \( \{z^\alpha\}, \alpha = 1, ..., n \) and the corresponding frame field

\[ \left\{ \left. \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\bar{\alpha}} \right| = \frac{\partial}{\partial z^\alpha} \right\}, \quad \alpha = 1, ..., n; \quad \bar{\alpha} = 1, ..., \bar{n} \]

for some calculations. The first Bianchi identity for the Hermitian curvature \( K \) with respect to local holomorphic coordinates gives

\[ K_{\alpha\bar{\beta}\gamma\bar{\lambda}} - K_{\gamma\bar{\alpha}\beta\lambda} = -D_{\beta\alpha} T_{\alpha\gamma\lambda}. \]

By the condition \( \delta \Omega = 0 \) from (3.7) it follows that [?]

\[ s(X, Y) = s(Y, X) = k(X, Y), \quad X, Y \in \mathcal{X}M. \]

It is immediate from (3.7) that on a balanced Hermitian manifold we have:

\[ \delta \omega = -\sum_{i=1}^{2n} (D_{e_i} \omega) e_i. \]

Now let \( a \) be a tensor of type \((0,2)\) and denote by \( a^t \) the tensor of type \((0,2)\) defined by \( a^t(X, Y) = a(Y, X), \quad X, Y \in \mathcal{X}M \). The symmetric part and the skew-symmetric part of the
tensor $a$ are given by

$$\text{Sym}(a) = \frac{1}{2}(a + a^t), \quad \text{Skew}(a) = \frac{1}{2}(a - a^t),$$

respectively. The induced by the metric $g$ scalar product in the vector space of $(0,2)$-tensors will be denoted by the same letter. For two tensors $a, b$ of type $(0,2)$ we have

$$g(a, b) = \sum_{i,j=1}^{2n} a(e_i, e_j)b(e_i, e_j); \quad g(a^t, b) = g(a, b^t) = \sum_{i,j=1}^{2n} a(e_i, e_j)b(e_j, e_i).$$

For a fixed vector field $X$ we obtain the following $(0,2)$-tensors $i_X T$ and $j_X T$ from the torsion tensor $T$:

$$(3.10) \quad i_X T(Y, Z) = T(X, Y, Z); \quad j_X T(Y, Z) = T(Y, Z, X), \quad Y, Z \in \mathcal{X}M.$$

The equalities (3.11) and (3.12) imply that the tensor $i_X T$ is $J$-invariant while the tensor $j_X T$ is $J$-antiinvariant, i.e.


The next statement, proved in [?], gives relations between the tensors $\rho$ and $\rho^*$.

**Proposition 3.1** [?] Let $(M, g, J)$ be a balanced Hermitian manifold. Then the Ricci tensors of the Riemannian and Hermitian curvature satisfy the following identities

$$(3.11) \quad \rho^*(X, Y) = \rho^*(JX, JY) = \rho^*(Y, X), \quad X, Y \in \mathcal{X}M;$$

$$(3.12) \quad \rho(X, Y) - \rho(JX, JY) = -g(i_X T, (i_Y T)^t), \quad X, Y \in \mathcal{X}M.$$

$$(3.13) \quad k(X, Y) - \rho^*(X, Y) = \frac{1}{4}g(j_X T, j_Y T),$$

$$(3.14) \quad k(X, Y) + k^*(X, Y) - \frac{1}{2}(\rho(X, Y) + \rho(JX, JY)) - \rho^*(X, Y) = \frac{1}{2}g(i_X T, i_Y T),$$

$$(3.15) \quad k(X, X) + k^*(X, X) - \rho(X, X) - \rho^*(X, X) = \|\text{Sym}(i_X T)\|^2,$$

where $X, Y \in \mathcal{X}M$, and $\|\cdot\|^2$ is the usual tensor norm.

We have

**Corollary 3.2** Let $(M, g, J)$ be a balanced Hermitian manifold. Then

i) $\tau = \tau^*$;

ii) $(M, g, J)$ is Kähler iff $\tau = u$.

**Proof:** Taking traces in (3.11) and (3.12) we find

$$u - \tau^* = \frac{1}{4}\|T\|^2, \quad 2u - \tau - \tau^* = \frac{1}{2}\|T\|^2.$$

Hence $\tau = \tau^*$ and $u - \tau = \frac{1}{4}\|T\|^2$. The last two equalities imply i) and ii). QED
Let η be a 1-form. Further we denote by \( d'\eta, D'\eta \) and \( \nabla'\eta \) the \((1,1)\)-part (with respect to the complex structure \( J \)) of the exterior derivative \( d\eta \), the covariant derivative \( D\eta \) with respect to the Chern connection and the covariant derivative \( \nabla\eta \) with respect to the Levi-Civita connection of \( \eta \), respectively. For the \(((2,0) + (0,2))\)-parts of \( d\eta, D\eta \) and \( \nabla\eta \) we use the denotations \( d''\eta, D''\eta \) and \( \nabla''\eta \), respectively. For example,

\[
d'\eta(X, Y) = \frac{1}{2}(d\eta(X, Y) + d\eta(JX, JY)); \quad d''\eta(X, Y) = \frac{1}{2}(d\eta(X, Y) - d\eta(JX, JY)).
\]

The next integral formulas are essential for the proof of our main results.

**Proposition 3.3** Let \((M, g, J)\) be a compact balanced Hermitian manifold. Then for any vector field \( X \in \mathcal{X} M \) we have

\[
\int_M 2\|\text{Skew}(D''\omega_X)^2\| dv = \int_M \left\{\|D''\omega_X\|^2 + k(X, X) - \frac{1}{2}(\delta\omega_X)^2 - \frac{1}{2}(\delta\omega_{JX})^2\right\} dv;
\]

\[
\int_M 2\|\text{Skew}(D''\omega_X)^2\| dv = \int_M \left\{\|D'\omega_X\|^2 + k(X, X) - k^*(X, X) - \frac{1}{2}(\delta\omega_X)^2 - \frac{1}{2}(\delta\omega_{JX})^2\right\} dv.
\]

**Proof.** Let \( \omega_X = \omega_\alpha dz^\alpha + \omega_\bar{\alpha} d\bar{z}^\bar{\alpha} \). We consider the following real 1-form

\[
\varphi = D_\alpha \omega_\beta X^\alpha dz^\beta + D_\bar{\alpha} \omega_\bar{\beta} X^\alpha d\bar{z}^\bar{\beta}
\]

and compute its co-differential \( \delta \varphi \). Here and further the summation convention is assumed.

Using (??) and taking into account the Ricci identity (??) for the Chern connection, (??), (??) and (??), we obtain

\[
-\delta \varphi = g(D''\omega_X, (D''\omega_X)^i) + k(X, X) - \frac{1}{2} X \delta \omega_X - \frac{1}{2} JX \delta \omega_{JX}.
\]

Integrating this equality over \( M \) we find

\[
\int_M \left\{g(D''\omega_X, (D''\omega_X)^i) + k(X, X) - \frac{1}{2} (\delta\omega_X)^2 - \frac{1}{2} (\delta\omega_{JX})^2\right\} dv = 0.
\]

On the other hand we have

\[
2\|\text{Skew}(D''\omega_X)^2\| = -g(D''\omega_X, (D''\omega_X)^i) + \|D''\omega_X\|^2.
\]

Then the last equality and (??) imply (??).

By similar calculations for the real 1-form

\[
(D_\alpha \omega_\beta X^\beta - D_\bar{\alpha} \omega_\bar{\beta} X^\beta) dz^\alpha + (D_\bar{\alpha} \omega_\bar{\beta} X^\beta + D_\alpha \omega_\beta X^\beta) d\bar{z}^\bar{\beta}
\]

we find

\[
\int_M \left\{\|D''\omega_X\|^2 - \|D'\omega_X\|^2 + k^*(X, X)\right\} dv = 0.
\]

Now (??) and (??) imply (??). QED
Proposition 3.4 Let $(M, g, J)$ be a compact balanced Hermitian manifold. Then for any vector field $X \in \mathcal{X}M$ we have

\begin{align}
(3.20) & \quad \int_M 2\|\text{Skew}(D'\omega_X)\|^2 \, dv = \\
& \quad \int_M \{\|D'\omega_X\|^2 - \frac{1}{2}(\delta\omega_X)^2 + \frac{1}{2}(\delta\omega_JX)^2 + g(j_X T, (D'\omega_X)^t)\} \, dv; \\
(3.21) & \quad \int_M g(i_X T, D'\omega) \, dv = \int_M \{k(X, X) - k^*(X, X) + g(j_X T, D''\omega_X)\} \, dv.
\end{align}

Proof. Let $\omega_X = \omega_\alpha dz^\alpha + \omega_\alpha dz^\bar{\alpha}$. We consider the real 1-form

$$D_\alpha \omega_\beta X^\alpha dz^\beta + D_\alpha \omega_\bar{\beta} X^\alpha dz^\bar{\beta}$$

and compute its co-differential. Integrating over $M$ the obtained equality we find that

\begin{align}
(3.22) & \quad \int_M \{g(D'\omega_X, (D'\omega_X)^t) - \frac{1}{2}(\delta\omega_X)^2 + \frac{1}{2}(\delta\omega_JX)^2 + g(j_X T, (D'\omega_X)^t)\} \, dv = 0.
\end{align}

On the other hand

$$2\|\text{Skew}(D'\omega_X)\|^2 = \|D'\omega_X\|^2 - g(D'\omega_X, (D'\omega_X)^t).$$

By virtue of the last equality and (??) we obtain (??).

To prove (??) we consider the real 1-form

$$T_{\alpha\beta\gamma} X^\beta X^\gamma d\omega_\alpha + T_{\alpha\beta\gamma} X^\beta X^\gamma d\omega_\bar{\alpha}$$

and compute its co-differential. Taking into account (??) after an integration over $M$ we get (??).

4 Proof of the theorems

Let $X$ be a real vector field in $\mathcal{X}M$ and $\omega_X = \omega_\alpha dz^\alpha + \omega_\alpha dz^\bar{\alpha} = \omega_X^{(1,0)} + \omega_X^{(0,1)}$ be its dual 1-form.

The $(1,0)$-form $\omega_X^{1,0} = \omega_\alpha dz^\alpha$ is $\partial$-harmonic iff

\begin{align}
(4.23) & \quad d\omega_{\alpha\beta} = D_\alpha \omega_\beta + D_\beta \omega_\alpha + T_{\alpha\beta\gamma} \omega_\gamma = 0, \quad \delta\omega_X = \delta\omega_JX = 0.
\end{align}

The real 1-form $\omega_X$ is harmonic iff

\begin{align}
(4.24) & \quad d\omega_{\alpha\beta} = D_\alpha \omega_\beta - D_\beta \omega_\alpha + T_{\alpha\beta\gamma} \omega_\gamma = 0, \quad d\omega_{\alpha\bar{\beta}} = D_\alpha \omega_{\bar{\beta}} - D_{\bar{\beta}} \omega_\alpha = 0, \\
(4.25) & \quad \delta\omega_X = 0.
\end{align}

The second equality of (??) implies that $\delta\omega_JX = 0$.

The $(1,0)$-form $\omega_X^{1,0} = \omega_\alpha dz^\alpha$ is holomorphic iff

\begin{align}
(4.26) & \quad D_{\alpha} \omega_\beta = 0.
\end{align}

It is immediate from this equality that $\delta\omega_X = \delta\omega_JX = 0$. 

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4.1 Proof of Theorem 1.1

i) Let \( \omega^{1,0}_X = \omega_\alpha dz^\alpha \) be a holomorphic (1,0)-form. Taking into account the condition (4.25) from (4.25) it follows that

\[
\int_M \{ \|
abla'' \omega_X \|^2 + k^*(X, X) \} \, dv = 0.
\]

Since

\[
\nabla'' \omega_X = D'' \omega_X + \frac{1}{2} j_X T,
\]

then

\[
\|
abla'' \omega_X \|^2 = \|
abla'' \omega_X \|^2 + g(D'' \omega_X, j_X T) + \frac{1}{4} \|j_X T\|^2.
\]

Under the condition (4.25) the equality (4.27) implies

\[
\int_M \{ g(j_X T, D'' \omega_X) + k(X, X) - k^*(X, X) \} \, dv = 0.
\]

By virtue of (4.25), (4.25), (4.25) and (4.25) it follows that

\[
\int_M \{ \|
abla'' \omega_X \|^2 + \rho^*(X, X) \} \, dv = 0.
\]

This formula proves a).

In order to prove b) we shall show that (4.26) is also true for any \( \partial \)-harmonic (1,0)-form. Indeed, let \( \omega^{1,0}_X = \omega_\alpha dz^\alpha \) be \( \partial \)-harmonic. Then (4.26) implies

\[
Skew(D'' \omega_X) = -\frac{1}{2} j_X T.
\]

Using this equality and (4.26) we find

\[
\|Skew(D'' \omega_X)\|^2 = \frac{1}{4} \|j_X T\|^2 = k(X, X) - \rho^*(X, X).
\]

The last equality, (4.26) and (4.26) imply

\[
\int_M \|D'' \omega_X\|^2 \, dv = \int_M (k(X, X) - 2 \rho^*(X, X)) \, dv.
\]

Since the tensor \( j_X T \) is skew-symmetric, then (4.26) leads to

\[
g(D'' \omega_X, j_X T) = g(Skew(D'' \omega_X), j_X T) = -\frac{1}{2} \|j_X T\|^2.
\]

We obtain from (4.26) that

\[
\|
abla'' \omega_X \|^2 = \|
abla'' \omega_X \|^2 - \frac{1}{4} \|j_X T\|^2 = \|
abla'' \omega_X \|^2 - k(X, X) + \rho^*(X, X).
\]

Integrating the last equality and taking into account (4.26), we obtain (4.26) which proves b).

The statement ii) follows from (4.26) by applying the Dolbeault theory to the \( \bar{\partial} \)-operator and the well known inequality (see e.g. [?], Section 3.5)

\[
b_1(M) \leq h^{1,0}(M) + h^{0,1}(M)
\]
To prove iii) and iv) let $\omega_X = \omega_\alpha dz^\alpha + \omega_\bar{\alpha} d\bar{z}^\bar{\alpha}$ be a harmonic 1-form. From (??) and (??) we have

$$g(i_X T, D''\omega_X) = -\frac{1}{2}||i_X T||^2 = 2\rho^*(X, X) - 2k(X, X).$$

Combining this equality with (??) we get

$$\int_M g(j_X T, D'\omega) dv = \int_M \{2\rho^*(X, X) - k(X, X) - k^*(X, X)\} dv$$

The last equality, (??) and (??) imply

$$\int_M \{||D'\omega_X||^2 + 2\rho^*(X, X) - k(X, X) - k^*(X, X)\} dv = 0.$$  

On the other hand we have

$$\nabla'\omega_X = D'\omega_X + Sym(i_X T)$$

and

$$||\nabla'\omega_X||^2 = ||D'\omega_X||^2 + 2g(D'\omega_X, Sym(i_X T)) + ||Sym(i_X T)||^2.$$

Integrating the last equality and taking into account (??), (??) and (??) we find

$$\int_M \{||\nabla'\omega_X||^2 + \rho(X, X) - \rho^*(X, X)\} dv = 0.$$  

Let $c$ be a positive constant. Combining (??) with (??) we obtain

$$\int_M \{c||\nabla'\omega_X||^2 + ||\nabla''\omega_X||^2 + c\rho(X, X) + (1-c)\rho^*(X, X)\} dv = 0.$$  

This formula implies immediately iii). The statement iv) also follows from (??) by using the Hodge theory. QED

4.2 Proof of Theorem 1.2

We define the tensor $H$ of type (0,2) by the equality

$$H(X, Y) = 2\rho^*(X, Y) - k(X, Y) - k^*(X, Y); \quad X, Y \in XM.$$  

Let $\omega_X = \omega_\alpha dz^\alpha + \omega_\bar{\alpha} d\bar{z}^\bar{\alpha}$ be a harmonic 1-form. By virtue of (??) we have

$$\int_M \left(||D'\omega_X||^2 + H(X, X)\right) dv = 0,$$

Now the equivalence i) $\Leftrightarrow$ iii) follows immediately from (??).

To prove the equivalence ii) $\Leftrightarrow$ iii) let $\omega_X^{1,0}$ be a holomorphic (1,0)-form. Since $d''\omega_X = 2\text{Skew}(D''\omega_X) + j_X T,$ then

$$||d''\omega_X||^2 = 4||\text{Skew}(D''\omega_X)||^2 + 4g(j_X T, D''\omega_X) + ||j_X T||^2.$$  

Taking into account (??) and (??) we find

$$\int_M \{2||\text{Skew}(D''\omega_X)||^2 + k^*(X, X) - k(X, X)\} dv = 0.$$
By virtue of the equalities (??), (??), (??) and (??) we obtain
\[ \int_M \left\{ \frac{1}{2} \| d'' \omega_X \|^2 + H(X, X) \right\} \, dv = 0. \]

The last equality implies the equivalence ii) ⇔ iii) which completes the proof of Theorem 1.2.

QED

In the next theorem we find obstructions to the existence of holomorphic (1,0)-forms in terms of the Ricci tensors of Chern connection. We have

**Theorem 4.1** Let \((M, g, J)\) be a compact balanced Hermitian manifold.

i) If the tensor \( k + k^* \) is nonnegative on \( M \), then any holomorphic \((1,0)\)-form \( \omega_X^{1,0} \) satisfies the conditions
\[ k(X, X) + k^*(X, X) = 0, \quad \text{Sym}(D'' \omega_X) = 0. \]

ii) If the tensor \( k + k^* \) is nonnegative on \( M \) and positive at least at one point in \( M \), then there are no holomorphic 1-forms other than zero and \( h^{1,0} = 0 \).

**Proof.** Let \( \omega_X^{1,0} \) be a holomorphic \((1,0)\)-form. The identity
\[ \| \text{Sym}(D'' \omega_X) \|^2 + \| \text{Skew}(D'' \omega_X) \|^2 = \| D'' \omega_X \|^2 \]
and the equality (??) give
\[ \int_M \left\{ 2\| \text{Sym}(D'' \omega_X) \|^2 - \| D'' \omega_X \|^2 + k(X, X) \right\} \, dv = 0. \]

Combining the last formula with (??) we obtain
\[ (4.42) \int_M \left\{ 2\| \text{Sym}(D'' \omega_X) \|^2 + k(X, X) + k^*(X, X) \right\} \, dv = 0. \]

Now the statements i) and ii) follow from formula (??). \( \text{QED} \)

We obtain as a corollary from the proof of Theorem 4.1 and formulas in Proposition ?? the following

**Proposition 4.2** Let \((M, g, J)\) be a compact balanced Hermitian manifold.

i) If the tensor \( \rho + \rho^* \) is nonnegative on \( M \), then any holomorphic \((1,0)\)-form \( \omega_X^{1,0} \) satisfies the conditions
\[ \rho(X, X) + \rho^*(X, X) = k(X, X) + k^*(X, X) = i_X T = 0 \]
and the vector field \( X \) is Killing.

ii) If \( \rho + \rho^* \) is nonnegative on \( M \) and positive at least at one point in \( M \), then there are no holomorphic 1-forms other than zero and the Hodge number \( h^{1,0} = 0 \).

**Proof.** We recall that a real vector field \( X \) is said to be Killing if \( L_X g = 0 \), where \( L_X \) denotes the Lie derivative with respect to \( X \). In terms of the Chern connection the Killing condition is expressed by the equalities
\[ (4.43) \text{Sym}(D'' \omega_X) = 0, \]
(4.44) \[ \text{Sym}(D'\omega_X) = -\text{Sym}(i\chi T). \]

Let \( \omega^{1,0}_X \) be a holomorphic \((1,0)\)-form. By virtue of (4.44) we can apply Theorem 1.3, which implies \( \text{Sym}(D'\omega_X) = 0 \) and \( k(X, X) + k^*(X, X) = 0 \). Taking into account (4.44) we find \( \rho(X, X) + \rho^*(X, X) = 0, i\chi T = 0 \). From (4.44) and (4.44) it follows that \( X \) is Killing.

The second statement follows immediately from (4.44) and Theorem 1.2. \( \text{QED} \)

5 Examples

Example 1. Let \((M^{4n}, g)\) be a compact \(4n\)-dimensional hyper-Kähler manifold, i.e. there are three anticommuting complex structures which are parallel with respect to the Levi-Civita connection of \( g \); for \( n = 1 \) \((M^{4}, g)\) means a self-dual Ricci flat manifold. It is well known that every hyper-Kähler manifold can be considered as a Ricci-flat quaternionic Kähler manifold. The twistor space of \( M^{4n} \) is a 2-sphere bundle \( Z \) over \( M^{4n} \) whose fibre at any point \( p \in M^{4n} \) consists of all complex structures on the tangent space \( T_pM^{4n} \) at \( p \) which are compatible with the given hyper-Kähler structure. There are two natural distributions on \( Z \), namely, the vertical 2-dimensional distribution \( V \) consisting of all vector fields tangent to the fibre and a horizontal \( 4n \)-dimensional distribution \( H \) induced by the Levi-Civita connection. The \((4n + 2)\)-dimensional twistor space \( Z \) admits a complex structure \( J \). There exists a natural 1-parameter family of hermitian metrics \( h_c, c > 0 \) on \((Z, J)\) such that the projection \( \pi : Z \to M^{4n} \) is a Riemannian submersion with totally geodesic fibres [1]. The twistor space \((Z, J, h_c), c > 0 \) is a compact balanced Hermitian manifold [2, 3]. The curvature of \((Z, h_c)\) for \( n = 1 \) has been calculated by many authors [2, 3, 4, 5, 6, 7]. The \(*\)-Ricci tensor \( \rho_c^* \) of \((Z, h_c, J)\) for \( n \geq 2 \) is given in [7] by formulas (3.12). The latter formulas are also valid when \( M^4 \) is an oriented self-dual Ricci-flat Riemannian manifold. Substituting \( s = 0 \) into (3.12) from [7], we obtain

\[ \rho_c^*(X^v, X^v) > 0, \quad \rho_c^*(Y^h, Y^h) = \rho_c^*(Y^h, X^v) = 0, \quad X^v \in V, \quad Y^h \in H. \]

The formula (4.44) shows that the tensor \( \rho_c^* \) is non-negative on \( Z \). An application of Theorem 1.3 leads to

Theorem 5.1 Let \((Z, J)\) be the twistor space of a compact hyper-Kähler manifold \( M \) endowed with the natural complex structure \( J \). Then we have

\[ h^{0,1}(Z) = \dim H^1(Z, \mathcal{O}_Z) = b_1(Z). \]

In particular, if the hyper Kähler manifold \( M \) is irreducible then \( H^1(Z, \mathcal{O}_Z) = 0 \)

Proof. Let \( \omega^{1,0}_X \) be a \( \partial \)-harmonic \((1,0)\)-form on \((Z, J, h_c)\). The condition \( \rho_c^*(X, X) = 0 \) of Theorem 1.3 together with (4.44) implies that the vector field \( X \) has to be horizontal i.e. \( X = X^h \). Using the general formula \( g(\nabla_j Y)Z, U) = g(T(Z, U), Y) \), which is a simple consequence of (4.44) and (4.44), we derive from (2.9) in [2] that \( jX^h T = 0, X^h \in H \). Then the condition \( \nabla^h \omega_X = 0 \) of
Theorem ?? together with the formula (??) implies \( D^p \omega_X = 0 \) which means that \( X \) is a (real) holomorphic vector field on \((Z, J)\). Hence, it generates a non-zero Killing vector field on \((M, g)\) (see e.g. [?, ?, ?]). The dimension of the space of Killing vector fields on \((M, g)\) is equal to \( b_1(M) \) since \((M, g)\) is Ricci flat. Applying Dolbeault theory, we obtain

\[
(5.47) \quad h^{0,1}(Z) \leq b_1(M).
\]

It is well known that \( h^{1,0}(Z) = 0 \) and \( b_1(M) = b_1(Z) \) [?]. The assertion follows from (??), (??) and the last two equalities. If \((M, g)\) is irreducible then it is simply connected [?]. Hence, \( b_1(M) = 0 \) and (??) implies \( H^1(Z, O_Z) = 0 \). Q.E.D.

The next example shows that the third condition in Theorem 1.3 is essential.

**Example 2.** Consider the complex Heisenberg group

\[
G = \left\{ \left( \begin{array}{ccc} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{array} \right) : z_1, z_2, z_3 \in \mathbb{C} \right\},
\]

with multiplication. The complex Iwasawa manifold is the compact quotient space \( M = G/\Gamma \) formed from the right cosets of the discrete group \( \Gamma \) given by the matrices whose entries \( z_1, z_2, z_3 \) are Gaussian integers. The 1-forms

\[
(5.48) \quad dz_1, \ dz_2, \ dz_3 - z_1 dz_2
\]

are left invariant by \( G \) and certain by \( \Gamma \). These 1-forms pass to the quotient \( M \). We denote by \( \alpha_1, \alpha_2, \alpha_3 \) the corresponding 1-forms on \( M \), respectively. Consider the Hermitian manifold \((M, g, J)\), where \( J \) is the natural complex structure on \( M \) arising from the complex coordinates \( z_1, z_2, z_3 \) on \( G \) and the metric \( g \) is determined by \( g = \sum_{i=1}^{3} \alpha_i \otimes \bar{\alpha}_i \). The Chern connection \( D \) is determined by the conditions that the 1-forms \( \alpha_1, \alpha_2, \alpha_3 \) are parallel. The torsion tensor of \( D \) is given by

\[
T(\alpha_i^#, \alpha_j^#) = -[\alpha_i^#, \alpha_j^#], \quad i, j = 1, 2, 3,
\]

where \( \alpha_i^# \) is the vector field corresponding to \( \alpha_i \) via \( g \). The nonzero term is only \( T(\alpha_1^#, \alpha_2^#) = -\alpha_3^\# \) and its complex conjugate. Thus, the space \((M, g, J)\) is a compact balanced Hermitian (non Kähler) manifold with a flat Chern connection.

It is easy to compute that

\[
H(\alpha_1^#, \alpha_1^#) = H(\alpha_2^#, \alpha_2^#) = 0, \quad H(\alpha_3^#, \alpha_3^#) = -2.
\]

The conclusions of Theorem 1.3 agree with the fact that the holomorphic \((1,0)\)-forms \( \alpha_1 \) and \( \alpha_2 \) are closed while the holomorphic \((1,0)\)-form \( \alpha_3 \) is not closed (indeed, from (??) it follows that \( d\alpha_3 = -\alpha_1 \wedge \alpha_2 \)).
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