ON THE SHAFAREVICH CONJECTURE
FOR SURFACES OF GENERAL TYPE

Egor Bedoulev

Department of Mathematics, Chair of Algebra, Moscow State University,
Lenin Hills, 119 899 Moscow, Russian Federation
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

In this note we consider the following Shafarevich conjecture, which in the case of curves was
proved by Parshin [?] and Arakelov [?] : the set of all families of surfaces of general type with
fixed $c_1^2$ and $c_2$ over a fixed curve with fixed degenerations is bounded. We show that in the case
when there are no degeneracies at all this conjecture is an easy consequence of the Miyaoka –
Yau inequality for smooth minimal threefolds.

MIRAMARE - TRIESTE

March 1999
1 Introduction

In 1962 in his talk at the International Congress in Stockholm Shafarevich raised the question of finiteness of the number of curves of fixed genus with a given set of points of bad reduction over some functional or number field $K$. Six years later, Parshin in his well-known paper [?] discovered the remarkable connection between this conjecture (which is now called the Shafarevich conjecture) and the famous Mordell conjecture in number theory. Moreover, he was able to prove partially the Shafarevich conjecture in the geometric case and to derive from it a new proof of the Mordell conjecture in this case (earlier proofs were given by Manin [?] and Grauert [?]). Soon afterwards, using some new ideas, Arakelov [?] completely proved the Shafarevich conjecture in the geometric case. The interest to this circle of ideas was greatly supported in 1983 by the appearance of the celebrated paper of Faltings [?], where he succeeded in proving the Shafarevich conjecture in the number case, thus obtaining the first known proof of the Mordell conjecture over number fields.

In this paper we are interested, in the case of a functional field, in the generalization of the Shafarevich conjecture for surfaces of general type.

More precisely, it seems natural to state the following conjecture.

**Conjecture.** Let $k$ be an algebraically closed field of characteristic 0; let $K = k(B)$ be the field of functions on some smooth projective curve $B$ over $k$. For a fixed finite subset $S_0 \subset B$ and two integers $c_1^2$ and $c_2$, define the set

$$\Phi = \Phi(S_0, c_1^2, c_2) = \{\text{The set of all smooth projective surfaces } X_K \text{ of general type with fixed Chern numbers } c_1^2 \text{ and } c_2 \text{ that are defined over } K \text{ and have good reduction everywhere outside of } S_0\}$$

(For the definition of a point of good (bad) reduction see section 1).

Then the set $\Phi$ is bounded. More precisely, there exists a quasi-projective variety $T$ and a family $\mathcal{X} \to T \times B$ with a surjective map $\mu : T \to \Phi$ that acts as follows: for $t \in T$ the image $\mu(t) \in \Phi$ is the generic fibre of the restricted family $\mathcal{X}_t \to B$.

This paper is devoted to the proof of the following result.

**Main Theorem.** The conjecture is true in the case $S_0 = \emptyset$.

For the proof see sections 2 – 4. In fact, this theorem is an almost immediate consequence of the Miyaoka – Yau inequality for smooth threefolds. The general case of the conjecture should follow from some generalization of this inequality to the non-smooth case.

It is also interesting to ask the following question, which corresponds to the “complete version” of Shafarevich conjecture for curves, as it was proved by Arakelov:

**Question.** Is the set $\Phi$ finite?

The answer can probably be negative, as it is suggested by the work of Faltings [?] on the analogous problem for abelian varieties.
2 The description of the situation

In this section we will describe the basic facts about the minimal and canonical models that we need. Before doing this, let us recall the usual definitions of minimal and canonical models.

**Definition 1.** A normal variety $X$ is called a minimal (resp. canonical) model if it satisfies the following conditions:

1. It has at worst $\mathbb{Q}$-factorial and terminal singularities (resp. at worst $\mathbb{Q}$-factorial and canonical singularities);
2. $K_X$ is nef (resp. ample).

We also have the following relative notion (to distinguish the previous one we will sometimes call it an absolute minimal (canonical) model, as opposed to these relative ones.)

**Definition 2.** Let $f : X \to Y$ be a morphism of normal varieties. A variety $X$ is called a relatively minimal (resp. canonical) model (with respect to $f$) if the following conditions are satisfied:

1. It has at worst $\mathbb{Q}$-factorial and terminal singularities (resp. at worst $\mathbb{Q}$-factorial and canonical singularities);
2. $K_X$ is $f$-nef (resp. $f$-ample).

Starting from here until the end of the paper we will consider the following basic situation. Let $B$ be a smooth projective curve over $k$ with the field of functions $K = k(B)$. Sometimes we will require that the genus $g = g(B)$ is greater than one, every time we need this assumption this will be mentioned.

Let $X_K$ be a minimal surface of general type defined over $K$; denote by $\pi_K : X_K \to \text{Spec } K$ the corresponding morphism.

**Definition 3.** A model of $X_K$ over $B$ is a (not necessarily smooth) threefold $X$ equipped with a flat and projective morphism $\pi : X \to B$ that induces the given morphism $\pi_K : X_K \to \text{Spec } K$ on the generic fiber.

**Definition 4.** We say that $X_K$ has good reduction at a point $b \in B$ if there exists a model $\pi : X \to B$ such that $\pi$ is smooth over $b$. Otherwise we say that $X_K$ has bad reduction at a point $b$.

We put $S_0 = \{\text{The set of points of bad reduction of } X_K\}$. It is obvious that $S_0$ is finite.

In the following proposition we summarize the main properties of minimal and canonical models related to our case.

**Proposition 1.** 1. There exist minimal models of $X_K$, i.e. such models $X = X_0$ that

(a) $X_0$ has (at worst) only normal, $\mathbb{Q}$-factorial and terminal singularities,
(b) $K_{X_0}$ is $\pi$-nef,
(c) for any other normal model $X$ of $X_K$ with only $\mathbb{Q}$-factorial and terminal singularities the natural rational map
\[
\begin{array}{ccc}
X & \dashrightarrow & X_0 \\
\downarrow & & \downarrow \\
B & = & B
\end{array}
\]

is a composition of (regular) divisorial contractions and rational maps that alterate the corresponding varieties only in codimension 2.

2. The set of all minimal models is finite; any two of them can be connected through a sequence of flops.

3. There exists a minimal model $X_0$ that is smooth outside of $S_0$.

4. There exists a unique canonical model $X_1$ of $X_K$, i.e. such that

(a) $X_1$ has (at worst) normal, $\mathbb{Q}$-factorial and canonical singularities,

(b) $K_{X_1}$ is $\pi$-ample,

(c) for any other normal model $X$ of $X_K$ with only $\mathbb{Q}$-factorial and terminal singularities

the natural rational map
\[
\begin{array}{ccc}
X & \dashrightarrow & X_1 \\
\downarrow & & \downarrow \\
B & = & B
\end{array}
\]

is a composition of (regular) divisorial contractions and rational maps that alterate the corresponding varieties only in codimension 2.

The natural rational $B$-map $f : X_0 \dashrightarrow X_1$ from any minimal model $X_0$ to $X_1$ is regular. Outside of $\pi^{-1}(S_0)$ its structure is as follows: it just contracts all $(-2)$-curves that lie in the fibers of $\pi$.

**Proof.** All assertions are well known (see, for example, [?]), except the third. To prove it, note at first that for every model $\pi : X \to B$ that is smooth (as a morphism) over $b \in B$, over some neighborhood of $b$ $X$ is smooth as a variety and $\pi : X \to B$ is a minimal model (of its generic fiber). Indeed, the fiber $X_b$ over $b$ cannot contain a $(-1)$-curve, for otherwise, by Kodaira’s theorem, this curve could be deformed over some neighborhood of $b$ to give a $(-1)$-curve in the generic fiber that is supposed to be minimal. From the classification of surfaces we now easily derive that $X_b$ should have general type and $K_{X_b}$ is nef. The same statement is true over some neighborhood of $b$.

Now it remains to check that the (finite) set of these ”locally-minimal” models can be glued together to get one smooth model over $S_0$. In fact, this can be done because of the ”locality” of the definition of a minimal model, see [?]. Alternatively, we can use explicit results of Rapoport and Burns [?] on the description of flops to obtain that all flops between minimal models over $S_0$ are accomplished in $(-2)$-curves contained in the fibers of $\pi$, so that we pass from one smooth model to another. •

The following proposition describes the connections between the notions of absolute and relative minimal models (in the case $g(B) \geq 2$).
Proposition 2. Let \( g(B) \geq 2 \). Then:

1. Every model \( X \) of \( X_K \) is of general type.

2. Every (relative) minimal model \( X_0 \) in the sense of the previous proposition is an absolute minimal model.

3. The (relative) canonical model \( X_1 \) in the sense of the previous proposition is the absolute canonical model.

Proof.

1. This is the case of the famous Iitaka conjecture that was proved in the works of Fujita – Viehweg [?].

2. By the famous result of Mori, if \( K_{X_0} \) is not nef, then there is a rational curve \( C \subset X_0 \) such that \( K_{X_0} \cdot C < 0 \). Since \( g(B) \geq 2 \), it has to lie in a fiber of \( \pi \), which contradicts the relative minimality of \( X_0 \).

3. Let us consider the canonical mapping \( \phi : X_0 \to X' \subset \mathbb{P}^N \) (\( X' \) being the image) induced by the pluricanonical system \( |8K_{X_0}| \). It is regular and birational [?] by claim 2. \( X' \) is the canonical model of \( X_0 \).

According to Miles Reid [?], all exceptional divisors in any resolution of \( X' \) are rational or ruled and, in particular, can be covered by rational curves. So all fibers of \( \phi \) can also be covered by rational curves. As in the proof of claim 2, it follows that we have a commutative diagram

\[
\begin{array}{ccc}
X_0 & \xrightarrow{\phi} & X' \\
\downarrow & & \downarrow \\
B & = & B,
\end{array}
\]

so \( X' \) is the relative canonical model. \( \Box \)

Corollary (of the proof). The canonical map \( \Phi = \Phi_{|8K_{X_0}|} : X_0 \to X_1 \subset \mathbb{P}^N \) separates the fibers of \( \pi : X_0 \to B \).

3 Application of Miyaoka – Yau inequality

Now we turn to the proof of our Main theorem. From now on we assume that \( S_0 = \emptyset \), even when it is not necessary for some of our statements.

Proposition 3. Let \( S_0 = \emptyset \). For every \( X_K \in \Phi(\emptyset, c_1^2, c_2) \) consider a corresponding minimal model \( \pi : X_0 \to B \) that is smooth everywhere (as a morphism). Then the following numerical invariants of \( X_0 \) are bounded by some constant that depends only on \( g(B), c_1^2 \) and \( c_2 \):

1. The Betti numbers \( b_i(X_0) \), \( i = 0 \ldots 6 \);

2. the arithmetic genus \( \chi(O_{X_0}) \);

3. the self-intersection index of the canonical class \( K_{X_0}^3 \).
Proof.

1. Consider the Leray spectral sequence:

\[ H^p(B, R^q\pi_* R) \Rightarrow H^{p+q}(X_0, R). \]

To show that the rank of the RHS is bounded we need to prove the boundedness of rank of the LHS. This follows from the following simple topological lemma.

**Lemma 1.** Let \( X \) be a CW-complex and \( R \) — a local system over \( X \). Then

\[ h_i(X, R) = \text{rk} H^i(X, R) \leq S_i \cdot \text{rk} R, \]

where \( S_i \) is the number of \( i \)-simplexes in arbitrary triangulation of \( X \).

**Proof.** In fact, this lemma immediately follows from the definition of the complex that computes the cohomology of a local system, see, for example, [?]. \( \square \)

Indeed, in our case the rank of \( R^q\pi_* R \) equals to \( b_q(X, R) \) and therefore is bounded.

2. Immediately follows from claim 1 using the Hodge decomposition.

3. Immediately follows from claim 2 using the Miyaoka inequality [?] and Riemann-Roch:

\[ K_{X_0}^3 \leq -3c_1(X_0)c_2(X_0) = -72\chi(O_{X_0}) \]

(we are using the fact that \( X_0 \) is smooth and \( K_{X_0} \) is nef). \( \square \)

4 Proof of the Main theorem

In this section we prove the Main theorem using proposition 3 (the boundedness of the invariants). All our arguments are fairly standard.

**The proof.** To begin with, we remark that the statement of our theorem is stable under base change, so we may assume that \( g(B) \geq 2 \).

Now consider an arbitrary surface \( X_K \in \Phi(0, c_1^2, c_2) \) and the corresponding smooth (as a morphism) minimal model \( \pi : X_0 \to B \). We have the canonical map \( \phi = \phi|_{8K_{X_0}} : X_0 \to \mathbb{P}^N \), which is a birational isomorphism onto its image – the canonical model \( i : X_1 \hookrightarrow \mathbb{P}^N \). Note that \( N = h^0(X_0, mK_{X_0}) - 1 \) is bounded in the sense of proposition 3 because of the Riemann-Roch theorem, Ramunajam vanishing and the boundedness of \( K_{X_0}^3 \) and \( \chi(O_{X_0}) \).

Let us fix \( N \). The degree of \( X_1 \) in \( \mathbb{P}^N \) is equal to \( s^8K_{X_0}^3 \) and is therefore bounded (this is in fact the main reason for the theorem to be true). The corollary of proposition 2 states that the map \( \phi \) separates the fibers \( X_{0,b} \) of \( \pi \). We will denote the images of these fibers by \( X_{1,b} \).

Now we claim that Hilbert polynomials \( P_b(n) = \chi(8nK_{X_{1,b}}) \) (that are of course independent of \( b \)) are bounded, i.e., form a finite set, provided \( B, c_1^2 \) and \( c_2 \) are fixed. This follows from the definition of canonical singularities (namely, \( P_b(n) = \chi(8nK_{X_{0,b}}) \)), the Riemann-Roch theorem for surfaces, adjunction formula \( K_{X_{0,b}} = K_{X_0}|_{X_{0,b}} \) and the boundedness of \( K_{X_0}^3 \) and \( \chi(O_{X_0}) \). Fix
one of these polynomials and denote by \( \mathcal{H} \) the corresponding Hilbert scheme. By the universal property of the Hilbert scheme we obtain a morphism \( \pi : B \to \mathcal{H} \), and the canonical model

\[
\begin{array}{c}
X_1 \xleftarrow{i \times \pi_1} P_B^N \\
\downarrow \pi_1 \quad \downarrow \\
B = B
\end{array}
\]

can obviously be reconstructed using this morphism. To prove our theorem we need to show that all morphisms \( \pi : B \to \mathcal{H} \) that occur in this way form a bounded family. For this we just need to prove that the degrees of graphs of these morphisms in \( B \times \mathcal{H} \) are bounded.

**Lemma 2.** Let \( \pi : B \to \mathcal{H} \) correspond to the canonical model

\[
\begin{array}{c}
X_1 \xleftarrow{i \times \pi_1} P_B^N \\
\downarrow \pi_1 \quad \downarrow \\
B = B
\end{array}
\]

Then

\[
\deg \Gamma_\pi = K_{X_1}^3 + O(1) = K_{X_0}^3 + O(1),
\]

where the left-hand side is the degree of the graph of \( \pi \) in some projective embedding of \( B \times \mathcal{H} \) (the constant in \( O(1) \) depends only on this projective embedding, \( B \), \( c_1 \) and \( c_2 \)).

This lemma gives us all that we need because of the boundness of \( K_{X_0}^3 \) (proposition 3). It follows (using the standard Chow embedding of \( \mathcal{H} \)) from the more general fact.

**Lemma 3.** Let \( i : X \hookrightarrow P^N \) be a subvariety and let \( \pi : X \to B \) be a morphism with connected fibers. Consider the Hilbert scheme \( \mathcal{H} \) corresponding to the Hilbert polynomial of fibers \( X_b \) of \( \pi \) and its embedding \( \mathcal{H} \subset P^M \) by the Chow coordinates. If \( \tilde{\pi} : B \to \mathcal{H} \) is the morphism that corresponds by the universality of Hilbert scheme to the commutative diagram

\[
\begin{array}{c}
X \xleftarrow{i \times \pi} P_B^N \\
\downarrow \pi \quad \downarrow \\
B = B
\end{array}
\]

then

\[
\deg \tilde{\pi}^*(O(1)) = \deg X,
\]

where \( O(1) \) is induced on \( \mathcal{H} \) by the standard hyperplane sheaf on \( P^M \).

The proof is almost tautological and immediately follows from the explicit description of Chow coordinates. \( \square \)
Acknowledgements

I want to express my gratitude to A.N. Parshin for proposing the problem and for constant attention to my work. I also want to thank V.A. Iskovskikh, Yu.G. Prokhorov, A. Pukhlikov, S. Nemirovski, M. Leenon and D. Osipov, who patiently answered my questions concerning some topics related to the content of this paper.

Part of this work was done during my stay (November – December 1998) at the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. I wish to thank Professor M. Narasimhan for his kind invitation.

References


