Available at: http://www.ictp.trieste.it/~pub_off

IC/99/7
RU-99-06

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

TWO LECTURES ON AdS/CFT CORRESPONDENCE

Michael R. Douglas[1]
Department of Physics and Astronomy, Rutgers University,
Piscataway, NJ 08855, USA
and
I.H.E.S., Le Bois-Marie, Bures-sur-Yvette France 91440

and

S. Randjbar-Daemi
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

This is a write-up of two lectures on AdS/CFT correspondance given by the authors at the 1998 Spring School at the Abdus Salam ICTP.

MIRAMARE – TRIESTE
January 1999

[1]Supported in part by DOE grant DE-FG05-90ER40559.

1 Introduction

The best candidate theories of quantum gravity, maximal supergravity and superstring theory, are formulated in more than four dimensions. Traditionally four-dimensional physics is derived using the Kaluza-Klein approach, which assumes space-time is a product of a non-compact manifold M and a compact manifold K. Einstein's equations in the full space-time then relate the curvature of these manifolds: if M is to be flat Minkowski space, K must be Ricci-flat (or related to a Ricci-flat manifold in a simple way). Massless fields on M arise as zero modes of differential operators on K.

These models are easily studied for K an n-torus, and candidate K's exist with reduced supersymmetry. However their Ricci-flat metrics are not known explicitly and studying them requires tricky, indirect methods. Rather, the next simplest possibility is to take K to be a homogeneous space such as the n-sphere S^n. The isometries of K lead to vector fields on M, so these compactifications produce gauged supergravity.

These models were much studied in the 1980's [?]; solutions exist but the positive curvature of K forces M to have equal and opposite negative curvature, making them at first sight phenomenologically irrelevant. The simplest possibility is M of constant negative curvature, i.e. anti-de Sitter space AdS_d.

These models also have maximal supersymmetry and thus are "too good" not to find a place in the web of theories known as M theory. A clear application emerged with the study of extremal black hole solutions [?]. Such solutions of supergravity in Minkowski space can preserve up to half of the global supersymmetry, but it was observed that in the near horizon limit they often preserve maximal supersymmetry. This comes about because the near horizon geometry is equivalent to an AdS compactification.

Black hole physics underwent a revolution with the advent of the D-brane as one could now find configurations with the same long range fields as an extremal black hole, but whose microscopic degrees of freedom are known explicitly. This led to the first generally accepted computation of the Bekenstein-Hawking entropy of a black hole, by Strominger amd Vafa [?]. Certain extremal black holes in type IIb superstring theory compactified on T^5 can be represented as bound states of D1 and D5-branes wrapped on the torus – both objects have exactly the same long-range fields, and are identified in string theory. On the other hand the D-brane system can also be described in world-volume terms – from open string theory one derives a two-dimensional conformal field theory, for which the entropy can be computed exactly. This entropy, certain Hawking emission rates, and many other observables agree exactly between the two descriptions.

The essential reason that conformal field theory appears in the discussion is to reflect the additional near-horizon symmetry visible in the space-time description. By D-brane considerations to be reviewed below, the near horizon limit in space-time is reflected in the low energy limit of the world-volume theory, which takes it to a fixed point with conformal symmetry. Such a d-dimensional conformal theory will then have $SO(d,2)$ space-time symmetry. This symmetry enhancement is exactly parallel to the enhancement of the black hole symmetry to the isometry group of AdS_{d+1}, $SO(d,2)$. The same relation holds in the superconformal case.

Another example of this type in type IIb superstring theory is the D3-brane. A configuration of N coincident D3-branes in Minkowski space is identified with another extremal black hole, as we discuss below. On the other hand, its low energy world-volume theory is $N = 4$ four-dimensional supersymmetric Yang-Mills theory. Thus we might conjecture that the entropy of this extremal black hole is equal to the entropy of $N = 4$ Yang-Mills theory.

Thus it was something of a surprise when it was found that – in distinction to the D1–D5 black hole – these entropies, computed in conventional perturbation theory, did not agree [**?**]. Further work found many quantities that do not agree, even in the D1–D5 system – for example, the metric as seen by a probe brane [**?**, **?**].

Of course the simplest resolution of all such issues is to argue that conventional perturbation theory is not accurate for these questions. In fact the gauge theory limit which must reproduce supergravity is large N and large 't Hooft coupling g^2N, where perturbation theory fails, and not much was known. So how does one proceed ?

Recently Maldacena has reversed this identification in a fruitful way [**?**]. Instead of trying to derive the properties of a black hole from the D-brane theory, one makes a precise conjecture stating that the D-brane theory is *equivalent to* the black hole and indeed all of the gravitational dynamics needed to describe it. Assuming this conjecture, one can derive results for the large 't Hooft coupling limit of gauge theory, by doing computations in AdS supergravity. Even better, by simple modifications of the background geometry, one can get results in gauge theories with reduced or no supersymmetry.

This relation has the same spirit as previously conjectured dualities: here there exists a single theory which reduces to perturbative gauge theory for weak coupling and to supergravity for strong coupling. There are a few observables constrained by supersymmetry to be independent of the coupling, and the basic test of the hypothesis is that these agree. Explicit computation has revealed that additional quantities agree in the two limits; no compelling statement has yet been made as to what should be expected to agree and why.

On the other hand the entropy and indeed almost all observables are expected to disagree[2] and thus one interprets these as non-trivial functions of the coupling for which one now has results in both limits. In general, one has as yet no direct way to test these predictions from either gauge theory or string theory; however even qualitative agreement can be regarded as evidence. In the case of the D3-brane, large N four-dimensional gauge theory was much studied over the years and numerous guesses made for its behavior. As it turns out the predictions from AdS/CFT contradict some of these guesses, but in ways which appear to form a new consistent picture of large N $\mathcal{N} = 4$ gauge theory; the better motivated guesses (e.g. for a deconfinement transition in pure Yang-Mills at finite temperature) are confirmed.

The aim of these lectures was to introduce this subject at an elementary level. We shall mostly be concerned with the case of the D3 brane in type IIb theory.

2 The 3-brane of type IIb

The massless sector of IIb string theory is IIb supergravity, which is described in many references including [**?**, **?**]. On general grounds a p-brane will be a source of a $p + 1$-form gauge potential

[2] The agreement for the D1-D5 system turns out to depend on a result special to two-dimensional conformal field theory; the invariance of central charge under marginal deformations.

and thus the minimal subsector of the theory required to describe a 3-brane will be the metric g_{MN} and the four-form potential C. This potential has a self-dual five-form field strength F for which the action is somewhat complicated but for our purposes we will only need the equations of motion

$$\frac{1}{\kappa^2}R_{MN} = \frac{1}{6}F_{MIJKL}F_N^{IJKL}$$
$$F = *F; \qquad dF = 0 \tag{1}$$

and supersymmetry transformations of the gravitino

$$\delta\psi_M = (D_M + \frac{i\kappa}{5!}F_{ABCDE}\Gamma^{ABCDE}\Gamma_M)\xi \tag{2}$$

where ξ and ψ_M are complex ten-dimensional Weyl spinors (representing the sum of two Majorana-Weyl spinors), and κ is the ten-dimensional Newton's constant.

We write the 3-brane ansatz

$$ds^2 = e^{2A(y)}dx^\mu dx^\nu \eta_{\mu\nu} + e^{-2B(y)}dy^2 \tag{3}$$

and

$$F = (1 + *)de^{4C(y)}dx^0 dx^1 dx^2 dx^3 \tag{4}$$

where $\eta_{\mu\nu}$, $0 \leq \mu, \nu \leq 3$ is the usual $3+1$ Minkowski metric, while y^i parameterize the six dimensions transverse to the brane. Ultimately the functions A, B and C will be taken to depend only on $r = |y|$ to describe a single 3-brane.

The brane solution will preserve supersymmetries corresponding to parameters ξ for which the gravitino variation is zero. We can decompose ξ into the product of four and six dimensional spinors of each chirality

$$\xi = f_+(y)\xi^{(4+)}\epsilon^{(6+)} + f_-(y)\xi^{(4-)}\epsilon^{(6-)} \tag{5}$$

and substituting the ansatz into (**??**) only Γ^{0123} appears, so up to 1/2 the bulk supersymmetry can be preserved, say the components $\xi^{(4+)}\epsilon^{(6+)}$ giving the equivalent of $N = 4$ supersymmetry in $d = 4$. Requiring (**??**) to have solutions we can derive the relation $A = B = C$ and substituting this into $dF = 0$ we find that $e^{-4C} \equiv f(y)$ must satisfy Laplace's equation in six flat transverse dimensions.

The final three-brane solutions are

$$ds^2 = f^{-1/2}dx^\mu dx^\nu \eta_{\mu\nu} + f^{1/2}dy^2 \tag{6}$$

and

$$F = (1 + *)d(f^{-1})dx^0 dx^1 dx^2 dx^3 \tag{7}$$

with

$$f(y) = 1 + \sum_{i=1}^{n} \frac{Q_i}{|y - y_i|^4}. \tag{8}$$

The charges Q_i are quantized as one can see by an extension of the Dirac argument: consider a second three-brane which extends in x^0 and (say) y^4, y^5 and y^6; from its point of view (**??**) is a magnetic gauge potential, and by moving it in the three remaining transverse dimensions we can rerun Dirac's argument. There is actually a subtle factor of two coming from the self-duality of F but the final quantization condition is

$$Q_i = 4\pi g_s N_i l_s^4 \tag{9}$$

with N_i integer.

Let us take $n = 1$ and $y_1 = 0$ to get a solution symmetric under $SO(6)$ rotations of y. Define $r = |y|$. The parameter $g_s N$ appearing in (**??**) will play a central role, so we define $\lambda = 4\pi g_s N$ and write

$$f(y) = 1 + \frac{\lambda l_s^4}{r^4}. \tag{10}$$

On general grounds in IIb string theory we believe the supergravity description of this solution only for $r >> l_s$, because at shorter distances we cannot justify ignoring the massive closed string modes. This is good as brane solutions in supergravity are typically singular (the extreme three-brane is an exception) but leads us to ask: How do we describe the regime $r << l_s$?

3 The 3-brane as a D3-brane

As is well-known by now, we can define the *same* solution of IIb string theory by introducing boundaries in our world-sheet theory, constrained to live on the plane $y = 0$. The charge N is represented by allowing an N-valued Chan-Paton factor; the multi-center solutions correspond to allowing several types of boundaries ending on the planes $y = y_i$. We can compute the long-range supergravity fields around this plane at leading order in g_s by doing world-sheet path integrals on a disk with a graviton or RR vertex operator inserted; we can also verify that the Dirichlet boundary conditions linearly relate the left and right-moving world-sheet operators generating space-time supersymmetry as

$$Q_L = \Gamma^{0123} Q_R \tag{11}$$

and thus preserve the same supersymmetries as the supergravity solution (**??**). These considerations lead us to *identify* the D-brane as the unique object in string theory corresponding to the field configuration (**??**).

This description is non-singular and in contrast to the supergravity description becomes simpler as we consider short transverse distance scales $r << l_s$. Fluctuations of the D-brane are described by exciting open strings ending on the D-brane, and effects described by open strings stretched to the radius r will be associated with the mass scale $m = T_s r = r/2\pi l_s^2$ (the usual string tension energy). In general excited open strings will also contribute but if $r << l_s$ these will be much heavier than the lightest open strings.

Thus we can simplify our brane theory by taking a scaling limit of short distances and large string tension: $r \to 0$ and $l_s \to 0$, and work in the energy units set by the stretched strings: $u = r/l_s^2$ fixed. This is the low energy limit of the world-volume theory, which is $N = 4$ $U(N)$ super Yang-Mills theory (SYM) with gauge coupling $g_{YM}^2 = 4\pi g_s$.[3]

We review some well-known facts about this theory. The $N = 4$ gauge multiplet contains besides the gauge field six real adjoint scalars Y^i and four Majorana gauginos $\chi^{I\alpha}$. The action

[3]Our conventions are such that S-duality is $g_s \to 1/g_s$.

can be obtained by dimensional reduction from $N = 1$, $d = 10$ SYM and contains the potential $V = \sum_{i<j}[Y^i, Y^j]^2$. It respects an $SO(6)$ R symmetry which is simply inherited from the original rotational symmetry in $d = 10$ SYM and acts on Y^i as a vector.

As a field theory on $\mathbf{R^{3,1}}$, the moduli space of supersymmetric vacua is parameterized by commuting matrices $[Y^i, Y^j] = 0$ up to gauge transformation; i.e. is $\mathbf{R^{6N}/S_N}$. This is identical to the parameter space for the most general harmonic function (**??**) with all $Q_i = 1$ so the moduli space is identical in the two descriptions.

If we take all $Y^i = 0$ we have an unbroken conformal invariance $SO(4,2)$ in the classical theory and (unusually) even after quantization, because the beta function vanishes. This directly implies scale invariance and indirectly implies conformal invariance in the quantum theory.[4] In fact we have a superconformal quantum theory in four dimensions. Now the superconformal algebra is generated by the product of the conformal and supersymmetry algebras, but is larger – we can apply conformal inversion to the supercharges $Q^{I\alpha}$, to obtain partners $S^{I\alpha}$, leading to a total of 32 fermionic generators. The full algebra is $SU(2,2|4)$ and given in [**?**]. We shall come back to this algebra in section 5 and give a brief descrption of some of its unitary irreducible representations.

Clearly this is a very interesting point in the moduli space, however it was never too clear what its physics might be. In particular, the usual particle and S-matrix interpretation for quantum field theory is problematic as Green's functions are not expected to have the required analytic structure.

On the other hand there are numerous qualitative consequences of conformal invariance which are well known in two dimensions and which we can expect to hold here. One of these is the operator-state correspondence. This is most clearly formulated by using radial quantization: we choose a point, say $x^\mu = 0$ in Euclidean $\mathbf{R^4}$, and define the state on surfaces of constant $|x|$. We then develop the canonical formalism with $|x|$ as time and quantize; the role of Hamiltonian is then played by the dilatation operator D. The conformal invariance prediction for the two-point function

$$\langle \phi^i(x)\phi^j(0)\rangle = \frac{C_i \delta^{ij}}{|x|^{2\Delta_i}} \tag{12}$$

then (by spectral decomposition) implies that each primary field $\phi^i(x)$ is associated with a distinct state $|i\rangle = \phi^i(0)|0\rangle$ of 'energy' $D\ |i\rangle = \Delta_i\ |i\rangle$.

Superconformal invariance also leads to constraints on short multiplets of supersymmetry. The generic $\mathcal{N} = 4$, $d = 4$ supersymmetry multiplet with 256 components must be annihilated by all of the $S^{I\alpha}$. This requires the multiplet to belong to the appropriate representation of $SO(6)$ to allow both sides of $\{Q, S\} = D + J$ (where J are the $SO(6)$ charges) to vanish. Since $SO(6)$ charges are quantized, the dimensions and spectrum of short multiplets are independent of continuous parameters such as the coupling constant. Thus "chiral operators of $\mathcal{N} = 4$" which create states in these multiplets can be completely enumerated in perturbation theory.

One way to do this is to pick an $\mathcal{N} = 1$ subalgebra of $\mathcal{N} = 4$ as chiral operators in $\mathcal{N} = 1$ are those which can be written as $\int d^2\theta$ in superspace; each $\mathcal{N} = 4$ multiplet will contain a unique

[4] Actually, when we are taking the low energy limit, any starting theory will flow to a conformal theory. Examples in which the theory only becomes conformal in this limit include the D1-D5 black hole, the 2-brane in M theory, and many others.

sub-$\mathcal{N}=1$ chiral multiplet of largest $U(1)_R$ charge.

The $\mathcal{N}=4$ theory in $\mathcal{N}=1$ superspace has three chiral adjoints Z^i and a field strength W^α, and superpotential $W=\mathrm{Tr}Z^1[Z^2,Z^3]$. A representative set of chiral operators (not all) is

$$O_{1n}^{i_1 i_2 \ldots i_n} = \mathrm{Tr}Z^{i_1}Z^{i_2}\ldots Z^{i_n} \tag{13}$$

Note that this set of operators is huge as we need to distinguish all possible orderings of the indices i_k in the large N limit. On the other hand we need to remove descendant operators such as those predicted by the equations of motion,

$$\frac{\partial W}{\partial Z^i} = \epsilon_{ijk}[Z^j,Z^k] = D\bar{D}Z^i \sim 0 \tag{14}$$

in terms of the chiral ring. In other words, any operator in (??) which includes a commutator is a descendant.

The result is that $\mathcal{N}=4$ superconformal theory, for any value of the Yang-Mills coupling constant, contains a sequence of chiral operators O_{1n} of dimension $\Delta=n$ which transform in the n-fold symmetric tensor of $SO(6)_R$. The complete spectrum of chiral operators can be worked out [?].

Note by contrast that we cannot make any statement about the spectrum of non-chiral operators at strong coupling. An operator such as $\sum_i \mathrm{Tr}Y^iY^i$ will have dimension 2 in the free theory but can gain an arbitrary anomalous dimension, presumably computable as a power series in g^2_{YM} and N. When these corrections are large, we have little direct control on these dimensions or indeed any generic observable from the gauge theory point of view.

4 Large N

Another limit which is believed to simplify the gauge theory is that of large N. Although we can imagine different such limits, the best studied (and probably best) is that of 't Hooft where we hold the 't Hooft coupling $g^2_{YM}N=\lambda$ fixed in the limit. As is by now classic (and reviewed in [?]) the perturbative expansion in this limit reduces to the sum of planar diagrams where a diagram with V vertices is weighed by the factor $(g^2_{YM}N)^V$. Furthermore, corrections in $1/N$ are organized in a topological expansion, with diagrams which can be drawn on a genus g surface weighed by N^{2-2g}.

This gives us a formal relation to string theory, in which each operator written as a single trace of a product of adjoints corresponds to an operator creating or destroying a single closed string, and $1/N$ plays the role of the closed string coupling constant g_s. Specifically,

$$\langle \mathrm{Tr}O_1\rangle = N[O_1]_{disk} + N^{-1}[O_1]_{punctured\ torus} + \ldots \tag{15}$$

$$\begin{aligned}\langle \mathrm{Tr}O_1\ \mathrm{Tr}O_2\rangle &= N^2[O_1]_{disk}[O_2]_{disk} + N^0[O_1O_2]_{annulus} \\ &+ N^0[O_1]_{disk}[O_2]_{punctured\ torus} + N^0[O_2]_{disk}[O_1]_{punctured\ torus} + \ldots\end{aligned} \tag{16}$$

where each term $[O_1\ldots]_{surface}$ corresponds to the contribution of a string world-sheet with that topology and the specified operators inserted at each boundary (or puncture).

This relation is very direct in weak coupling perturbation theory, so when this series converges, as was true for the "old matrix models" [?], we can confidently say that large N field theory is equivalent to a string. We might hope to prove (??) for general g^2N by analytic continuation.

However, we should recognize that the story for most field theories is not so simple – the weak coupling perturbation theory is more typically asymptotic. This leads to many potential difficulties with the string interpretation. The simplest appears in asymptotically free theories, where g^2N is dimensionful, and the dynamically generated mass gap is known to be non-analytic at $g^2N = 0$. In the solvable example of $\mathcal{N} = 2$ SYM, (??) can be seen explicitly to fail [?].

A different approach to string theory starts with a strong coupling expansion around $g^2N \sim \infty$. (This is a subject with a long history; see [?] and references there.) This can at present be made precise only in two dimensions or on the lattice but in these cases leads directly to a string with finite string tension, computable in an expansion with finite radius of convergence. However these expansions generically predict large N transitions and a critical g_c^2N below which the string expansion breaks down.

The situation may well be better in a superconformal theory and there is no strong argument against analyticity on the positive g^2N axis in this case (but see [?]). One could then assume (??) to obtain a non-perturbative description of the theory, as we explain in the last section.

5 Near-horizon geometry and AdS_5 supergravity

The p-brane solutions were originally found as generalizations of the extreme Reissner-Nordstrom solution of Einstein-Maxwell theory and this leads us to ask whether the solutions have event horizons and should be considered as black holes. The story is different for different solutions but what we need to do is consider the limit $r = 0$ and understand the behavior of the metric there.

This limit of the (single center) metric (??) is

$$ds^2 = \left(\frac{r^2}{\lambda^{1/2} l_s^2}\right) dx^2 + \left(\frac{\lambda^{1/2} l_s^2}{r^2}\right) \left(dr^2 + r^2 d\Omega_5^2\right) \tag{17}$$

where we have written $y^i = r\hat{y}^i$; $\hat{y}^i$ is a unit vector in R^6 parameterizing the sphere S^5, and $d\Omega_5^2$ is the round metric on S^5. This limit will be justified (we can drop the "1" term in (??)) when

$$r << R \equiv \lambda^{1/4} l_s. \tag{18}$$

The first thing to notice is that the r dependence cancels out in the S^5 metric, leaving us with a solution $M \times S^5$. Flux quantization for the five-form field strength tells us that its restriction to S^5 must also be independent of r, while the equation of motion tells us it will be a harmonic form on S^5. The stress tensor for this special case is easily computed and is invariant under $SO(4,1) \times SO(6)$, leading to a constant curvature solution in both S^5 and in M. The $SO(d-1,1)$ Lorentzian metric of constant negative curvature is anti-de Sitter space AdS_d and in fact admits an action of $SO(d-1,2)$, as we will see below.

Now that we have obtained $\mathrm{AdS}_5 \times S^5$ as a near-horizon limit we can also think in five-dimensional terms, by making the Kaluza-Klein reduction of IIb supergravity on S^5. This was worked out in detail in [?, ?]; we summarize here.

According to the Kaluza-Klein program we must expand all the ten dimensional fields in harmonics of the isometry group $SO(6)$ of S^5. In this way we generate an infinite number of AdS_5 fields with spins ranging from 0 to 2. Each field has a definite $SO(6)$ content. One should then solve the linearized type IIb supergravity equations for these modes to determine the mass spectrum of the physical states and their behaviour under the isometry group $SO(4,2)$ of AdS_5. For the $AdS_5 \times S^5$ solution the analysis has been caried out in detail in [?].

An alternative route to obtain information about the spectrum is to use the supersymmetry algebra $SU(2,2|4)$ of the $\mathrm{AdS}_5 \times S^5$ background. Let M_{AB}, $A, B = 1,2,3,4$ or $0,-1$ denote the generators of its $SO(4,2)$ bosonic subgroup of isometries of AdS_5, and let B^M_N with $1 \leq M, N = 4$ denote generators of $SU(4) \cong SO(6)$, the isometry group of S^5. Supersymmetry generators correspond to solutions of the Killing spinor equations (which set (??) to zero). Half of these were discussed in section 2: the D3-brane solution preserves an $SO(3,1)$ subgroup of $SO(4,2)$, generated by the subset M_{ab}, $0 \leq a, b \leq 3$, and 16 real supersymmetries, a 4-plet of Weyl spinors of $SO(3,1)$ which we denote by Q^M with $M = 1,..4$ a vector of $SU(4)$.

The commutators of Q^M with $K_a = M_{a4} - M_{a,-1}$ now produce another 4-plet of Weyl spinorial charges S^M, enlarging the total number of unbroken supersymmetries to 16 complex. The set M_{AB}, B^M_N, Q^M, S^M generates $SU(2,2|4)$.[5] As we commented in section 3 this symmetry group follows from combining $\mathcal{N} = 4$ supersymmetry with the enlargement $SO(4,2) \supset SO(3,1)$ and is also the superconformal symmetry of $\mathcal{N} = 4$ SYM.

Being the symmetry group of the background manifold, the KK spectrum will be a discrete unitary (reducible) representation of $SU(2,2|4)$. We will discuss this representation theory in some detail shortly. The masses of 5-dimensional modes will be given in terms of the eigenvalues of the generator $E = M_{0,-1}$. This operator generates translations along a global timelike Killing vector field of AdS and therefore is a useful choice for the AdS energy operator.

The simplest way [?, ?] to construct the lowest weight unitary irreducible representation (UIR) of $SU(2,2|4)$ is to introduce a set of superoscillators $\xi^A = \{a^i, \alpha^\mu\}$ and $\eta^M = \{b^r, \beta^x\}$, where i, μ, r and $x =$ 1,2. They satisfy the usual algebra of the creation and annihilation operators, viz, $[a^i, a_j] = \delta^i_j$, $[b^r, b_s] = \delta^r_s$, $\{\alpha^\mu, \alpha^\nu\} = \delta^\mu_\nu$ and $\{\beta^x, \beta^y\} = \delta^x_y$. All other commutators or anticommutators are zero. Here we have denoted the hermitian conjugate of a^i by a_i, etc. The $SU(2,2|4)$ generators are constructed as bilinears in these oscillators. For example the 16 generators B^M_N of $U(4)$ are given by $\alpha^\mu \alpha_\nu, \beta^x \beta_y, \alpha^\mu \beta^x$ and $\alpha_\mu \beta_y$.

The Fock space of the oscillators provides the vector space on which one particular class of lowest weight UIR of $SU(2,2|4)$ are realized. These representations are called the doubleton represention, because only one pair of superoscillators are used in their construction. Furthermore, although the doubleton representations are not part of the KK spectrum of the IIb supergravity on the $\mathrm{AdS}_5 \times S^5$ background, some of them can be given an interpretation of massless states in a conformal field thery in 4 dimensions. They are massless because the entire set of doubleton representations are in the kernel of the 4-dimensional mass operator $m^2 = P^a P_a$, [?] where the

[5] This group has an extra $U(1)$ factor which should be factored out.[?]

momentum is defined by $P_a = M_{a,0} + M_{a,-1}, a = 0, 1, 2, 3$.

The simplest doubleton representation is built using the Fock vacuum $|0>$ as the lowest weight vector of $SU(2,2|4)$. It can be shown that the multiplet contains 2^4 physical states: 6 real scalars, 4 Weyl spinors and a vector potential, transforming respectively in the vector, spinor and the singlet representations of $SO(6)$. This is the $\mathcal{N} = 4$ super Yang Mills multiplet in $d = 4$.

As another example we consider the direct sum of the two doubleton representations built on the lowest weight vectors $\xi\xi|0>$ and $\eta\eta|0>$. The resulting states correspond to the spectrum of the $\mathcal{N} = 4$ superconformal gravity in $d = 4$. The full list of the doubleton representations have been given in [?].

By taking tensor products of doubletons one builds other repesntations of $SU(2,2|4)$. An equivalent way to do this is to affix an index $K = 1, ..., p$ to the oscillators and write them as $a^i(K), b^r(K), ...$ The commutators then become $[a^i(K), a_j(K')] = \delta^i_j \delta_{KK'}$, etc. The generators of $SU(2,2|4)$ are again bilinear invariants of the $O(p)$ group acting on the index K. For example, the $U(4)$ generators are given by $\alpha^\mu(K)\alpha_\nu(K), \beta^x(K)\beta_y(K), \alpha^\mu(K)\beta^x(K)$ and $\alpha_\mu(K)\beta_y(K)$, where we sum over the repeated K label from 1 to p.

By choosing different values of p and different lowest weight spaces, one can obtain many UIRs of $SU(2,2|4)$. Each of these will contain a finite number of UIRs of $SO(4,2) \times SO(6)$, and each mass shell physical mode in AdS_5 will come in such a UIR. However, states in the KK spectrum of IIb supergravity on $\mathrm{AdS}_5 \times S^5$ will correspond to using the Fock vacuum $|0>$ as the lowest weight vector, and live in short multiplets of $SU(2,2|4)$. This follows simply because KK reduction will only produce states with spins not exceeding 2. Other choices of the lowest weight vectors such as $\xi\xi...\xi|0>$ will produce a longer spin range and thereby also longer multiplets.[6]

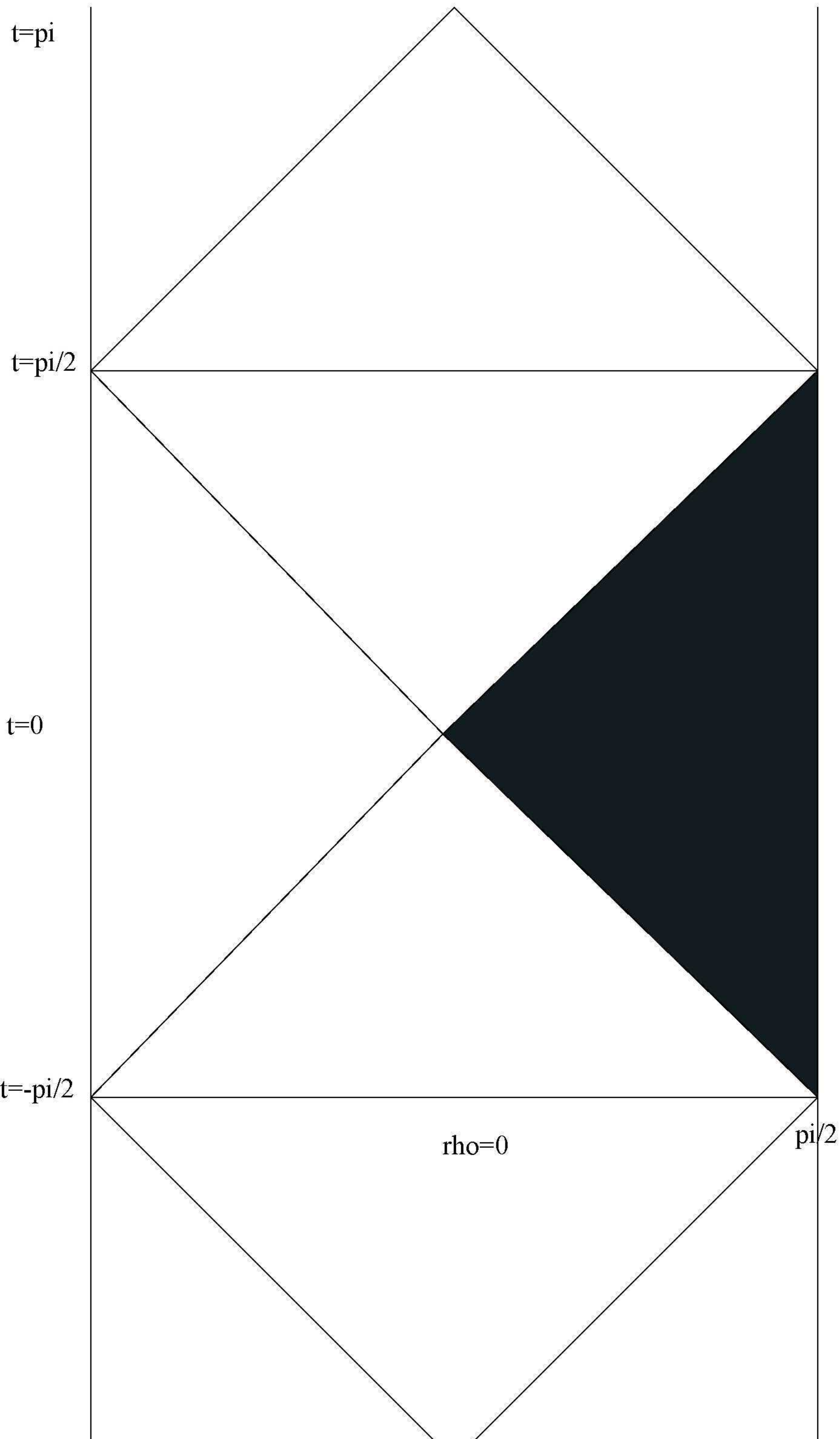
Restricting attention to this choice, the multiplets are conveniently characterized by their lowest weights (J_L, J_R, E) under the subgroup $SO(4) \times SO(2) \subset SO(4,2)$. For example, for each $p > 1$, the $SO(4,2)$ representations defined by the lowest weight vector $(1, 1, 2p + 4)$ contain all spin 2 modes in AdS_5 transforming in the representations characterized by the Dynkin label $(0, p - 2, 0)$ of $SO(6)$. The complete multiplet contains $128\frac{p^2(p^2-1)}{12}$ fermionic and the same number of bosonic states (see Table I of [?]).

For $p = 2$ the spin 2 mode is the 5-dimensional graviton which is a singlet of $SO(6)$. In addition this multiplet (the massless supergraviton multiplet) contains four complex gravitini, 15 vector fields in the adjoint of $SO(6)$, plus spin zero and spin 1/2 objects for a total of 128+128 physical states.

The $p > 2$ representations will correspond to the massive KK towers. In fact, the complete KK spectrum on $\mathrm{AdS}_5 \times S^5$ is obtained by taking a single copy of each $p \geq 2$ representation. Again with each $SO(4,2)$ lowest weight we can associate a particular mass shell mode in AdS_5 in a definite $SO(6)$ representation.

To close this section we remark that since the supersymmetries transform under $SO(6)$, so will the gravitinos. The presence of vector fields in the adjoint of $SO(6)$ thus means that the 5-dimensional theory is one of the known gauged supergravities [?].

[6] For $p = 2$ a complete classification of these has been given in [?].


t=pi
t=pi/2
t=0
t=-pi/2
rho=0
pi/2