ABOUT THE EXISTENCE TIME OF SOLUTIONS FOR FIELD EQUATIONS IN ONE SPACE DIMENSION

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Abstract
In this paper, we study the lower bounds problem for the existence time of solutions to the different massive Dirac-Klein-Gordon equations and with different massive Klein-Gordon equations, in one space dimension, for weakly decaying Cauchy data, of size \( \varepsilon \). The result asserts that the existence time is (almost) larger than \( \varepsilon^{-4} \).

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One of the important nonlinear interaction encountered in field theory is the following Dirac-Klein-Gordon equations coupled through a Yakawa interaction,

\[
\begin{align*}
-\imath \gamma^\mu \partial_\mu \psi + M \psi &= \phi V \psi \\
\Box \phi + m^2 \phi &= ig_0 \psi \gamma^0 \gamma^5 \phi + g_1 \psi \gamma^0 \psi
\end{align*}
\]

where \( V \) is a complex 4 \times 4 matrix such that \( \tilde{V} = \gamma^0 V \), \( M \) and \( m \) are nonnegative real constants, and \( g_0 \) and \( g_1 \) are real constants. The Dirac matrices \( \gamma^\mu, \mu = 0, 1, 2, 3, 5 \), are defined by

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} id & 0 \\ 0 & -id \end{pmatrix}, \\
\gamma^j &= \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, \quad j = 1, 2, 3, \\
\gamma^5 &= -\imath \gamma^0 \gamma^1 \gamma^2 \gamma^3,
\end{align*}
\]

such that

\[
\gamma^5 = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix}, \quad \text{and} \quad \gamma^0 \gamma^5 = \begin{pmatrix} 0 & id \\ -id & 0 \end{pmatrix}
\]

where

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

are the Pauli matrices. \( \psi \) is a complex 4-dimensional vectors (called spinors [?]), \( \tilde{\psi} \) denotes the conjugate transpose of \( \psi \), and \( \phi \) is a real scalar field. This system comes from physics, the fundamental example is the pseudoscalar Yukawa model of nuclear forces.

There exists a global solution for (??) with small, smooth, and decay rapidly enough at infinity initial data in the following cases:

For three space dimensions case, if the massive is not zero (\( M \neq 0, m \neq 0 \)), the system (??) is equivalent to a system of Klein-Gordon equations with quadratic nonlinearities studied by S. Klainerman [??]. The key point is the \( L^{\infty}(\mathbb{R}^3) \) norm decay as \(|t|^{-1-\varepsilon}, \varepsilon > 0 \). For one space dimension case, it was studied by J. M. Chadam [??].

If the massive \( M = m = 0 \), the system (??) is conformal invariance and the existence of the global solution is established by Y. Choquet-Bruhat[?] (see [?] and [?] also).

If the massive \( M \neq 0, m = 0 \), the global Cauchy problem is well posed, proved by A. Bachelot [??], if the nonlinearities satisfy some algebraic conditions related to the Lorentz invariance, the null condition and the compatibility of a sesquilinear form with the Dirac system.

Another of important equations is Klein-Gordon equations,

For Klein-Gordon equations:

\[
\begin{align*}
\Box u_1 + m_1 u_1 &= F_1(u, u', u''), \\
\Box u_2 + m_2 u_2 &= F_2(u, u', u''),
\end{align*}
\]

where \( m_1 \) and \( m_2 \) are two massive constants. \( u = (u_1(x, t), u_2(x, t)) \) is a function in \( \mathbb{R} \times \mathbb{R}^d \), \( u' \) (resp. \( u'' \)) are the derivatives of order 1 (resp. 2) of \( u \) with respect to their arguments, \( F = (F_1, F_2) \) is a regular function, vanishes of second order at \( O \), and \( \Box \) is a d’Alembert operator.
operator. For this equations, when $d \geq 3$, Klainerman [?] and Shatah [?] have proved that there exists a global solution of the Cauchy problem to (7) for above condition’s initial data. For $d = 2$ (resp. 1), Hörmander, in his monograph [?], has proved that the Cauchy problem with data in $C^\infty$, of size $\varepsilon$, admits a solution in $[-T_\varepsilon, T_\varepsilon]$ with $\liminf_{\varepsilon \to 0} (\varepsilon \log T_\varepsilon) = +\infty$ (resp. $\liminf_{\varepsilon \to 0} \sqrt{T_\varepsilon} = +\infty$), and there is a conjecture for dimensional 2 the solution exists globally. The conjecture has been proved by Geogiev-Popivanov [?] for special nonlinearities, and then, Kosecki [?], Simon and Taflin [?], and Ozawa, Tsutaya and Tsutsumi [?] with null condition nonlinearities.

If the conditions of initial data is replaced by weakly decaying, of size $\varepsilon$, the only results can be found is obtained by Delort in [?] for one dimension, and in [?] for multidimension with periodic initial data. However, we mainly deal with the first problem here, for the second, one can extend the conclusion to the different massive case by using the conclusion in [?] and the idea used here. The conclusion will be stated in Section 2, the detail will be omitted.

In this paper, we consider above two problems, in one space dimension, for weakly decaying Cauchy data, of size $\varepsilon$. The main result asserts the existence time is (almost) larger than $\varepsilon^{-4}$. The same result as that in J. M. Delort’s [?] for the same massive Klein-Gordon equations. About the method, just as Delort said in [?]: The weak decay of the data prevents one from using classical methods employed to treat such kind of problems. Here, we use the method of J. M. Delort established in [?], that is, we exploit the curvature of the characteristic variety of the equation, through the use of 2-microlocal ellipticity.

2. STATEMENT OF THE RESULT AND REDUCTION TO A LOCAL PROBLEM

2.1. Statement of the Result. Denote $(x, t)$ the coordinate of $\mathbb{R}^2$, $\phi$ is a real function on $\mathbb{R}$, and $\psi = (\psi_1(x, t), \psi_2(x, t))$ is a $C^2$-valued function on $\mathbb{R}^2$.

After a normalization to obtain mass equal to one, and In a noninvariant hyperbolic form, the Dirac system can be written as

$$\partial_t \psi - iA(D_x)\psi + i\gamma^0 \psi = 0, \quad (2.1)$$

where

$$A(\xi) = \begin{pmatrix} 0 & \xi \\ \xi & 0 \end{pmatrix}.$$

We would like to solve the coupled massive Dirac-Klein-Gordon system

$$\begin{cases} \partial_t \psi - iA(D_x)\psi + i\gamma^0 \psi = \phi V \psi, \\
\Box \phi + m^2 \phi = ig_0 \bar{\gamma} \gamma^5 \psi + g_1 \bar{\psi} \gamma^0 \psi,
\psi|_{t=0} = \varepsilon \psi_0, \phi|_{t=0} = \varepsilon \phi_0, \phi_1|_{t=0} = \varepsilon \phi_1
\end{cases} \quad (2.2)$$

where $\psi_0$ is a given $C^2$ valued function on $\mathbb{R}$, $m \neq 1$, we will suppose $0 < m < 1$, for the convenience of the statement in the context, for $m > 1$, it can be dealt with in a similar way. $\phi_0$ and $\phi_1$ are given real functions on $\mathbb{R}$, and $\varepsilon > 0$ is a small parameter. And we will prove the following theorem
Theorem 2.1. Let $N \geq 3$ be a fixed integer. There exists a constant $c > 0$ such that for all triple $(\psi_0, \phi_0, \phi_1)$ in a unit ball of $H^{N-1}(\mathbb{R}) \times H^N(\mathbb{R}) \times H^{N-1}(\mathbb{R})$, the problem (??) admits a unique solution

$$\psi \in C^0([-T\varepsilon, T\varepsilon], H^{N-1}(\mathbb{R})),\quad \text{and} \quad \phi \in C^0([-T\varepsilon, T\varepsilon], H^N(\mathbb{R})) \cap C^1([-T\varepsilon, T\varepsilon], H^{N-1}(\mathbb{R})), \quad T\varepsilon > ce^{-4 | \log \varepsilon |^{-6}}.$$  

For the Klein Gordon equations, we consider the following different massive case

$$\begin{align*}
\Box U_1 + U_1 &= F_1(U, \partial_U, \nabla x U), \\
\Box U_2 + m^2 U_2 &= F_2(U, \partial_U, \nabla x U),
\end{align*}\quad (2.3)$$

where $F_i$ is a polynomial of its arguments, $m > 0 (\neq 1)$ is a massive number, $U_i = (u_i^1(x, t), \cdots, u_i^n(x, t)), i = 1, 2, n = n_1 + n_2$, and $U = (U_1, U_2)$,

We suppose that the nonlinear terms $F_k$ have the form

$$F_k(U, \partial_U, \partial_x U) = \Gamma_{ij}^0(U) + \sum_{1 \leq i \leq j \leq n} \Gamma_{ij}^{1,k}(U) Q_1(\partial_U u_i, \partial_x u_j; \partial_U u_j, \partial_x u_j)$$

$$+ \sum_{1 \leq i < j \leq n} \Gamma_{ij}^{2,k}(U) Q_2(\partial_U u_i, \partial_x u_j; \partial_U u_j, \partial_x u_j)$$

with the $Q_i, (i = 1, 2)$ having the following symbols

$$q_1(\tau, \xi, \tau', \xi') = \tau \tau' - \xi \xi',$$

$$q_2(\tau, \xi, \tau', \xi') = \tau \xi' - \tau' \xi,$$

where $\Gamma_{0}^k, \Gamma_{ij}^{1,k}$, and $\Gamma_{ij}^{2,k}, k = 1, 2$ are polynomials of $U$, $\Gamma_0$ is vanishing at $U = 0$ of order 2.

Theorem 2.2. Let $N \geq 3$ be a fixed integer. There exists a constant $c > 0$ such that for all pairs $(U_0, U_1)$ in a unit ball of $H^N(\mathbb{R}^2) \times H^{N-1}(\mathbb{R}^2)$, the problem (??) admits a unique solution

$$U \in C^0([-T\varepsilon, T\varepsilon], H^N(\mathbb{R}^2)) \cap C^1([-T\varepsilon, T\varepsilon], H^{N-1}(\mathbb{R}^2)),$$

the existence time $T\varepsilon$ of the solution satisfies $T\varepsilon > ce^{-4 | \log \varepsilon |^{-6}}$.

2.2. Reduction to a Local Existence Theorem. In this subsection, we use a rescalling transformation, and some function spaces similar to that in [?], reduce the problem (??) to a local one. To this end, we assume that $h > 0$ is a real parameter, $U(x, t) = (\psi(x, t), \phi(x, t))^t$ is a solution of (??), and let

$$U^h(x, t) = U(x/h, t/h), \psi_0^h(x) = \psi_0(x/h), \phi_j^h(x) = \phi_j(x/h), j = 0, 1.$$
Then one has:

\[
\begin{align*}
\partial_t \psi^h - iA(D_x)\psi^h + ih^{-1}\gamma_0^0 \psi^h &= h^{-1}\phi^h V\psi^h, \\
\Box \phi^h + m^2 h^{-2} \phi^h &= h^{-2}(i\varphi_0 \psi^h \gamma_0^0 \gamma^5 \psi^h + g_1 \psi^h \gamma_0^0 \psi^h), \\
\psi^h|_{t=0} &= \varepsilon \psi_0^h, \phi^h|_{t=0} = \varepsilon \phi_0^h, \psi^h|_{t=0} = h^{-1} \phi_1^h
\end{align*}
\] (2.7)

**Definition 2.1.** Assume that \( s \in \mathbb{R} \) and \( N \in \mathbb{N} \). We say that a family of functions \((U^h)_{h \in [0,1/2]}\) is in \( H_N^s(\mathbb{R}) \) if \( U^h \) is in \( L^2 \) for each fixed \( h \), and satisfies

\[
|| (U^h)_{h} ||_{H_N^s}^2 = \sup_{h \in [0,1/2]} \left( \int |U^h(\xi)|^2 (1 + h|\xi|)^2 \, d\xi \right) < \infty. \tag{2.8}
\]

We denote the multipliers of Fourier by \( \Delta_0 = 1_{|\xi|<1} \), and \( \Delta_j = 1_{2^{j-1} < |\xi| < 2^j} \) for \( j \in \mathbb{N}^* \). Then, \((U^h)_{h} \in H_N^s \) if and only if \( U^h \in L^2(\mathbb{R}) \) for each fixed \( h \), and there exists a sequence \((c_j(h))_j\) in a unit ball of \( l^2(\mathbb{N}) \), and a constant \( C > 0 \) such that

\[
|| \Delta_j U^h || \leq C c_j(h) h^j \log h |^\frac{3}{2} (1 + 2^j h)^{-N}, \tag{2.9}
\]

holds for all \( j \in \mathbb{N} \). The best constant in (2.9) is a equivalent norm of (2.8) in \( H_N^s \).

If we take \( \varepsilon = \alpha h^{1/4} \log h |^{-3/2} \), one can see that \( U^h |_{t=0} \in H^3/4(\mathbb{R}) \times H^{3/4}(\mathbb{R}) \times H^{-1/4}(\mathbb{R}) \) admits a unique solution \( U^h \) satisfying

\[
\psi^h \in C^0(] - 1, 1[; H_N^{3/4}),
\]

and

\[
\phi^h \in C^0(] - 1, 1[; H_N^{-3/4}) \cap C^1(] - 1, 1[; H_N^{-1/4}).
\]

**Theorem 2.3.** Assume that \( N \geq 3 \) is a fixed integer. There exists a real \( \delta > 0 \) such that, if for all \((V^h_1, W^h_1, W^h_2)_{h \in [0,1/2]} \in H^3/4(\mathbb{R}) \times H^{3/4}(\mathbb{R}) \times H^{-1/4}(\mathbb{R}) \) the norms can be controlled by \( \delta \), then the problem

\[
\begin{align*}
\partial_t \psi^h - iA(D_x)\psi^h + ih^{-1}\gamma_0^0 \psi^h &= h^{-1}\phi^h V\psi^h, \\
\Box \phi^h + m^2 h^{-2} \phi^h &= h^{-2}(i\varphi_0 \psi^h \gamma_0^0 \gamma^5 \psi^h + g_1 \psi^h \gamma_0^0 \psi^h), \\
\psi^h|_{t=0} &= V^h_1, \phi^h|_{t=0} = W^h_1, \phi^h|_{t=0} = W^h_2
\end{align*}
\] (2.10)

admits a unique solution \( U^h = (\psi^h, \phi^h)^t \) satisfying

\[
\psi^h \in C^0(] - 1, 1[; H_N^{3/4}),
\]

and

\[
\phi^h \in C^0(] - 1, 1[; H_N^{-3/4}) \cap C^1(] - 1, 1[; H_N^{-1/4}).
\]

3. Nonlinear Estimates

To estimate the nonlinear terms of (2.1), we introduce some spaces. To this end, we let, for \( j, k \in \mathbb{N} \),

\[
\Phi_{j,k}^\pm(\tau, \xi) = 1_{(\tau > 0)} 1_{(2^{j-1} < |\xi| < 2^j)} 1_{(2^{k-1} < |\tau| < 2^k)} 1_{\frac{m^2}{4\pi} + \varepsilon^2 < 2^k} \tag{3.1}
\]

if \( j > 0, k > 0 \).
\[
\phi_{0k}^{\pm,m}(\tau,\xi) = 1_{\{\pm \tau > 0\}} 1_{\{\xi < 1\}} 1_{\{2^{k-1} < \tau \leq 2^k\}} \sqrt{\tau^2 + \xi^2} < 2^k)
\] if \(k > 0\),

\[
\phi_{j0}^{\pm,m}(\tau,\xi) = 1_{\{\pm \tau > 0\}} 1_{\{\xi < 1\}} 1_{\{\tau \leq \sqrt{\frac{m^2}{n^2} + \xi^2} \}} < 1\]
\] if \(j > 0\), and

\[
\phi_{00}^{\pm,m}(\tau,\xi) = 1_{\{\pm \tau > 0\}} 1_{\{\xi < 1\}} 1_{\{\tau \leq \sqrt{\frac{m^2}{n^2} + \xi^2} \}}< 1\]
\] such that \(1 = \sum_{j,k \geq 0} \phi_{jk}^{+m}(\tau,\xi) + \sum_{j,k \geq 0} \phi_{jk}^{-m}(\tau,\xi)\).

For \(u \in L^2(\mathbb{R}^2)\), Let

\[
\Delta_{jk}^{\pm m} u = \mathcal{F}^{-1}(\phi_{jk}^{\pm m}(\tau,\xi) \hat{u}(\tau,\xi))
\]

**Definition 3.1.** Assume \(s \in \mathbb{R}, s' \in \mathbb{R}, N \in \mathbb{N}\). It is said that a family of \(L^2\) functions \((u^h)_{h \in [0,1/2]}\) in \(\mathbb{R}^2\) valued in \(\mathbb{R}\) or \(\mathbb{C}^m\) is in \(H^{s,s'}_N(\mathbb{R}^2)\) if there exists a \(C > 0\), a sequence \((c_{jk}(h))_{jk}\) satisfying \(\sum_{j,k \geq 0} (\sum_{j,k \geq 0} |c_{jk}|)^2 \leq 1\), and the inequality

\[
\|\Delta_{jk}^{\pm m} u^h\| \leq C c_{jk}(h) h^s |\log h|^{-\frac{3}{2}} 2^{-k s'} (m + 2^j h + 2^k h)^{-N}
\]

holds for all \(j, k\). The best constant \(C \geq 0\) in (??) is defined as a norm in \(H^{s,s'}_N\).

Let \(J = (j,j',j''), K = (k,k',k''),\) and \(E = (e,e',e'')\) be in \(\mathbb{N}^3, \mathbb{N}^3, \{+,+\}^3\) respectively, denote by \((j_1,j_2,j_3),(k_1,k_2,k_3),\) and \((e_1,e_2,e_3)\) the images of each triple by a same exchange of symmetric group such that \(k_3 = \max\{k,k',k''\}\).

\[
A(J,K,E) = \{(\tau,\xi,\tau',\xi');(\tau,\xi) \in \text{Supp} \Phi_{j,k}^{s',m},\]

\[
(\tau',\xi') \in \text{Supp} \Phi_{j',k'}^{s'',m},(\tau - \tau',\xi - \xi') \in \text{Supp} \Phi_{j''}^{s,m}\}.
\]

Assume \(b(\tau,\xi,\tau',\xi')\) is a local bounded function on \(\mathbb{R}^2 \times \mathbb{R}^2\). If \(u\) and \(v\) are both functions with compact spectra, we define \(B(u,v)\) by

\[
B(u,v) = \int \hat{u}(\tau - \tau',\xi - \xi') \hat{v}(\tau',\xi') b(\tau,\xi,\tau',\xi') d\tau' d\xi'.
\]

**Theorem 3.1.** There exists a constant \(C > 0\) such that, for all \(J, K, E\) as before and \(u, v \in L^2(\mathbb{R}^2)\), we have

i) If \(k_3 = k''\) and \(e_1 \neq e_2\) or \(k_3 \neq k''\) and \(e_1 = e_2\), we have

\[
\|\Delta_{jk}^{s',1} B \left(\left(\Delta_{jk}^{s,m} u, \Delta_{jk}^{s,m} v\right)\right)\| \leq C \inf \left(2^{j_1/2}, h^{-1/4} 2^{j_1/2}, h^{-1/4} \right) \times \frac{2^{j_1/2}}{2^{j_1/2}} \left\| \Delta_{jk}^{s',1} v \right\| \|\Delta_{jk}^{s,m} u\| \left\| b \right\|_{L^\infty(A(J,K,E))}. \]
and

\[ \| \Delta^{\varepsilon_m}_{J^m_{k'}} \left( B \left( \left( \Delta^{\varepsilon_{m-1}}_{J^m_{k'}}, \left( \Delta^{\varepsilon_{m-1}}_{J^m_{k'}} v \right) \right) \right) \| \leq C \inf \left( 2^{j/2}, h^{-1/4}2^{\text{sup}(k_1,k_2)/4}(1 + h2^{3/4}) \right) \times 2^{\text{inf}(k_1,k_2)/2}\| \Delta^{\varepsilon_{m-1}}_{J^m_{k'}} v \| \| \Delta^{\varepsilon_{m-1}}_{J^m_{k'}} u \| \| b \| \| \|L^\infty(A(J,K,E)) \| . \]

(3.8)

ii) In the other case, we have

\[ \| \Delta^{\varepsilon_{m-1}}_{J^m_{k'}} \left( B \left( \left( \Delta^{\varepsilon_{m-1}}_{J^m_{k'}}, \left( \Delta^{\varepsilon_{m-1}}_{J^m_{k'}} v \right) \right) \right) \| \leq C \inf \left( 2^{j/2}, h^{-1/4}2^{\text{sup}(k_1,k_2)/4}(1 + h2^{3/4}) \right) \times 2^{\text{inf}(k_1,k_2)/2}\| \Delta^{\varepsilon_{m-1}}_{J^m_{k'}} v \| \| \Delta^{\varepsilon_{m-1}}_{J^m_{k'}} u \| \| b \| \| \|L^\infty(A(J,K,E)) \| . \]

(3.9)

and

\[ \| \Delta^{\varepsilon_{m-1}}_{J^m_{k'}} \left( B \left( \left( \Delta^{\varepsilon_{m-1}}_{J^m_{k'}}, \left( \Delta^{\varepsilon_{m-1}}_{J^m_{k'}} v \right) \right) \right) \| \leq C \inf \left( 2^{j/2}, h^{-1/4}2^{\text{sup}(k_1,k_2)/4}(1 + h2^{3/4}) \right) \times 2^{\text{inf}(k_1,k_2)/2}\| \Delta^{\varepsilon_{m-1}}_{J^m_{k'}} v \| \| \Delta^{\varepsilon_{m-1}}_{J^m_{k'}} u \| \| b \| \| \|L^\infty(A(J,K,E)) \| . \]

(3.10)

The conclusion of this theorem can be easy obtained from the following proposition. The inequalities (3.11) and (3.12) can be gotten by a similar argument to that corresponding analogue (3.8) and (3.9). So, here we only give the proof for the inequality (3.11) and (3.12) respectively.

**Proposition 3.2.** There exists a constant \( C > 0 \) such that for all triple indexes \( J = (j,j',j'') \), \( K = (k,k',k'') \) in \( N^3 \), \( k' \leq k \leq k'' \) and \( u,v \in L^2(\mathbb{R}^3) \), and

\[ \| \Delta^{\pm 1}_{J^m_{k'}} \left( B \left( \left( \Delta^{\pm 1}_{J^m_{k'}}, \left( \Delta^{\pm 1}_{J^m_{k'}} v \right) \right) \right) \| \leq C \inf \left( 2^{j/2}, h^{-1/2}2^{3/4}(1 + h2^{3/4}) \right) \times 2^{k'/2}\| \Delta^{\pm 1}_{J^m_{k'}} v \| \| \Delta^{\pm 1}_{J^m_{k'}} u \| \| b \| \| \|L^\infty(A(J,K,E)) \| . \]

(3.11)

holds, and \( E = (+, +, \pm) \).

The proof of the analogue in [?] can be adapted. So, we omit the detail, but we need giving the analogous lemmas used in there. To this end, we consider a function

\[ F(\xi,\xi') = \sqrt{\frac{m^2}{h^2} + (\xi - \xi')^2 + \sqrt{h^{-2} + \xi'^2}}, \]

(3.12)

and establishing the following lemma

**Lemma 3.3.** For a small positive real number \( \delta \ll 1 \), we have, for all \( h \in [0,1], \)

i)

\[ \frac{\partial F}{\partial \xi'} \geq \frac{ch|\xi - \xi'|}{1 + h|\xi - \xi'|} \]

if \( \xi \xi' < 0 \).
if $0 < \xi'/\xi < 1$ and $|\xi' - \lambda \xi| > \delta|\xi|$, where $\lambda = \frac{1}{1+m}$.

\[
\left| \frac{\partial F}{\partial \xi'} \right| \geq \frac{ch|\xi'|}{1 + h|\xi'|} (1 + h \inf(|\xi'|, |\xi - \xi'|))^{-2}
\]  
(3.14)

if $\xi' > 0$ and $|\xi'| \geq |\xi|$.

Therefore, when $|\xi' - \lambda \xi| > \delta|\xi|$, we have,

\[
\left| \frac{\partial F}{\partial \xi'} \right| \geq \frac{ch \sup(|\xi|, |\xi'|, |\xi - \xi'|)}{(1 + h \sup(|\xi|, |\xi'|, |\xi - \xi'|)) (1 + h \inf(|\xi|, |\xi'|, |\xi - \xi'|))^{1/2}}
\]  
(3.16)

Proof. We can assume $\xi \neq 0$, and $\xi' \neq 0$. Let $a = \frac{1}{h|\xi|}$, $\sigma = \frac{1}{h|\xi|}$, and $\xi' = \lambda \xi - \theta \xi$.

\[
\frac{\partial F}{\partial \xi'}(\xi, \xi') = -\frac{\xi - \xi'}{\sqrt{\sigma^2 + (\xi - \xi')^2}} + \frac{\xi'}{\sqrt{h^2 + \xi'^2}}
\]  
(3.17)

When $\xi' > 0$, or $\theta > \lambda$, we have

\[
\left| \frac{\partial F}{\partial \xi'}(\xi, \xi') \right| \geq \frac{1 - \lambda + \theta}{(\sigma_m^2 + (1 - \lambda + \theta)^2)^{1/2}},
\]

it follows (??). When $\xi' < 0$, or $\theta > \lambda$, we have

\[
\left| \frac{\partial F}{\partial \xi'}(\xi, \xi') \right| \geq \frac{\lambda - \theta}{(\sigma_m^2 + (1 - \lambda + \theta)^2)^{1/2}} = \frac{h|\xi'|}{\sqrt{1 + h^2 |\xi'|^2}}.
\]

Suppose $\delta < \theta < \lambda$, $0 < \delta < 1$,

\[
\left| \frac{\partial F}{\partial \xi'}(\xi, \xi') \right| \geq \left( \frac{1 - \lambda + \theta}{(\sigma_m^2 + (1 - \lambda + \theta)^2)^{1/2}} - \frac{\lambda - \theta}{(\sigma^2 + (\lambda - \theta)^2)^{1/2}} \right),
\]

\[
= \left( \frac{1}{\left( 1 + \frac{\sigma^2}{(\lambda - \theta)^2} \right)^{1/2}} - \frac{1}{\left( 1 + \frac{\sigma^2}{(\lambda - \theta)^2} \right)^{1/2}} \right)
\]  
(3.18)

\[
\frac{\sigma^2 (1 - \lambda + \theta)^2 - m^2 (\lambda - \theta)^2}{(1 - \lambda + \theta) ((1 - \lambda + \theta)^2 + \sigma^2)^{1/2} ((\lambda - \theta)^2 + \sigma^2)},
\]

and if $1 - \lambda + \theta \leq \lambda - \theta$, it can be bounded from below by
\[
\frac{\sigma^2 (m^2(\lambda - \theta)^2 - (1 - \lambda + \theta)^2)}{(\lambda - \theta)((\lambda - \theta)^2 + \sigma^2)^{1/2}((1 - \lambda + \theta)^2 + \sigma^2)}.
\]

Hence, we have (??) can be controlled from below by

\[
\frac{\sigma^2 \theta (1 + m) \left( \frac{2m}{1+m} + (1-m)\theta \right)}{\max\{1 - \lambda + \theta, \lambda - \theta\}((\max\{1 - \lambda + \theta, \lambda - \theta\})^2 + \sigma^2)^{1/2} \times \left( \min\{\lambda - \theta, 1 - \lambda + \theta\}^2 + \sigma^2 \right)}.
\]

Note that \(|\xi - \xi'| = |\xi| \cdot |1 - \lambda + \theta| \leq |\xi|\), one has (??).

In the same way, we can prove the case of \(\lambda - 1 < \theta < -\delta\) for \(\delta < \theta < \lambda\). \(\square\)

When \(\xi'\) approaches to \(\lambda \xi\), we have

**Lemma 3.4.** With the same \(\delta\) as above, there exists a \(C > 0\) such that for all \(\eta, \eta'\) with the same sign verifying \(|\eta| \leq \delta|\xi|\), and \(|\eta'| \leq \delta|\xi|\), one has

\[
|F(\xi, \lambda \xi + \eta) - F(\xi, \lambda \xi + \eta')| \geq \frac{c(m + 1)h}{(1 + h|\xi|)^3} |\eta^2 - \eta'^2|.
\]

The proof of this lemma is almost the same with the analogue of that in [?].

**Proposition 3.5.** There exists a constant \(C > 0\) such that for all triple indexes \(J = (j, j', j'')\), \(K = (k, k', k'')\) in \(\mathbb{N}^3\), \(k' \leq k \leq k''\) and \(u, v \in L^2(\mathbb{R}^3)\), one has

\[
\|\Delta^{\pm 1}_{j'k''} \left( B \left((\Delta^{\pm m}_{jk}u), (\Delta^{-1}_{j'k''}v)\right) \right) \| \leq C inf \left( 2^{j}, h^{-1/4}2^{k/4}(1 + h2^{j})^{3/4} \right) \times 2^{k/2} \|\Delta^{-1}_{j'k''}v\| \|\Delta^{m}_{jk}u\| \|\delta\|_{L^\infty(A(J,K,E))} (3.19)
\]

holds, and \(E = (+, -, ±)\).

One can prove this proposition by using the following lemmas, and the same methods as that for the proof of the Proposition ??.

**Lemma 3.6.** Let \(G(\xi, \xi')\) be a function \(\sqrt{\frac{m^2}{k^2} + (\xi - \xi')^2} - \sqrt{h^{-2} + \xi'^2}\). There exists a \(C > 0\) such that for all \(\eta, \eta'\) with the same sign verifying \(|\eta| \leq \delta|\xi|\), and \(|\eta'| \leq \delta|\xi|\), where \(\delta < 1\) is a small number, one has

\[
|G(\xi, \lambda \xi + \eta) - G(\xi, \lambda ' \xi + \eta')| \geq \frac{ch}{(1 + h|\xi|)^3} |\eta^2 - \eta'^2|.
\]

where \(\xi' = \xi/\lambda' + \eta, \lambda' = \frac{1}{1-m}\).
Lemma 3.7. For above $\delta$, there exists a constant $C > 0$ such that for all $h \in \mathbb{R}$, we have

$$
\left| \frac{\partial G}{\partial \xi'} \right| \geq \frac{ch|\xi|}{(1 + h \sup \{ |\xi|, |\xi'|, |\xi - \xi'| \})(1 + h \inf \{ |\xi'|, |\xi - \xi'| \})^2} \tag{3.20}
$$

if $|\xi' - \xi| \geq \delta|\xi|$.

Proof. Assume $\xi, \xi' \neq 0$, and let $u = \xi' / \xi$, $\sigma_m = \frac{m}{h|\xi|}$, and $|\sigma| = \frac{1}{h|\xi|}$. From

$$
\frac{\partial G}{\partial \xi'}(\xi, \xi') = -\frac{\xi - \xi'}{\sqrt{\frac{\sigma_m^2}{h^2} + (\xi - \xi')^2}} - \frac{\xi'}{\sqrt{h^{-2} + \xi'^2}}
= -\frac{\xi}{|\xi|} \left[ \frac{1 - u}{(\sigma_m^2 + (1 - u)^2)^{1/2}} + \frac{u}{(\sigma^2 + u^2)^{1/2}} \right]. \tag{3.21}
$$

If $0 < u = \xi' / \xi < 1$, it obvious that

$$
|\frac{\partial G}{\partial \xi'}(\xi, \xi')| \geq \frac{ch|\xi|}{1 + h|\xi|}.
$$

If $u = \xi' / \xi < 0$, and $|\xi'| \leq |\xi|$, from (3.21), we have

$$
|\frac{\partial G}{\partial \xi'}(\xi, \xi')| \geq \frac{\sigma^2(1 - 2u)}{u^2(1 - u)^2} \left( 1 + \frac{\sigma_m^2}{(1 - u)^2} \right)^{-1/2} \left( 1 + \frac{\sigma^2}{u^2} \right)^{-1/2}
\geq \frac{1}{8} \frac{\sigma^2}{(\sigma^2 + u^2)^{1/2}} \geq \frac{ch|\xi|}{1 + h|\xi|} \frac{(1 + h|\xi'|)^{-2}}{1 + h|\xi|}.
$$

If $\xi, \xi' > 0$, and $|\xi'| > |\xi|$, we have to investigate $1 < u < \lambda - \delta$, and $u > \lambda + \delta$ respectively. From (3.21), we can get

$$
|\frac{\partial G}{\partial \xi'}(\xi, \xi')| \geq \frac{|\sigma^2(1 - 2u)(1 - u)^2|}{u^2(1 - u)^2} \left( 1 + \frac{\sigma_m^2}{(1 - u)^2} \right) \left( 1 + \frac{\sigma^2}{u^2} \right)^{-1/2}
\geq \frac{\sigma^2}{8(\sigma^2 + u^2)^{1/2}} \geq \frac{ch|\xi|}{1 + h|\xi|} \frac{(1 + h|\xi'|)^{-2}}{1 + h|\xi|}.
$$

For $1 < u < \lambda - \delta$, (3.22) can be bounded from below by

$$
\delta \sigma^2(1 - \frac{1}{1 + m})(1 - m^2)
\frac{1}{(\lambda - \delta)\sqrt{u^2 + \sigma^2(\sigma^2 + (1 - u)^2)}},
$$

and we have the desired conclusion. For $u > \lambda + \delta$, (3.22) can be bounded from below by

$$
\frac{(1 - m^2)\delta \sigma^2(1 - \frac{1}{1 + m})}{u \sqrt{u^2 + \sigma^2(\sigma^2 + (1 - u)^2)}} \geq
\frac{\sigma^2}{(1 - m^2)\delta \left(1 - \frac{1}{(\lambda + \delta)(1 - m)}\right) \sqrt{u^2 + \sigma^2(\sigma^2 + (1 - u)^2)},}
$$

and it follows the desired result.

If $\xi, \xi' < 0$ and $|\xi'| \geq |\xi|$, we let $u = \xi / \xi' \in [-1, 0], \sigma = \frac{1}{h|\xi|}$ and write
\[ \frac{\partial G}{\partial \xi'}(\xi, \xi') = -\frac{\xi}{|\xi|} \left[ \frac{u - 1}{(\sigma_m^2 + (1 - u)^2)^{1/2}} + \frac{1}{(\sigma^2 + 1)^{1/2}} \right]. \]

Thus,
\[ \left| \frac{\partial G}{\partial \xi'}(\xi, \xi') \right| \geq \left( 1 + \frac{\sigma^2}{(1 - u)^2} \right)^{-1/2} - (1 - \sigma^2)^{-1/2} \]
\[ \frac{\sigma^2(u^2 - 2u + 1 - m^2)}{2(1 - u)^2(1 + \sigma^2)^{3/2}} \]
\[ C|u| \left( \frac{\sigma^2}{(1 + \sigma^2)^{3/2}} \right) \geq \frac{ch|\xi|}{(1 + h|\xi'|)^3}. \]

Combine all above obtained inequalities, the desired inequality follows. \( \square \)

**Theorem 3.8.** Assume that \((u^h)_{h\in [0,1/2]} \in H^{3/4 - \frac{1}{2}}_{N-1,1,1}, \) and \((v^h)_{h\in [0,1/2]} \in H^{3/4 - \frac{1}{2}}_{N,1,m}(\mathbb{R}^2) \) with \( N \geq 2. \) Then, \( h^{-1}u^hv^h \in H^{3/4 - \frac{1}{2}}_{N-1,1,1}, \) the bilinear form is continuous from \( H^{3/4 - \frac{1}{2}}_{N-1,1,1} \times H^{3/4 - \frac{1}{2}}_{N,1,m} \) to \( H^{3/4 - \frac{1}{2}}_{N-1,1,1}. \)

We can use the same strategy as that cited in [?] to prove this theorem. To this end, we introduce some lemmas

**Lemma 3.9.** (Delort [?]) Let \( \rho > 0 \) and sequence \((c_j)_j \) and \((c'_j)_j \) are in \( l^2(\mathbb{R}). \) Define a new sequence \((c''_j)_j \) by
\[ c''_j = \sum_{j,j'} \left( 1 + h \inf(2^j, 2^j') \right)^{-\rho} c_j c'_j, \]
where \( j, j' \in \{ j, j'| j \ll j'' \text{ or } j' \ll j \sim j'' \text{ or } j'' - 5 \leq j \sim j' \}. \) Then \((c''_j)_j \in l^2 \) and there exists a constant \( C > 0 \) such that
\[ ||(c''_j)_j||_\rho \leq C ||(c_j)_j||_\rho ||(c'_j)_j||_\rho. \]

The following one was proved in [?] for the same mass, but it is easy to improved to our desired conclusion.

**Lemma 3.10.** There exists a constant \( c > 0 \) such that for all triple indexes \( J, K \) and \( E, \)
\[ \Delta^{e_m,1}_{(J,K)} \left( (\Delta^{e_m,1}_{J,K}) (\Delta^{e_m,1}_{J,K}) \right) \neq 0, \] we have
\[ 2^{\sup(k,k',k'')} \geq \frac{c}{h} \left( 1 + h \inf(2^j, 2^j', 2^j'') \right)^{-1} \]
\[ \left( 1 + h \sup(2^j, 2^j') \right) \]
when \( k_3 \geq k_1 + 3, \) and \( k_3 \geq k_2 + 3, \) where \( C \) is a constant.
Proof of Theorem ??

Decompose \( u = \sum_{jk} \left( \Delta_{jk}^{+1} u + \Delta_{jk}^{-1} u \right), \) \( v = \sum_{j'k'} \left( \Delta_{j'k'}^{+m} v + \Delta_{j'k'}^{-m} v \right). \)

Thus, to prove the desired result, it suffices to obtain the following inequality for \( e, e', e'' \in \{+,-\}, \)

\[
\sum_{j,j' \in E} \sum_{k,k'} \left\| \Delta_{j'k'}^{e''} \left( h^{-1} (\Delta_{jk}^{e,1} u) (\Delta_{j'k'}^{e',m} v) \right) \right\| \leq C h^{3/4} |\log h|^{-3/2} \times 2^{k''/2} (1 + 2^{k''} h + 2^{k''} h)^{-N+1} c_{j'k'}^{e''} \| u \|_{H_{N-1,1}^{3/4}} \| v \|_{H_{N,m}^{3/4}} \tag{3.26}
\]

for a sequence \( (c_{j'k'}^{e''})_{j'k'} \) satisfying \( \sum_{j'k'} \left( \sum_{k''} \| c_{j'k'}^{e''} \|^2 \right)^{1/2} \leq 1. \)

We will divide the estimate into three cases \( k_3 = k'', k_3 = k, k_3 = k'. \)

1) The estimate of

\[
\sum_{j,j' \in E} \sum_{k,k' \leq k''} \left\| \Delta_{j'k'}^{e''} \left( h^{-1} (\Delta_{jk}^{e,1} u) (\Delta_{j'k'}^{e',m} v) \right) \right\|. \tag{3.27}
\]

From Theorem ??, we have

\[
\left\| \Delta_{j'k'}^{e''} \left( h^{-1} (\Delta_{jk}^{e,1} u) (\Delta_{j'k'}^{e',m} v) \right) \right\| \leq \frac{C}{h} \inf \left( 2^{j/2}, h^{-1/4} 2^{2\sup(k_1,k_2)/4} (1 + h \inf(2^j, 2^{k'}))^3/4 \right) \times 2^{2\sup(k_1,k_2)/2} \| \Delta_{j'k'}^{e',m} v \| \| \Delta_{jk}^{e,1} u \| \tag{3.28}
\]

when \( k \leq k'' \) and \( k' \leq k'' \), where \( j = \min\{j,j',j''\} \). By using (??), we know that the left hand side of (??) can be controlled by

\[
C (h^{3/4})^2 h^{-5/4} \left( 1 + h \inf(2^j, 2^{k'}) \right)^3/4 (1 + 2^{j} h + 2^{k'} h)^{-N+1} \times (m + 2^{j} h + 2^{k'} h)^{-N} c_{jk}(h)c_{j'k'}(h)\| u \|_{H_{N-1,1}^{3/4}} \| v \|_{H_{N,m}^{3/4}} \tag{3.29}
\]

with \( \sum_{j} (\sum_{k} |c_{jk}|)^2 \leq 1 \), and \( \sum_{j'} \left( \sum_{k''} \| c_{j'k''} \|^2 \right)^{1/2} \leq 1. \)

To get the conclusion, one only need to prove (??) is bounded by

\[
C (h^{3/4})^2 h^{-3/4} 2^{k''/2} (1 + 2^{j''} h + 2^{k''} h)^{-N+1} d_{j''k''} \| u \|_{H_{N-1,1}^{3/4}} \| v \|_{H_{N,m}^{3/4}} \tag{3.30}
\]

where

\[
d_{j''k''} = \sum_{j,j'} \sum_{k,k' \leq k''} 2^{-k''/2} h^{-1/2} (1 + 2^{j''} h + 2^{k''} h)^{N-1} \times \left( 1 + h \inf(2^j, 2^{j'}) \right)^3/4 (1 + 2^{j} h + 2^{k'} h)^{-N+1} (m + 2^{j} h + 2^{k'} h)^{-N} \times c_{jk} c_{j'k'} \Theta (J, K, h), \tag{3.31}
\]

where \( J, K, \) and \( \Theta \) are bounded and vanishing if \( 2^{j''} h < c(1 + h2^j)^{-1} \), or \( 2^{k''} h > C \left( 1 + h \sup(2^j, 2^{j'}) \right) \) if \( k'' \geq k + 3, k'' \geq k' + 3 \). Now, it remains to estimate the above \( d_{j''k''} \). Note that


we have

\[
(1 + 2^j h + 2^k h)^{-N+1} \leq m^{-N-1} C (1 + 2^{j'} h + 2^{k'} h)^{-N+1} \left(1 + h \inf(2^j, 2^{j'})\right)^{-N+1}
\]

\[
(1 + 2^{j'} h + 2^{k'} h)^{-1} \leq (1 + h \inf(2^j, 2^{j'}))^{-N},
\]

(3.32)

By using the inequality \(2^{k''} \geq \frac{c}{h}(1 + h2^j)^{-1}\), the \(k, k', \text{ and } k''\) summation in (3.32) can be controlled by

\[
\sum_{k''} d_{j'k''} \leq C \sum_{j,j',k,k'} 2^{-k''/2} h^{-1/2} \times \left(1 + h \inf(2^j, 2^{j'})\right)^{-N+3/4} c_{j'k'} \odot (J, K, h).
\]

(3.33)

where \((c_j)_j\), and \((c'_{j'})_{j'}\) are both in the unit ball of \(l^2\), and

\[
j, j' \in \{j, j'|j \ll j' \sim j''\}, \text{ or } j' \ll j \sim j'', \text{ or } j'' - 5 \leq j \sim j'.
\]

From (3.33), (3.32) can be controlled by \(C|\log h|^{1/2}c''_{j''}\) for a new sequence \((c''_{j''})_{j''}\) in the unit ball of \(l^2\). Hence, the conclusion in this part is obtained.

2) The estimate of

\[
\sum_{j,j'} \sum_{k,k'} \|\Delta_{jk'}^{1/2} \left(h^{-1}(\Delta_{jk}^{1/2} u)(\Delta_{jk'}^{1/2} v)\right)\|
\]

(3.35)

for \(e' \neq e''\).

From (3.35), we know that the general terms of (3.35) can be controlled by

\[
C(h^{3/4}2^j h^{-3/4}2^{k''}/2^j(1 + 2^{j'} h + 2^{k''} h)^{-N+1}d_{j'k''} \|u\|_{H^{3/4}_{N-1,1}} \|v\|_{H^{3/4}_{N,0}} (3.36)
\]

where

\[
d_{j'k''} = \sum_{j,j'} \sum_{k,k',k''} 2^{-k''/2} h^{-1/2}(1 + 2^{j''} h + 2^{k''} h)^{-N+1}
\]

\[
(1 + h2^{j'/4}(1 + 2^{j'} h + 2^{k'} h)^{-N+1}(m + 2^{j'} h + 2^{k'} h)^{-N} \times \]

\[
c_{j'k''} \odot (J, K, h),
\]

(3.37)

where \(\odot\) is a function which is vanishing if \(2^k h < c(1 + 2^j h)^{-1}\), and the support of it in \(j \ll j' \sim j'',\text{ or } j' \ll j \sim j'',\text{ or } j'' - 5 \leq j \sim j'.\) As before, it is not difficult to know that

\[
\sum_{k''} d_{j'k''} \leq C \sum_{j,j'} \sum_{k,k',k''} 2^{-k/2} h^{-1/2} \times \left(1 + h \inf(2^j, 2^{j'})\right)^{-N+3/4} c_{j'k'} \odot (J, K, h)
\]

(3.38)

In view of \(2^k h \geq c(1 + 2^j h)^{-1}\) on the support of \(\odot\). we have
\[ \sum_{\{k',k'' \leq k\}} 2^{-k/2} \Theta \leq (k + 1)2^{-k/2} \Theta \leq C|\log h|h^{-1/2}(1 + 2^{2} h)^{1/2}. \]

Hence, in (??), the summation on \( k, k' \), and \( k'' \) can be estimated by
\[ C|\log h|\sum_{j,j'} \left(1 + h \inf(2^j, 2^{j'})\right)^{-N+5/4} c_j c_{j'} \]
with \((c_j)_j, (c_{j'})_{j'} \in l^2\), and we obtained the result in this part as well.

3) The estimate of
\[ \sum_{j,j'} \sum_{\{k,k';k'<k,k'' \leq k\}} \left\| \Delta_{jk}^{\epsilon,1} \left( h^{-1}(\Delta_{jk}^{\epsilon} u)(\Delta_{jk'}^{\epsilon} v) \right) \right\| \]
for \( \epsilon' = \epsilon'' \).

It is easy to know that the contribution of (??) can be controlled by
\[ C(h^{3/4})^{2} h^{-3/4} 2^{k''/2} (1 + 2^{2} h + 2^{k''} h)^{-N+1} d_{j,k''} \| \| u \|_{H^{3/4}_{N-1,1}} \| v \|_{H^{3/4}_{N-1,1}} \]
with
\[ d_{j,k''} = \sum_{j,j'} \sum_{\{k,k';k'' \leq k\}} 2^{-k/2} h^{-1/2} \left(1 + h \inf(2^j, 2^{j'})\right)^{3/4} \times \]
\[ \left(1 + 2^j h + 2^{k''} h\right)^{-N+1} \Theta c_{j,k''} \Theta \leq \left(1 + 2^{j} h + 2^{k''} h\right)^{-N+3/4} \Theta c_{j,k''} \Theta. \]

In this expression, the \( l^1 \) norm on \( k'' \) can be estimated by (??), and using the same argument as there we can obtain the result immediately.

The study of the summation
\[ \sum_{j,j'} \sum_{\{k,k';k'' \leq k\}} \left\| \Delta_{jk'}^{\epsilon,1} \left( h^{-1}(\Delta_{jk}^{\epsilon} u)(\Delta_{jk'}^{\epsilon} v) \right) \right\| \]
can be deduced to cases 2) and 3) by interchange the index. Hence, we complete the proof. \( \square \)

Using a similar argument to above, we can improve the similar result in [?] to the following one for our needness,

**Theorem 3.11.** Assume that \((u^h)_{h \in [0,1/2]} \in H^{3/4}_{N,1} \), and \((v^h)_{h \in [0,1/2]} \in H^{3/4}_{N,1} (\mathbb{R}^2) \) with \( N \geq 2 \).
Then, \( (h^{-2}(u^h v^h))_h \in H^{-1/2-1/2}_{N,m}, \) the bilinear form is continuous from \( H^{3/4}_{N,1} \times H^{3/4}_{N,1} \) to \( H^{-1/2-1/2}_{N,m} \).
4. Local Existence of the Low Regularity Local Solutions

4.1. Proof of the Theorem ??

To prove Theorem ??, we establish some lemmas at first,

**Lemma 4.1.** (J. M. Delort [?]) Let $s \in \mathbb{R}, N \in \mathbb{N}$ and $\chi \in C_0^\infty(\mathbb{R})$. There exists a constant $C > 0$, only depends on finite numbers of the norms $\|\partial^\alpha \chi\|_{L^1}$, such that $\chi(t)u \in H^{s,\frac{3}{2}}_N$ for all $u \in H^{s,\frac{3}{2}}_N$, and $\|\chi u\|_{H^{s,\frac{3}{2}}_N} \leq C\|u\|_{H^{s,\frac{3}{2}}_N}$.

**Lemma 4.2.** Let $s \in \mathbb{R}, N \in \mathbb{N}^*$, $\psi_0 \in H^s_N(\mathbb{R})$, $f = (f_1, f_2)^t \in l^2(\mathbb{R}^2)$, and $\psi$ be a solution of

\[
\begin{cases}
\partial_t \psi - iA(D)\psi + ih^{-1}\gamma^0 \psi = f, \\
\psi|_{t=0} = \psi_0.
\end{cases}
\] (4.1)

Then, we have $\chi(t)\psi \in H^{s,\frac{3}{2}}_N$ and it depends on $\psi_0$ and $f$ continuously if $\chi(t) \in C_0^\infty(\mathbb{R})$.

**Proof.** Notice that

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } A(\xi) = \begin{pmatrix} 0 & \xi \\ \xi & 0 \end{pmatrix},
\]

for any $\xi \in \mathbb{R}\setminus 0$, we have that the eigenvalues for $A(\xi) - h^{-1}\gamma^0$ are $\pm \sqrt{\frac{1}{12\tau} + \xi^2}$ for $\xi \in \mathbb{R}$.

For initial data $\hat{\psi}_0 = (\hat{\psi}_1^0, \hat{\psi}_2^0)$ it is easy to check that for each $t$ the Fourier transformation of the solution to the Cauchy problem of massive Dirac system (??) with respect to $x$ is given by

\[
2\psi_1(\xi, t) = e^{it\sqrt{\frac{1}{12\tau} + \xi^2}} \left( \psi_1^0 + \frac{\xi}{\sqrt{\frac{1}{12\tau} + \xi^2} + \frac{1}{h^2}} \psi_2^0 \right) + e^{-it\sqrt{\frac{1}{12\tau} + \xi^2}} \left( \psi_1^0 - \frac{\xi}{\sqrt{\frac{1}{12\tau} + \xi^2} + \frac{1}{h^2}} \psi_2^0 \right)
\]

\[
-\frac{1}{2\pi} \int \frac{e^{it\tau} - e^{it\sqrt{\frac{1}{12\tau} + \xi^2}}}{\tau - \sqrt{\frac{1}{12\tau} + \xi^2}} \hat{f}_1(\tau, \xi) d\tau
\]

\[
+ \frac{1}{2\pi} \int \frac{e^{it\tau} - e^{-it\sqrt{\frac{1}{12\tau} + \xi^2}}}{\tau + \sqrt{\frac{1}{12\tau} + \xi^2}} \hat{f}_1(\tau, \xi) d\tau
\] (4.2)
We say that \( \Omega = T \). In fact, the Fourier transform of the considered function is \( \chi(t)e^{i\sqrt{1/\hbar^2 + \xi^2} \psi_1(t)} \). One has
\[
\|Aiw\| < C\|\psi_1\|^{N+1} \text{ for all } M.
\]
Thus, one can easily obtain that the previous two terms of (4.3) or (4.4) is in \( H^{N+1}_{N+1} \).

To get the result for the last two terms of (4.3) or (4.4), we denote \( \Psi(\tau, \xi) = \frac{1}{\sqrt{\hbar^2 + \xi^2}} \psi_1(\xi) \) and rewrite the last two terms of (4.4), for example, to
\[
\frac{1}{2\pi} \int e^{it\tau} \frac{1 - \Psi^-}{\tau + \sqrt{1/\hbar^2 + \xi^2}} \hat{f}_1(\tau, \xi) d\tau
\]
\[
-\frac{1}{2\pi} \int e^{it\tau} \frac{1 - \Psi^+}{\tau - \sqrt{1/\hbar^2 + \xi^2}} \hat{f}_1(\tau, \xi) d\tau
\]
\[
-\frac{1}{2\pi} e^{-it\sqrt{1/\hbar^2 + \xi^2}} \int \frac{1 - \Psi^+}{\tau + \sqrt{1/\hbar^2 + \xi^2}} \hat{f}_1(\tau, \xi) d\tau
\]
\[
+\frac{1}{2\pi} e^{it\sqrt{1/\hbar^2 + \xi^2}} \int \frac{1 - \Psi^-}{\tau - \sqrt{1/\hbar^2 + \xi^2}} \hat{f}_1(\tau, \xi) d\tau
\]
\[
(4.4)
\]
\[
+ \frac{1}{2\pi} \int \left( e^{it\tau} - e^{-it\sqrt{1/\hbar^2 + \xi^2}} \right) \frac{\Psi^-}{\tau + \sqrt{1/\hbar^2 + \xi^2}} \hat{f}_1(\tau, \xi) d\tau
\]
\[
- \frac{1}{2\pi} \int \left( e^{it\tau} - e^{-it\sqrt{1/\hbar^2 + \xi^2}} \right) \frac{\Psi^+}{\tau - \sqrt{1/\hbar^2 + \xi^2}} \hat{f}_1(\tau, \xi) d\tau
\]
\[
= I + II + III + IV + V + VI.
\]

We can draw the conclusion, for terms III and IV, from the previous discussion. For terms I and II we take the Fourier transform of them with respect to \( t \) and have a summation
\[
\begin{align*}
\left[ \frac{1 - \Psi^-}{\tau + \sqrt{\frac{1}{h^2} + \xi^2}} - \frac{1 - \Psi^+}{\tau - \sqrt{\frac{1}{h^2} + \xi^2}} \right] \hat{f}_1(\tau, \xi) \\
= G_1 + G_2 + G_3
\end{align*}
\] (4.5)
\[
G_1 = -2\sqrt{\frac{1}{h^2} + \xi^2}(1 - \Psi^-)(1 - \Psi^+) \frac{\hat{f}_1(\tau, \xi)}{\tau^2 - (\frac{1}{h^2} + \xi^2)}
\]
\[
G_2 = \frac{(1 - \Psi^-)\Psi^+}{\tau + \sqrt{\frac{1}{h^2} + \xi^2}} \hat{f}_1(\tau, \xi)
\]
\[
G_3 = \frac{(1 - \Psi^+)\Psi^-}{\tau - \sqrt{\frac{1}{h^2} + \xi^2}} \hat{f}_1(\tau, \xi)
\] (4.6)

One has
\[
\|\Phi_{-jk}G_1(\tau, \xi)\| \leq Ch^s |\log h|^{-3/2}2^{-k/2}(1 + 2^2h + 2^k h)^{-N} c_{jk}
\] (4.7)

immediately. Noting that the support of $G_3$ is in $|\tau + \sqrt{\frac{1}{h^2} + \xi^2}| < 1$, and $|\tau| < 1 + \sqrt{\frac{1}{h^2} + \xi^2}$, we have the same estimate as (??) for $\|\Phi_{-jk}G_3\|$. Using the similar arguments, we can get the same estimate for $\Phi_{-jk}G_2$. Therefore, the Fourier inverse transform of $I + II$ is in $H^{s, \frac{1}{2}}_N$.

For $V$ and $VI$, we discuss $VI$ for example. To this end, we write $VI$ as
\[
\int \left( e^{it\tau} - e^{-it\sqrt{h^2 + \xi^2}} \right) \frac{\Psi^+}{\tau - \sqrt{\frac{1}{h^2} + \xi^2}} \hat{f}_1(\tau, \xi) d\tau
\]
\[
= \sum_{l=1}^{+\infty} e^{it\sqrt{\frac{1}{h^2} + \xi^2}} \chi(t)(it)^l I_l(t)
\] (4.8)

with $I_l(\xi) = \int (\tau - \sqrt{\frac{1}{h^2} + \xi^2})^{l-1} \Psi^+ \hat{f}_1(\tau, \xi) d\tau$. Hence, $\mathcal{F}_\xi^{-1} I_l \in H^s_N$, and easy to know that the right hand side of (??) is in $H^{s, \frac{1}{2}}_N$ also. 

\[\Box\]

Lemma 4.3. (Delort[?]) Let $s \in \mathbb{R}, N \in \mathbb{N}^*$, $\phi_0 \in H^s_N(\mathbb{R}), \phi_1 \in H^{-1}_{s-1}(\mathbb{R})$, $f \in H^{-1}_{s-\frac{1}{2}}(\mathbb{R}^2)$, and $\phi$ be a solution of
\[
\begin{align*}
\Box \phi + \frac{m^2}{h^2} \phi &= f \\
|\phi|_{l=0} &= \phi_0 \\
|\phi|_{l=0} &= \phi_1.
\end{align*}
\] (4.9)

Then, if $\chi \in C^\infty_0(\mathbb{R})$, we have that $\chi(t)\phi \in H^{s, \frac{1}{2}}_N$ depends continuously on $\phi_0, \phi_1$, and $f$ in the indicated spaces.

Proof of Theorem ?? Let $\chi \in C^\infty_0(\mathbb{R}), \chi \equiv 1$ on $[0, 1]$. define a function sequence $U^\alpha = (\Psi^\alpha, \Phi^\alpha)^t$ by
\[
\begin{aligned}
\begin{array}{l}
\partial_{t}\Psi^{n+1} - iA(D_{x})\Psi^{n+1} + ih^{-1}\gamma^{0}\Psi^{n+1} = h^{-1}\Phi^{n}V\Psi^{n}, \\
\square\Phi^{n+1} + m^{2}h^{-2}\Phi^{n+1} = h^{-2}(ig_{0}\Psi_{1}\gamma^{0}\gamma^{5}\Psi_{1} + g_{1}\Psi_{1}\gamma^{0}\Phi^{n}), \\
\Psi^{n+1}|_{t=0} = V_{1}, \Phi^{n+1}|_{t=0} = W_{1}, \Phi^{n+1}|_{t=0} = W_{2}.
\end{array}
\end{aligned}
\]

for \( n \geq 0 \), \( U^{0} \) is a solution of
\[
\begin{aligned}
\begin{array}{l}
\partial_{t}\Psi^{0} - iA(D_{x})\Psi^{0} + ih^{-1}\gamma^{0}\Psi^{0} = 0, \\
\square\Phi^{0} + m^{2}h^{-2}\Phi^{0} = 0, \\
\Psi^{0}|_{t=0} = V_{1}, \Phi^{0}|_{t=0} = W_{1}, \Phi^{0}|_{t=0} = W_{2}.
\end{array}
\end{aligned}
\]

Notice that \( Q(\Psi_{1}, \Psi_{2}) = \Psi_{1}, \gamma^{0}\gamma^{5}\Psi_{2} = \bar{X}_{1}T - Y^{t}Z \) if \( \Psi_{1} = (X, Y)^{t}, \Psi_{2} = (T, Z)^{t} \), for \( \bar{\Psi}^{t}\gamma^{0}\Psi \) can be discussed in a similar way. From Theorem ??, and Theorem ??, we have that the right hand side of (??) is in \( H_{N-1}^{-\frac{1}{2}} \) if \( \chi(t)\Psi^{n}_{1} \in H_{N-1}^{\frac{3}{2}} \) and \( \chi(t)\Phi^{n}_{1} \in H_{N-1}^{\frac{3}{2}} \). From Lemma ??, Lemma ??, and Lemma ??, we have \( \chi(t)\Psi^{n+1}_{1} \in H_{N-1}^{\frac{3}{2}}, \chi(t)\Phi^{n+1} \in H_{N-1}^{\frac{3}{2}}, \) and they can be controlled by
\[
C(||V_{1}||_{H_{N-1}^{\frac{3}{2}}} + ||W_{1}||_{H_{N}^{\frac{3}{2}}} + ||W_{2}||_{H_{N-1}^{-\frac{1}{2}}}) + \\
C||\chi(t)\Psi^{n}||_{H_{N-1}^{\frac{3}{2}}}^{2}P(||\chi(t)\Psi^{n}||_{H_{N-1}^{\frac{3}{2}}}) + \\
C||\chi(t)\Phi^{n}||_{H_{N}^{\frac{1}{2}}}^{2}Q(||\chi(t)\Phi^{n}||_{H_{N-1}^{\frac{3}{2}}} + ||\chi(t)\Psi^{n}||_{H_{N}^{\frac{3}{2}}}).
\]

Since \( \chi(t)\Psi^{0} \in H_{N-1}^{\frac{3}{2}}, \chi(t)\Phi^{0} \in H_{N-1}^{\frac{3}{2}}, \) and the norm can be controlled by the first term of (??), we know that the iteration sequence \( \{U^{n}\} \) is bounded in \( H_{N-1}^{\frac{3}{2}} \times H_{N-1}^{\frac{3}{2}} \) uniformly in \( n \) if \( ||V_{1}||_{H_{N-1}^{\frac{3}{2}}} + ||W_{1}||_{H_{N}^{\frac{3}{2}}} + ||W_{2}||_{H_{N-1}^{-\frac{1}{2}}} \leq \delta \) small enough.

Noting that
\[
\begin{aligned}
\begin{array}{l}
\partial_{t}(\Psi_{1} - \Psi_{2}) - iA(D_{x})(\Psi_{1} - \Psi_{2}) + ih^{-1}\gamma^{0}(\Psi_{1} - \Psi_{2}) \\
= h^{-1}(\Phi_{1} - \Phi_{2})V\Psi_{1} + \Phi_{2}V(\Psi_{1} - \Psi_{2}), \\
\square(\Phi_{1} - \Phi_{2}) + m^{2}h^{-2}(\Phi_{1} - \Phi_{2}) \\
= h^{-2}(ig_{0}(\bar{\Psi}_{1} - \bar{\Psi}_{2})\gamma^{0}\gamma^{5}\Psi_{1} + g_{1}(\bar{\Psi}_{1} - \bar{\Psi}_{2})\gamma^{0}\Phi_{1}), \\
+ ig_{0}(\bar{\Psi}_{2})\gamma^{0}\gamma^{5}(\bar{\Psi}_{1} - \bar{\Psi}_{2}) + g_{1}(\bar{\Psi}_{2})\gamma^{0}(\bar{\Psi}_{1} - \bar{\Psi}_{2}).
\end{array}
\end{aligned}
\]

It is not difficult to show that the sequence is convergence also by a similarly argument to above, and the limit functions are in the indicated spaces of the theorem for \( t \in ]-1, 1[ \), this is because that \( H_{N-1}^{\frac{3}{2}} \subset C^{0}(R, H_{N-1}^{\frac{3}{2}}) \) and \( \Phi \in H_{N}^{\frac{3}{2}} \subset C^{0}(R, H_{N}^{\frac{3}{2}}) \) implies \( \Phi_{t} \in H_{N-1}^{-\frac{1}{2}} \subset C^{0}(R, H_{N-1}^{\frac{1}{2}}) \). \( \square \)
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