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DISORDER-INDUCED PHASE TRANSITION
IN A ONE-DIMENSIONAL MODEL OF RICE PILE

M. Bengrine
Laboratoire de Magnetisme et de Physique des Hautes Energies,
Departement de Physique, Faculte des Sciences, Rabat, Morocco
and
The Abdus Salam International Centre For Theoretical Physics, Trieste, Italy,

A. Benyoussef
Laboratoire de Magnetisme et de Physique des Hautes Energies,
Departement de Physique, Faculte des Sciences, Rabat, Morocco,

F. Mhirech¹
Laboratoire de Magnetisme et de Physique des Hautes Energies,
Departement de Physique, Faculte des Sciences, Rabat, Morocco
and
The Abdus Salam International Centre For Theoretical Physics, Trieste, Italy

and

S.D. Zhang²
Institute of Low Energy Nuclear Physics, Beijing Normal University,
Beijing 100875, People’s Republic of China,
Beijing Radiation Center, Beijing 100088, People’s Republic of China
and
The Abdus Salam International Centre For Theoretical Physics, Trieste, Italy.

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¹ E-mail: mhirech@fsr.ac.ma
² E-mail: zhangsd@bnu.edu.cn
Abstract

We propose a one-dimensional rice-pile model which connects the 1D BTW sandpile model (Phys. Rev. A 38, 364 (1988)) and the Oslo rice-pile model (Phys. Rev. Lett. 77, 107 (1997)) in a continuous manner. We found that for a sufficiently large system, there is a sharp transition between the trivial critical behaviour of the 1D BTW model and the self-organized critical (SOC) behaviour. When there is SOC, the model belongs to a known universality class with the avalanche exponent $\tau = 1.53$. 
Bak et al. proposed "self-organized criticality" (SOC) as a framework to understand the dynamics of driven, dissipative systems [1]. These systems, via self-organization, reach the steady state which is characterized by a power-law distribution of the sizes of avalanches. The authors of Ref. [1] originally used a cellular automata, now referred to as the BTW sandpile model, to illustrate their ideas. Moreover, the SOC behaviour was also found in some biological [2] and economical [3] systems. Different variations of the BTW sandpile were proposed and studied [4-8]. The behaviours of these sandpiles are somewhat different depending on whether the rules of evolution are based on the absolute sand heights of the pile [6-8], the local slopes [4,6], or the Laplacians of the sand height function [6]. Besides, the numerical simulations many different methods were also used to treat the SOC problems. Dynamical mean field theory [9] gives a unified description of some stochastic SOC systems including the BTW sandpile model and the forest fire model [10]. Langevin-type approaches [11] have been used on a phenomenological basis. Furthermore, a real space renormalization group method [12] provided good estimates of the avalanche exponents. Finally, it has been shown [5] that a large class of sandpiles were Abelian and this property leads to a particularly simple equiprobable partitioning in configuration space that allows to extract some exact results. Numerical simulations of high dimensional BTW model [13] were recently performed to determine the upper critical dimension where the avalanche distributions are characterized by the mean-field exponents. The idea of SOC also stimulated much interest in the granular matter and some experiments [14] were done to investigate whether real sandpile display SOC behaviour. In order to make a comparison with theories and models, a group of researchers in Oslo did experiments on real rice piles [15] and showed that under some conditions a real rice pile displays SOC behaviours. In fact, for grains with a large aspect ratio the system self-organizes into a critical state. They explained this result with the increased friction and different packing possibilities. By measuring the transit time after the pile has reached the stationary critical state, they found that the distribution of the transit times follows the form:

\[ P(T, L) = L^{-\beta} F(T/L^\nu), \] (1)

where \( T \) is the transit time, \( L \) is the system size. The scaling function \( F(x) \) is constant for small \( x \) and decays as power law with a slope \( \alpha = 2.4 \pm 0.2 \) for larger \( x \).

To take into account the changes in the local slopes observed in the rice pile experiment, Christensen et al. proposed a rice-pile model, hereafter called the Oslo model [16-18], where the critical slope for each site is a dynamical variable. The Oslo model is based on a linear array of cells labelled by \( i \), where \( i = 1, 2, ..., L \), and an integer variable \( h(i) \) assigned to each of them, with a wall at \( i = 0 \) and an open boundary at \( i = L + 1 \). Here \( h(i) \) is called the local height of the rice pile at site \( i \). The local slope assigned to each site \( i \) is defined as
\[ z(i) = h(i) - h(i + 1) \] for \( i = 1, 2, \ldots, L \). Initially, the system is empty, i.e., \( h(i) = 0 \) \( \forall i \). The dynamics of the model consists of deposition and relaxation, and the relaxation process is considered to be fast compared to the deposition time scale. At each time step one grain is added to a column \( i \) which increases the height of \( i \) by 1, i.e., \( h(i) \rightarrow h(i) + 1 \). With the dropping of rice grains, a rice pile is built up. Whenever there is an active column, i.e., \( z(i) > z_c(i) \), where \( z_c(i) \) is a slope threshold, one grain of rice will be transferred from this column to its right neighbor, \( h(i) \rightarrow h(i) - 1 \) and \( h(i + 1) \rightarrow h(i + 1) + 1 \), and all the unstable sites topple in parallel. The critical slope \( z_c \) of a site remains unchanged if the site is stable but assumes a new value 1 or 2 randomly every time a rice grain on this site has toppled. The toppling of one or more sites is called an avalanche event, and during the avalanche no grains are added to the pile. The avalanche stops when the system reaches a stable state with \( z(i) \leq z_c(i) \) \( \forall i \). The internal randomness in the critical slopes makes the Oslo model different from the 1D BTW model. For the Oslo model, with arbitrary initial conditions, the system reaches a stationary state where the avalanche sizes are power-law distributed, while for the 1D BTW model the size distribution of avalanche is generally not power-law. Since the only difference between the 1D BTW model and the Oslo model is the presence of internal disorder in the latter, it is natural to consider that the criticality in the Oslo model is induced by the disorder (randomness in the critical slopes). One motivation of this letter is to investigate the transition between the 1D BTW model and the Oslo model.

We modify the Oslo model as follows: When \( z(i) \leq z_c(i) \) \( \forall i \), the pile is stable and there isn’t a diffusion of particles. If, at site \( i \), \( z(i) > z_c(i) \), where \( z_c(i) \) takes randomly a value 1 or 2, then this site topples with a probability which depends on its slope \( z(i) \), namely: if \( 1 < z(i) \leq 2 \), the site \( i \) topples with a probability \( p_1 \) and if \( z(i) > 2 \), it topples with probability \( p_2 \). Notice that if we set \( p_1 = 0 \) and \( p_2 = 1 \), the model becomes the BTW model with the critical slope \( z_c = 2 \), and if we set \( p_1 = p_2 = 1 \), it is just the Oslo model.

In a previous paper [19], one of us generalized the Oslo model in a different way, in which the critical local slope can assume \( r \) different values, from 1 to \( r \), where \( r \) is an integer. It is possible to show that the exponents for the avalanche-size and transit-time distributions are insensitive to the level of medium disorder (different values of \( r > 1 \)) when the grain is dropped at a fixed position. So in the present model we will only consider the case \( r = 2 \), i.e., the critical slope takes randomly the value 1 or 2. In this study, we will restrict ourselves to the case where a grain is added to the site \( i = 1 \). When a grain is dropped on the left-end site of the pile, it may make the site unstable. The site topples and transfers a grain of rice to its right neighbour and so on. And in this way avalanches occur. As in the literature, we define the size of an avalanche as the number of toppling. In this letter, we will take \( p_2 \) equal to 1 and \( p_1 \leq p_2 \) because the higher is the slope, the higher is the jump probability.
So by varying $p_1$ from 0 to 1, we can change the model from the 1D BTW sandpile model to the Oslo model in a continuous manner.

We have performed extensive numerical simulations and investigated the effect of $p_1$ on the behaviours of the model.

Let us first study the transport properties of the model. The transit time of a grain is defined as the time it spent in the pile, and the time is measured in the unit of additions of grains. When a grain slips out of the pile, we can measure its transit time $T = T_{out} - T_{in}$, where $T_{out}$ and $T_{in}$ denote the input and output time of the grain. For the case $p_1 = 0$ (the 1D BTW model) when the stationary state is reached every newly-added grain will slip out of the pile instantly, thus the transit time is $T = 0$, and the average transportation velocity of grains, defined as $<v> = L / <T>$, is infinite for this case. On the other hand, for the case $p_1 = 1$ (the Oslo model) previous studies show that the average transportation velocity scales with the system size as $<v> \propto L^{-\gamma}$ with $\gamma = 0.30 \pm 0.10$, indicating that $<v> \rightarrow 0$ in the limit of infinite system size. It would be interesting to see how $<v>$ changes from $\infty$ to 0 when $p_1$ is varied from 0 to 1. In fact, we found that there is a sharp transition at $p_1 = 0$. The general behaviour of $<v>$ as a function of $p_1$ is shown in Fig. 1. One can see from Fig. 1 that for a given system size $L$, the velocity $<v>$ becomes constant when $p_1$ is larger than some value, say $p_1^c$, where $p_1^c$ itself is dependent on the system size and becomes closer to 0 when $L$ becomes larger. The numerical results make us consider that $p_1^c \rightarrow 0$ as $L \rightarrow \infty$. When $p_1 \rightarrow 0$, $<v> = 1$, independent of the system size. So there is a sharp transition from $<v> = \infty$ for $p_1 = 0$ to $<v> = 1$ for $p_1 = 0^+$. At first, the sharp transition may be surprising. But it can be understood by the following argument. It is clear that when $p_1$ is exactly 0, no newly-added grain will stay in the pile as long the stationary state is reached. So $T = 0$ for every grain, and hence $<T> = 0$. For $p_1 = 0^+$ however, the situation is different, and there is the possibility that some grains will be buried in the surface layer of the pile. These grains will stay in the pile for a very long time. Once they slip out of the pile, these grains, although very few in number, will make a significant contribution to $<T>$ since their transit times are extremely large. It is the existence of these grains that makes $<T>$ assume a finite value for $p_1 = 0^+$. Thus the sharp transition here is induced by the tiny disorder. Between $p_1 = 0^+$ and $p_1 = p_1^c$, there is crossover behaviour of $<v>$, which is due to finite size effects. Since we expect $p_1^c \rightarrow 0$ when $L \rightarrow \infty$, we can also expect that for an infinite system the transition takes place at $p_1 = 0$ form $<v> = \infty$ to $<v> = 0$.

In what follows we shall show that the critical behaviours of the model belong to the same universality class of the Oslo model when $p_1$ is larger than $p_1^c$. In Fig. 2, we plot the average velocity as a function of the system size for different values of $p_1$ greater than $p_1^c$. It
is clear that for large enough system \((L > 100)\), the average velocity \(<v>\) scales as \(L^{-\gamma}\), where \(\gamma \approx 0.23\). Therefore the average velocity decreases with the system size, which is due to the increase in the active zone depth with system size, as explained by Christensen et al [16].

We have also studied the avalanche size and the transit time distributions for different values of the probability \(p_1 > p_1^c\). In a previous study on the rice pile models [17] it was shown that the avalanche size distribution is of the form:

\[
P(S, L) = S^{-\tau} G(S/L^D),
\]

with \(\tau = 1.53 \pm 0.05\) and \(D = 2.20 \pm 0.05\). We report in Fig. 3.a our simulation data for the distribution of avalanche size for different values of system size. The distribution is a power law with the presence of a peak close to the cutoff size \(S_c \propto L^D\). This peak is due to the finite-size effects which leads the system into a supercritical state, followed by a massive avalanche. Since on average each site must topple exactly once during an avalanche to transport a grain out of the pile when the stationary state is reached, the average size of avalanche is thus \(<S> = L\). And this leads to the scaling relation \(D(2 - \tau) = 1\) [18].

The numerical results of the exponents, namely \(\tau \approx 1.53\) and \(D \approx 2.20\) are in quite good agreement with the scaling relation. The transit-time distribution can be described by the scaling form (1) with the same exponents as the Oslo model \((p_1 = 1)\), namely, \(\beta \approx 1.25\) and \(\nu \approx 1.25\). The power-law exponent \(\alpha\) for the large transit time is obtained as \(\alpha \approx 2.4\).

(See Fig. 4 for an example.) Fig. 5.a shows the avalanche-size distribution for a pile of size \(L = 400\) and for several values of jumping probability \(p_1\). It is clear that the size exponent \(\tau\) is insensitive to the value of the probability \(p_1\) greater than a critical value \(p_1^c\). Fig. 5.b gives the corresponding transit time distribution, which is nearly a constant for small transit time, and decays as a power law for larger transit time. As in Fig. 5.a, the exponents remain the same for several values of \(p_1\) greater than \(p_1^c\). Therefore, we can conclude that for \(p_1\) greater than \(p_1^c\) the size and transit time exponents are the same as the Oslo rice-pile model, thus the SOC state in our model is not affected by the fact that some grains topple with different jumping probability \(p_1\), when \(p_1 > p_1^c\).

The crossover behaviours were also investigated. We collect statistics on the size of the avalanche and transit time of the grains for a value of \(p_1\) very close to zero \((p_1 = 10^{-2})\). As illustrated in Fig. 6.a, the avalanche-size distribution does not exhibit a power-law behaviour. The data can be described by the following form:

\[
P(S, L) = L^{-\sigma} G(S/L^\sigma)
\]

which is confirmed by the good data collapse obtained with the exponents \(x = 0.94\) and \(\sigma = 0.94\) and where the scaling function \(G(x)\) decays exponentially (Fig. 6.b).
form of Eq.(3) is very different from the distribution form for the case $p_1 = 0$, which is $P(S,L) = \delta(S - L)$, where $\delta(x)$ is the Dirac function. This confirms our statement that there is a sharp transition at $p_1 = 0$. In Fig. 7.a, the transit time distribution is plotted. It is visually apparent that the distribution peaks at the vicinity of the system size. A data collapse is shown in Fig. 7.b, which is obtained by rescaling the original data according to Eq. (1) with $\beta = 1.2$ and $\nu = 1.2$. The exponents used here are a little smaller than that for $p_1 > p_c^1$. We expect that when $p_1 \to 0$ the exponents $\nu$ and $\beta$ shall become even smaller and finally approach 1. In this case the scaling for average velocity $\langle v \rangle \sim L^{-\gamma} = L^{\nu-1}$ will give $\langle v \rangle \sim L^{0} = 1$, which is inconsistent with the results that $\langle v \rangle = 1$ for finite system at $p_1 = 0^+$.  

In summary, we have investigated a one-dimensional model of a rice pile where the sites with higher slopes have more chance to topple (with a probability $p_2 = 1$) while the sites with lower slopes topple with a probability $p_1 \leq p_2$. Depending on the value of $p_1$, our model exhibits three behaviours: the trivial critical behaviour for $p_1 = 0$ (1D BTW model), the crossover behaviour due to the finite size of the rice-pile and the self-organized critical behaviour for $p_1$ greater than a certain critical value $p_c^1$. In fact, for a sufficiently large pile, $p_c^1$ goes to zero and therefore, there is a sharp transition between the trivial behaviour and the SOC behaviour at $p_1 = 0$. When the system exhibits the SOC behaviour, the exponents did not depend on the value of the probability $p_1$ and our model belongs to the same universality class as the Oslo rice-pile model.

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REFERENCES


FIG. 1. The average velocity $<v>$, defined in the text, as a function of the probability $p_1$ for several values of the system size.
FIG. 2. The average velocity $<v>$ as a function of the system size $L$, for two values of the probability $p_1$. 

FIG. 3.
FIG. 3. (a) Log-Log plot of the avalanche size distributions for several values of the system size $L$ with $p_1 = 0.8$. (b) Data collapse of the curves displayed in (a) according to Eq. (2) with the exponents $\tau = 1.53$, $D = 2.2$. 
FIG. 4. (a) Log-Log plot of the transit-time distribution for several values of the system size with $p_1 = 0.8$. (b) Data collapse of the curves displayed in (a) according to Eq. (1). The best fit to the numerical data gives the slope $\alpha = 2.4$. 

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FIG. 5. (a) Log-Log plot of the avalanche size distribution. The best fit gives the slope $\tau = 1.53$
(b) Log-Log plot of the transit time distribution. The best fit gives the slope $\alpha = 2.4$ In both (a) and (b) the system size is $L = 400$. 
FIG. 6. (a) Log-Log plot of the avalanche size distribution with $p_1 = 0.01$. (b) Semi-log plot of the data rescaled according to Eq. (3). The curve is quite straight line, indicating the exponential form of the scaling function $G(x)$. 

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FIG. 7. (a) Log-Log plot of the transit-time distribution with $p_1 = 0.01$. (b) Log-Log plot of the rescaled data with the exponents $\beta = 1.20$ and $\nu = 1.20$. 
Fig. 7

![Graph](image-url)
Fig. 7

Graph showing the relationship between $P(T,L)$ and $T$ for different values of $L$:
- $L=200$
- $L=400$
- $L=600$

The x-axis represents $T$ on a logarithmic scale from 10 to 1000, and the y-axis represents $P(T,L)$ on a logarithmic scale from $10^{-6}$ to $10^{-2}$. The curves indicate the probability distribution for different lengths $L$. 
Fig. 6

$P(S, L)$

- $L=100$
- $L=200$
- $L=400$

$S$

$10^{-2}$ $10^{-3}$ $10^{-4}$ $10^{-5}$ $10^{-6}$

$100$ $1000$
Fig. 5

- $p_i = 1$
- $p_i = 0.8$
- $p_i = 0.5$
- $p_i = 0.3$

$P(T, p_i)$

$T$
Fig. 5

$P(S, p_i)$

- $p_i = 1$
- $p_i = 0.8$
- $p_i = 0.5$
- $p_i = 0.3$

slope = -1.53
Fig. 4

$L^p(T/L)$ vs. $T/L^\gamma$

- $L=80$
- $L=40$
- $L=20$

Slope = 2.4
Fig. 4

The graph shows the probability distribution function $P(T,L)$ plotted on a logarithmic scale on both axes. The curves represent different values of $L$: $L=20$, $L=40$, and $L=80$. The x-axis represents $T$, ranging from 1 to 1000, while the y-axis represents $P(T,L)$, ranging from $10^{-1}$ to $10^{-7}$. The curves indicate a decreasing trend as $T$ increases for each value of $L$. The labels on the y-axis are indicated as $10^{-1}$, $10^{-2}$, $10^{-3}$, $10^{-4}$, $10^{-5}$, and $10^{-6}$. The x-axis is marked with major ticks at 1, 10, 100, and 1000, with minor ticks in between.
Fig. 3

The graph shows a logarithmic plot of $P(S, L)$ vs. $S$. The curves are labeled $L=20$ and $L=40$. The $y$-axis represents the probability density, with values ranging from $10^{-7}$ to $10^0$. The $x$-axis represents $S$, ranging from 1 to 1000.
Fig. 2

slope = -0.23