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COFINITE MODULES

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Abstract

The $R$–module $M$ is called a $\mathcal{F}A$ module if there exists finite submodule $N$ such that $M/N$ is Artinian, and is called an $\mathcal{A}F$ module if there exists Artinian submodule $A$ such that $M/A$ is finite. We show that if $M$ and $K$ are $\mathcal{F}A$ (or $\mathcal{A}F$) modules with $\text{Supp}(K) \subseteq \mathbb{V}(\alpha)$ then $\text{Ext}_R^i(K, H_\alpha^j(M))$ is finite for all $i \geq 0$ and $j > 0$, if $R$ is local and $\alpha$ is an ideal of dimension one or principal.

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0. Introduction

Throughout this note the ring $R$ is commutative Noetherian with non-zero identity. It is a well-known result that if $R$ is a complete local ring with maximal ideal $m$ then the $R$–module $M$ is Artinian if and only if $\text{Supp}(M) \subseteq V(m)$ and $\text{Ext}_R^i(R/m, M)$ is finite (it means finitely generated) for all $i \geq 0$, cf. [Ha; Proposition 1.1]. Grothendieck proposed the following:

**Grothendieck's conjecture:** For any ideal $a$ and any finite $R$–module $M$, the module $\text{Hom}(R/a, H_a^j(M))$ is finite for all $j \geq 0$, where $H_a^j(M)$ is the $j^{th}$ local cohomology module of $M$ with support in $a$.

In [Ha], Hartshorne refined this conjecture, and posed the following:

**Hartshorne's conjecture:** For any ideal $a$ and any finite $R$–module $M$, the module $\text{Ext}_R^i(R/a, H_a^j(M))$ is finite for all $i \geq 0$ and all $j \geq 0$.

Hartshorne showed that, in general, the answer is NO, even if $R$ is a regular local ring. But he used the derived category and showed that the conjecture is true if $R$ is a complete regular local ring for two cases;

(i) $a$ is a non–zero principal ideal.

(ii) $a$ is a prime ideal with dimension one.

In [HK], Huneke and Koh extended Hartshorne’s second case for a complete Gorenstein domain. The result was further extended by Delfino in [D]. She was able to replace the Gorenstein property of $R$ by some weaker conditions. Recently Delfino and Marley in [DM] and Yoshida in [Yo] have eliminated the complete domain hypothesis entirely and they proved the following:

**Theorem 0.1.** Let $R$ be a Noetherian local ring, let $a$ be an ideal of $R$ with $\text{dim}(R/a) = 1$ and let $M$ be a finite $R$–module then $\text{Ext}_R^i(R/a, H_a^j(M))$ is finite for all $i \geq 0$ and $j \geq 0$.

Concerning result (i), the present author extended it for a Gorenstein ring and also for Noetherian local ring when $M$ has finite projective dimension, by using generalized section functors, cf. [Ya]. Recently Kawasaki in [K1] proved the following:

**Theorem 0.2.** Let $R$ be a Noetherian ring, let $a$ be a principal ideal of $R$ and let $M$ be a finite $R$–module then $\text{Ext}_R^i(R/a, H_a^j(M))$ is finite for all $i \geq 0$ and $j \geq 0$.

The $R$–module $M$ is called $\mathcal{FA}$ module if there exists finite submodule $N$ of $M$ with $M/N$ Artinian and is called $\mathcal{AF}$ module if there exists Artinian submodule $A$ of $M$ with $M/A$ finite. In [E; Proposition 1.3], Enochs showed that if $R$ is local then $\mathcal{FA}$ modules contain the Matlis reflexive modules and there is equality if $R$ is complete with respect to the $m$–adic topology (we have strict inequality if $R$ is not complete). It is also easy to see that $\mathcal{FA}$ and $\mathcal{AF}$ modules contain finite and Artinian modules.

In the first section we bring some characterization of cofinite modules.

In section 2, we show that for a local ring $R$ if $M$ and $K$ are in $\mathcal{FA}$ or $\mathcal{AF}$ with $\text{Supp}(K) \subseteq V(a)$ then $\text{Ext}_R^i(K, H_a^j(M))$ is finite for all $i \geq 0$ and $j > 0$ if $a$ is an ideal with dimension one or principal. This is a generalization of [BSW; Theorem 5].
1. Cofinite modules

In this section we bring some characterizations of cofinite modules.

**Definition 1.1.** Let $a$ be an ideal of $R$. The $R$-module $M$ is called an $a$-cofinite module if the support of $M$ is contained in $V(a)$ and $\text{Ext}_R^i(R/a, M)$ is finite for all $i \geq 0$.

In the next theorem we characterize $a$-cofinite modules. Some parts of this theorem appear in [HK], [Ya], [D] and [K1], but we have put these parts together with further new equivalent conditions.

**Theorem 1.2.** Let $M$ be an $R$-module and let $a$ be an ideal of $R$. Then the following are equivalent:

(i) $M$ is an $a$-cofinite module.

(ii) $\text{Ext}_R^i(N, M)$ is finite for all $i \geq 0$ and all finite $R$-modules $N$ such that the annihilator of $N$ contains $a$.

(iii) $\text{Ext}_R^i(N, M)$ is finite for all $i \geq 0$ and all finite $R$-modules $N$ such that the support of $N$ is contained in $V(a)$.

(iv) $M$ is an $a^n$-cofinite module for all $n \in \mathbb{N}$.

(v) For any ideal $b$ if $a \subseteq \sqrt{b}$ then $M$ is a $b$-cofinite module.

(vi) For any $p \in \text{Min}(a)$ the $R$-module $M$ is a $p$-cofinite.

(vii) For any $m \in \text{Max}(R)$ the $R$-modules $H_m^0(M)$ and $M/H_m^0(M)$ are $a$-cofinite.

(viii) There exists $m \in \text{Max}(R)$ such that the $R$-modules $H_m^0(M)$ and $M/H_m^0(M)$ are $a$-cofinite.

**Proof.** (ii)$\Rightarrow$(i), (iii)$\Rightarrow$(i), (iv)$\Rightarrow$(i), (v)$\Rightarrow$(i) and (vii)$\Rightarrow$(viii) are clear.

(i)$\Rightarrow$(ii). We use induction on $i$. Since $\text{Ann}(N) \supseteq a$ we know that $N$ is finite as an $R/a$-module. Thus there exists an exact sequence $(R/a)^n \rightarrow N \rightarrow 0$ with $n \in \mathbb{N}$. Therefore $0 \rightarrow \text{Hom}(N, M) \rightarrow \text{Hom}(R/a, M)^n$ is exact and hence $\text{Hom}(N, M)$ is a finite module. Suppose that $\text{Ext}^n(X, M)$ is finite for all finite $R$-module $X$ with $\text{Ann}(X) \supseteq a$. There is an exact sequence $0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0$ such that $F$ is a finite free $R/a$-module. Note that $\text{Ann}(K) \supseteq a$. There is a long exact sequence

$$\cdots \rightarrow \text{Ext}^0_R(K, M) \rightarrow \text{Ext}^1_R(N, M) \rightarrow \text{Ext}^i_R(F, M) \rightarrow \text{Ext}^1_R(K, M) \rightarrow \cdots$$

Since $\text{Ext}^0_R(K, M)$ and $\text{Ext}^1_R(F, M)$ are finite we have that $\text{Ext}^n+1(N, M)$ is finite.

(i)$\Rightarrow$(iv). We prove it by induction on $n$. For $n = 0$ we have nothing to prove. Let us assume $M$ is an $a^n$-cofinite module. We know that $a^n/a^{n+1}$ is a finite $R$-module and $\text{Ann}(a^n/a^{n+1}) \supseteq a$. Thus by (i), $\text{Ext}_R^i(a^n/a^{n+1}, M)$ is finite for all $i \geq 0$. The exact sequence

$$0 \rightarrow a^n/a^{n+1} \rightarrow R/a^{n+1} \rightarrow R/a^n \rightarrow 0$$
induces the long exact sequence

\[ \cdots \to \text{Ext}^i_R(R/a^n, M) \to \text{Ext}^i_R(R/a^{n+1}, M) \to \text{Ext}^i_R(a^n/a^{n+1}, M) \to \cdots \]

Since \( \text{Ext}^i_R(R/a^n, M) \) and \( \text{Ext}^i_R(a^n/a^{n+1}, M) \) are finite for all \( i \geq 0 \) we have that the module \( \text{Ext}^i_R(R/a^{n+1}, M) \) is finite for all \( i \geq 0 \).

(i) \( \Rightarrow \) (v). Since \( a \subseteq \sqrt{b} \) there exists \( n \in \mathbb{N} \cup \{0\} \) such that \( a^n \subseteq b \). By (iv) we know that \( M \) is an \( a^n \)-cofinite module and \( \text{Ann}(R/b) \supseteq a^n \). Therefore by (i), \( \text{Ext}^i_R(R/b, M) \) is finite for all \( i \geq 0 \), and hence \( M \) is a \( b \)-cofinite.

(v) \( \Rightarrow \) (iii). Induction on the length of a prime filtration for \( N \) shows that it suffices to prove it for \( R/p \), where \( p \in \text{Supp}(N) \). Now the assertion follows from the fact \( a \subseteq \sqrt{b} \).

(i) \( \Rightarrow \) (vi). Assume \( p \in \text{Min}(a) \). We have \( \text{Ann}(R/p) = p \supseteq a \) and so by (i) \( \Rightarrow \) (v) we have that \( M \) is a \( p \)-cofinite.

(vi) \( \Rightarrow \) (v). Assume that \( \text{Min}(a) = \{p_1, p_2, \ldots, p_n\} \) and \( b \) is an ideal of \( R \) such that \( a \subseteq \sqrt{b} \). Since \( \cup_{i=1}^n p_i \subseteq \sqrt{b} \) there exists \( 1 \leq j \leq n \) such that \( p_j \subseteq \sqrt{b} \). Since \( M \) is a \( p_j \)-cofinite we have that \( M \) is a \( b \)-cofinite module by (i) \( \Rightarrow \) (v).

(i) \( \Rightarrow \) (vii). Since \( M \) is an \( a \)-cofinite module we have that \( H^0_m(M) \) is an \( a \)-cofinite for all \( m \in \text{Max}(R) \), cf. [M; Corollary 1.8]). The short exact sequence of \( R \)-modules \( 0 \to H^0_m(M) \to M \to M/H^0_m(M) \to 0 \) induces the long exact sequence

\[ \cdots \to \text{Ext}^i_R(R/a, M) \to \text{Ext}^i_R(R/a, M/H^0_m(M)) \to \text{Ext}^{i+1}_R(R/a, H^0_m(M)) \to \cdots \]

Since \( \text{Ext}^i_R(R/a, M) \) and \( \text{Ext}^{i+1}_R(R/a, H^0_m(M)) \) are finite for all \( i \geq 0 \) we have that the module \( \text{Ext}^i_R(R/a, M/H^0_m(M)) \) is finite for all \( i \geq 0 \).

(viii) \( \Rightarrow \) (i). Use the long exact sequence involving Ext induced from the short exact sequence \( 0 \to H^0_m(M) \to M \to M/H^0_m(M) \to 0 \).

**Corollary 1.3.** Let \( (R, m) \) be a local ring with maximal ideal \( m \) and let \( a \) be an ideal of \( R \) with dimension one or principal. Let \( A \) be an Artinian \( R \)-module and let \( M \) be a finite \( R \)-module. Then \( \text{Ext}^i_R(A, H^0_m(M)) \) is finite for all \( i \geq 0 \) and \( j \geq 0 \).

**Proof.** Since \( A \) is an Artinian \( R \)-module we have that \( \text{Supp}(A) = \{m\} \subseteq V(a) \). Now the assertion follows from theorems (0.1) and (1.2) or theorems (0.2) and (1.2).

### 2. \( \mathcal{FA} \) and \( \mathcal{AF} \) modules

In this section suppose \( (R, m) \) is a local ring with maximal ideal \( m \) and \( a \) is an ideal of \( R \) with dimension one or principal.

**Definition 2.1.** The \( R \)-module \( M \) is called an \( \mathcal{FA} \) module if there exists a finite submodule \( N \) of \( M \) with \( M/N \) Artinian and is called an \( \mathcal{AF} \) module if there exists an Artinian submodule \( A \) of \( M \) with \( M/A \) finite.
Lemma 2.2. If $K$ is an $\mathcal{FA}$ or $\mathcal{AF}$ module such that $\text{Supp}(K) \subseteq V(\mathfrak{a})$ and $M$ is a finite $R$-module then $\text{Ext}^j_R(K, H^i_\mathfrak{a}(M))$ is finite for all $i \geq 0$ and $j \geq 0$.

Proof. If $K$ is an $\mathcal{FA}$ module, then there exists a short exact sequence

$$0 \to N \to K \to A \to 0$$

with $N$ finite and $A$ Artinian $R$-modules. This induces a long exact sequence

$$\cdots \to \text{Ext}^j_R(A, H^i_\mathfrak{a}(M)) \to \text{Ext}^j_R(K, H^i_\mathfrak{a}(M)) \to \text{Ext}^j_R(N, H^i_\mathfrak{a}(M)) \to \cdots$$

By Corollary (1.3) we know that the left-hand side is finite. Since $N$ is a finite module and $\text{Supp}(N) \subseteq V(\mathfrak{a})$ we have that the right-hand side is finite by theorems (0.1) and (1.2) or (0.2) and (1.2). Thus the assertion holds.

If $K$ is an $\mathcal{AF}$ module, there exists a short exact sequence

$$0 \to A \to K \to N \to 0$$

with $A$ Artinian and $N$ finite $R$-modules. This induces a long exact sequence

$$\cdots \to \text{Ext}^j_R(N, H^i_\mathfrak{a}(M)) \to \text{Ext}^j_R(K, H^i_\mathfrak{a}(M)) \to \text{Ext}^j_R(A, H^i_\mathfrak{a}(M)) \to \cdots$$

By Corollary (1.3) we know that the right-hand side is finite. Since $N$ is finite and $\text{Supp}(N) \subseteq V(\mathfrak{a})$ we have that the left-hand side is finite by theorems (0.1) and (1.2) or theorems (0.2) and (1.2). Thus the assertion holds.

Theorem 2.3. If $M$ is an $\mathcal{FA}$ or $\mathcal{AF}$ module then $H^i_\mathfrak{a}(M)$ is $\mathfrak{a}$-cofinite for all $j > 0$.

Proof. If $M$ is an $\mathcal{FA}$ module then follows the proof of [BSW; Theorem 2].

If $M$ is an $\mathcal{AF}$ module, then there exists an exact sequence $0 \to A \to M \to N \to 0$ with $A$ Artinian and $N$ finite $R$-module. Since $A$ is Artinian we have $H^j_A(A) = 0$ for all $j > 0$ and hence we obtain from the induced long exact sequence of local cohomology modules that $H^j_\mathfrak{a}(M) \cong H^j_\mathfrak{a}(N)$ for all $j > 0$. Now the assertion follows from theorem (0.1) or (0.2).

Corollary 2.4. If $A$ is an Artinian $R$-module and $M$ is an $\mathcal{FA}$ or $\mathcal{AF}$ module then the module $\text{Ext}^i_R(A, H^j_\mathfrak{a}(M))$ is finite for all $i \geq 0$ and $j > 0$.

Proof. Use theorem (2.3) and the proof of corollary (1.3).

Theorem 2.5. The following holds.

(a) If $M$ and $K$ are $\mathcal{FA}$ modules and $\text{Supp}(K) \subseteq V(\mathfrak{a})$ then the module $\text{Ext}^i_R(K, H^j_\mathfrak{a}(M))$ is finite for all $i \geq 0$ and $j > 0$.

(b) If $M$ and $K$ are $\mathcal{AF}$ modules and $\text{Supp}(K) \subseteq V(\mathfrak{a})$ then the module $\text{Ext}^i_R(K, H^j_\mathfrak{a}(M))$ is finite for all $i \geq 0$ and $j > 0$. 

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Proof (a): There exists a short exact sequence $0 \to N \to K \to A \to 0$ with $N$ finite and $A$ Artinian $R$-modules. This sequence induces a long exact sequence

$$\cdots \to \Ext^i_R(A, H^d_a(M)) \to \Ext^i_R(K, H^d_a(M)) \to \Ext^i_R(N, H^d_a(M)) \to \cdots$$

By corollary (1.3) we have that the left-hand side is finite. Since $\Supp(N) \subseteq V(\mathfrak{a})$ we have that the right-hand side is also finite by theorems (2.3) and (1.2). Now the assertion holds.

(b): There exists a short exact sequence $0 \to A \to K \to N \to 0$ with $A$ Artinian and $N$ finite $R$-modules. This sequence induces a long exact sequence

$$\cdots \to \Ext^i_R(N, H^d_a(M)) \to \Ext^i_R(K, H^d_a(M)) \to \Ext^i_R(A, H^d_a(M)) \to \cdots$$

By corollary (1.3) we have that the right-hand side is finite. Since $\Supp(N) \subseteq V(\mathfrak{a})$ we have that the left-hand side is also finite by theorems (2.3) and (1.2). Now the assertion holds.

If $N$ is an $R$-module then the $i$th Bass number of $N$ with respect to the prime ideal $\mathfrak{p}$ is defined to be $\mu_i(\mathfrak{p}, N) = \dim_{k(\mathfrak{p})} \Ext^i_{R_{\mathfrak{p}}}(k(\mathfrak{p}), N_{\mathfrak{p}})$, where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, cf. [B]. If $M$ is finite then the Bass numbers of $H^i_m(M)$ are finite since $H^i_m(M)$ is Artinian. However, Hartshorne showed that this does not hold in general, even over a complete regular local ring. In the special case that $M = R$, Huneke and Sharp proved that if $R$ is a regular local ring of characteristic $p$ and $\mathfrak{a}$ is an ideal of $R$, then the Bass numbers of $H^i_\mathfrak{a}(R)$ are finite for all $j \geq 0$. [HS; Theorem 2.1]. In [L], Lyubeznik proved the same result in the case $R$ is a regular local ring containing a field of characteristic $0$. There are some other attempts in this direction for example, [DM], [K2] and [BSW]. Now we extend this result for $\mathcal{FA}$ and $\mathcal{AF}$ modules.

**Theorem 2.6.** If $M$ is an $\mathcal{FA}$ (resp. $\mathcal{AF}$) module, then the Bass numbers of $H^i_\mathfrak{a}(M)$ are finite for all $j > 0$.

**Proof.** We know that $\Ext^i_{R_{\mathfrak{p}}}(R/\mathfrak{m}, H^i_\mathfrak{a}(M))$ is finite by theorem (2.4). If $\mathfrak{p}$ is not maximal then $M_{\mathfrak{p}}$ is a $\mathcal{FA}$ (resp. $\mathcal{AF}$) $R_{\mathfrak{p}}$-module. Now if $\mathfrak{p} \supseteq \mathfrak{a}$ then we have that $(\Ext^i_{R_{\mathfrak{p}}}(R/\mathfrak{p}, H^i_\mathfrak{a}(M)))_{\mathfrak{p}} \cong \Ext^i_{R_{\mathfrak{p}}}(k(\mathfrak{p}), H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}))$ by [BS; 4.3.3], that is finite. If $\mathfrak{p}$ does not contain $\mathfrak{a}$ then $\mu_i(\mathfrak{p}, H^i_\mathfrak{a}(M)) = 0$.

Let $M$ and $N$ be $R$-modules, and let $\mathfrak{a}$ be an ideal of $R$. Then the generalized local cohomology module $H^i_\mathfrak{a}(M, N)$ was introduced by Herzog in [He] by defining

$$H^i_\mathfrak{a}(M, N) = \lim_{\to} \Ext^i_R(M/\mathfrak{a}M, N).$$

If $M = R$, then $H^i_\mathfrak{a}(M, N) = H^i_\mathfrak{a}(N)$, the usual local cohomology module. Now it is natural to ask the following question:

**Question 2.7.** Assume $M$ and $N$ are finite $R$-modules and $\mathfrak{a}$ is an ideal of $R$ with dimension one or principal. Is the module $H^i_\mathfrak{a}(M, N)$, $\mathfrak{a}$-cofinite for all $j \geq 0$?
Here we give a positive answer for the principal ideal over Gorenstein ring.

**Theorem 2.8.** Let $R$ be a Gorenstein ring and let $M$ and $N$ be finite $R$–modules. Then for any non–zero principal ideal $a$ the $R$–module $H^j_a(M, N)$ is an $a$–cofinite module for all $j \geq 0$.

**Proof.** Use [Ya; 3.4], [Ya; 4.3] and then [Ya; 4.8].

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