Abstract

An algebra $A$ with identity $(a \circ b) \circ c - a \circ (b \circ c) = (a \circ c) \circ b - a \circ (c \circ b)$, is called right-symmetric. Cohomology and deformation theory for right-symmetric algebras are developed. Cohomologies of $gl_n$ and half-Witt algebras $W_{n+p}^{sym}$, $p = 0$, $W_{n+p}^{sym}(m)$, $p > 0$, are calculated. In particular, one right-symmetric central extension of $W_1^{sym}$ is constructed.
Contents
1. Introduction.

An algebra $A$ over a field $K$ of characteristic $p \geq 0$ is called right-symmetric [1], [2], if for any $a, b, c \in A$, the following condition takes place

$$a \circ (b \circ c) - (a \circ b) \circ c = a \circ (c \circ b) - (a \circ c) \circ b.$$  

Any associative algebra is right-symmetric. For example, $gl_n$ under usual multiplication of matrices is right-symmetric. An algebra of vector fields $K[[x_1, \ldots, x_n]]$ under multiplication $u \partial_i \circ v \partial_j = v \partial_j (u \partial_i)$ gives us a less trivial example of right-symmetric algebras. It is not associative. Since its Lie algebra is isomorphic to Witt algebra $W_n$, we call it a half-Witt algebra and denote it as $W_n^{sym}$. If $n = 1$, this algebra satisfies one more identity

$$a \circ (b \circ c) = b \circ (a \circ c).$$

Such algebras are called Novikov [1], [2]. The generalisation of Novikov structure for the case $n > 1$ is possible, if we consider half-Witt algebra not with one, but with two multiplications. If we endow $K[[x_1, \ldots, x_1]]$ with a second multiplication $u \partial_i * v \partial_j = \partial_i (u) v \partial_j$, then we obtain algebra with the following identities

$$a \circ (b \circ c) - (a \circ b) \circ c - a \circ (c \circ b) + (a \circ c) \circ b = 0,$$

$$a * (b * c) - b * (a * c) = 0,$$

$$a \circ (b * c) - b * (a \circ c) = 0,$$

$$(a \circ b - b \circ a - a \circ b + b \circ a) * c = 0,$$

$$(a \circ b - b \circ a) * c + a * (c \circ b) - (a * c) \circ b - b * (c \circ a) + (b * c) \circ a = 0.$$  

These two multiplications are useful in the construction of right-symmetric, Chevalley-Eilenberg and Leibniz cocycles of such algebras.

We develop cohomology theory for right-symmetric algebras. We endow right-symmetric cochain complex $C^*_{sym}(A, M) = \oplus_k C^*_{sym}(A, M)$, where $C^k_{sym}(A, M) = Hom(A \otimes A^k(A), M)$, $k > 0$, by a pre-simplicial structure. Corresponding cohomologies can be "almost" obtained by derived functor formalism. The exact meaning of the word "almost" can be found in section ???. Roughly speaking, this means that one should be more careful in considering small degree cohomologies. If we take $C^0_{sym}(A, M)$ as $M$, then we should consider the operator $d_{sym}$ with cubic condition $d_{sym}^3 = 0$. We prefer taking $C^0_{sym}(A, M)$ as a $ker d_{sym}^2$ on $M$, i.e., $C^0_{sym}(A, M) := M^{lass} := \{ m \in M : (m, a, b) = 0, \forall a, b \in A \}$. Then for any $m \in M$, we can correspond 2-right-symmetric cocycles, $\nabla(m) : (a, b) \mapsto (m, a, b)$. We call such cocycles as standard. If $m \in M^{lass}$, then $\nabla(m) = 0$. If $m \in M$, then the cohomological class $[\nabla(m)] = 0$, because of $\nabla(m) = d\omega$, where $\omega(a) = [a, m]$. Moreover it is true, if $M$ is a submodule of some right-symmetric $A$-module $\bar{M}$, and $m \in \bar{M}$, such that $[a, m] = a \circ m - m \circ a \in \bar{M}, \forall a \in A$. If $\bar{m} \in \bar{M}$, such that $d_{sym} \bar{m}(a) \notin \bar{M}$, then $\nabla(\bar{m})$ can give a nontrivial class of 2-right-symmetric cocycles in $H^2_{sym}(A, M)$. For example, Osborn 2-right-symmetric cocycles for $A = W_1^{sym}(m), p > 0$, that appear in constructing simple Novikov algebras,

$$(u \partial, v \partial) \mapsto x^{m-1} uv \partial,$$

$$(u \partial, v \partial) \mapsto x^{m-2} uv \partial,$$

are $\nabla(x^{m-1} \partial)$, and $\nabla(x^{m-2} \partial)$, correspondingly.

If $k > 0$, right-symmetric cohomologies $H^k_{sym}(A, M)$ are isomorphic to Chevalley-Eilenberg cohomologies $H^k_{lie}(A, C^l(A, M))$, where $A^{lie}$-module structure on $C^l(A, M)$ is given by a special way: $[a, f] = -d_{sym} f (b, a)$. We endow also right-symmetric universal enveloping algebra by a Hopf algebraic structure. It allows us to consider cup products, that are very useful in coycle constructions.

Second cohomology space $H^2_{sym}(A, A)$ is interpreted as a space of right-symmetric deformations. We calculate right-symmetric cohomologies of matrix algebra $gl_n, p = 0$. We prove that, in the category of irreducible antisymmetric $gl_n^{sym}$-modules, nontrivial cohomologies appear
only in the case of $M = (gl_n)_{anti}$. Moreover, right-symmetric cohomology of $gl_n^{sym}$ in $(gl_n)_{anti}$ can be reduced to Chevalley-Eilenberg cohomology of Lie algebra $gl_n$ with coefficients in trivial module:

$$H^{k+1}_{rsym}(gl_n, (gl_n)_{anti}) \cong H^k_{lie}(gl_n, K), \quad k > 0.$$  

In particular, $H^{k+1}_{rsym}(gl_n, K) = 0$, $k \geq 0$. We calculate also right-symmetric cohomologies of $gl_n$ with coefficients in regular module. These results show that $gl_n$ has $(n^2 - 1)$--parametrical nontrivial right-symmetric deformations. Any formal right-symmetric deformation of $gl_n$ is equivalent to the deformations given by the rule

$$(a, b) \mapsto a \circ b + t tr b [X, a], \quad X \in sl_2.$$  

One can choose the prolongation in another way:

$$(a, b) \mapsto a \circ b + t X \circ ((tr a)b - tr(a \circ b) + (tr b)a)$$

$$+ t^2 \{tr a tr b - (tr a \circ b)^2 X^2 - (tr a tr(X \circ b))X - (tr(a \circ X) tr b)X\} + \cdots .$$

We prove that right-symmetric cohomologies of $A = W_n^{sym}$ with coefficients in antisymmetric modules can also be reduced to Chevalley-Eilenberg cohomologies of the Lie algebra $W_n$. As it turned out, $H^2_{rsym}(A, A)$ for $A = W_n, p = 0$, or $A = W_n(m), p > 0$, is too large and this happens mainly because of largeness of a space of right-symmetric derivations. There is an imbedding

$$Z^1_{rsym}(A, A) \otimes H^1_{lie}(A, U) \hookrightarrow H^2_{rsym}(A, A).$$

We prove that $Z^1_{rsym}(A, A)$ has a basis consisting of two types of right-symmetric derivations: $\partial_i, i = 1, \ldots, n$, if $p > 0$, one should consider also derivations $\partial_i^{p_k}, 0 \leq k_i < m_i$; and $x_i \partial_j, i, j = 1, \ldots, n$. So, any right-symmetric derivation of $A$ has a form $\sum_{i=1}^n u_i \partial_i + \delta(p > 0) \sum_{i=1}^n \sum_{k_i=0}^{m_i-1} \lambda_i \partial_i^{p_k}$, such that $\partial_i \partial_j(u_s) = 0, i, j, s = 1, \ldots, n, \lambda_i \in K$. We formulate a result about local deformations of $W_n, p = 0$, or $W_n(m), p > 3$. The space $H^2_{rsym}(W_n, W_n), p = 0$, is generated by classes of cocycles of four types. In the case of $p > 3$ Steenrod Squares also appears. We prove that $W_1^{rsym}$ has exactly one right-symmetric central extension. It can be given by cocycle

$$(e_i, e_j) \mapsto (j + 1)j \delta_{i+j,-1}, \quad p = 0,$$  

$$(e_i, e_j) \mapsto (-1)^j \delta_{i+j,p^m-1}, \quad p > 0.$$  

For $n > 1$, $H^2_{rsym}(W_n, K) = 0$.

For right-symmetric algebras, Novikov algebras and some cohomology calculations see also [7], [14], [2], [7], [2], [7], [1].

2. Right-symmetric algebras and (co)modules.

2.1. Right-symmetric algebras. An algebra $A$ over a field $K$ with multiplication $(a, b) \mapsto a \circ b$, is called Lie-admissible, if the vector space $A$ under commutator $[a, b] = a \circ b - b \circ a$ can be endowed by a structure of Lie algebra. An algebra $A$ is right-symmetric, if it satisfies the following identity:

$$(a \circ b) \circ c - a \circ (b \circ c) = (a \circ c) \circ b - a \circ (c \circ b), \quad \forall a, b, c \in A.$$  

Let $(a, b, c) = a \circ (b \circ c) - (a \circ b) \circ c$ be the associator of elements $a, b, c \in A$. In terms of associators the right-symmetric identity is

$$(a, b, c) = (a, c, b), \quad \forall a, b, c \in A.$$  

Right-symmetric algebra $A$ is Lie-admissible. Similarly, one can define left-symmetric algebra by identity

$$(a, b, c) = (b, a, c), \quad \forall a, b, c \in A.$$  

Categories of left-symmetric algebras and right-symmetric algebras are equivalent. Any left(right)-symmetric algebra under new multiplication $(a, b) \mapsto b \circ a$. 


An element $e$ of right-symmetric algebra is called left unit, if $e \circ a = a$, for any $a \in A$. Denote by $Q_l(A)$ a space of left units. Let $Z_l(A) = \{ z \in A : z \circ a = 0, \forall a \in A \}$ be left center of $A$. Call a space $N_l(A) = Z_l(A) \oplus Q_l(A)$ as a semi-center of $A$. Then $[N_l(A), N_l(A)] \subseteq Z_l(A)$.

An algebra $A$ is called (left) unital, if it has nontrivial left units.

Any associative algebra is a right-symmetric algebra. In such cases, we will use notations like $A^{ess}$, if we consider $A$ as associative algebra and $A^{rsym}$, if we consider $A$ as right-symmetric algebra. Similarly, for right-symmetric algebra $A$ notation $A^{sym}$ means that we use only right-symmetric structure on $A$ and $A^{lie}$ stands for a Lie algebra structure under commutator $(a, b) \mapsto [a, b]$.

Matrix algebras $gl_n$ gives us examples of unital right-symmetric algebras.

Less trivial examples appear in the consideration of Witt algebras. The algebra $W_n, p = 0$, and $W_n(m)$ defined below has not only right-symmetric multiplication $(a, b) \mapsto a \circ b$, but also one more multiplication $(a, b) \mapsto a * b$, that satisfies the following identities

\[
(a \circ (b \circ c) - (a \circ b) \circ c - a \circ (c \circ b) + (a \circ c) \circ b = 0, \\
(\alpha + \beta) \frac{\alpha}{\alpha} = \left(\begin{array}{c}
\frac{n}{l} \\
\frac{n}{l}
\end{array}\right) = \frac{n!}{l!(n - l)!}, n, l \in \mathbb{Z}_+
\]

Let $U = k[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]] = \{ x^\alpha = \prod_{i=1}^{k} x_i^{\alpha_i} : \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbb{Z}, i = 1, \ldots, n \}$ be an algebra of Laurent power series, if the main field $k$ has characteristic 0 and $U = O_n(m) = \{ x^{(\alpha)} = \prod_{i} x_i^{(\alpha_i)} : \alpha = (\alpha_1, \ldots, \alpha_n), 0 \leq \alpha_i < p, i = 1, \ldots, n \}$ be a divided power algebra if $char k = p > 0$. Recall that $O_n(m)$ is $p^m$ -dimensional and the multiplication is given by

\[x^{(\alpha)} x^{(\beta)} = \left(\frac{\alpha + \beta}{\alpha}\right) x^{(\alpha + \beta)},\]

where $m = \sum_i m_i$, and

\[
\left(\frac{\alpha + \beta}{\alpha}\right) = \prod_i \left(\frac{\alpha_i + \beta_i}{\alpha_i}\right), \left(\begin{array}{c}
n \\
l
\end{array}\right) = \frac{n!}{l!(n - l)!}, n, l \in \mathbb{Z}_+.
\]

Let $\epsilon_i = (0, \ldots, 1, \ldots, 0)$. Define $\partial_i$ as a derivation of $U$,

\[\partial_i(x^{(\alpha)}) = \alpha_i x^{(\alpha - \epsilon_i)}, p = 0, \]
\[\partial_i(x^{(\alpha)}) = x^{(\alpha - \epsilon_i)}, p > 0.
\]

Endow a space of derivations $Der U = \{ \sum_i u_i \partial_i : u_i \in U \}$ by multiplications:

\[u \partial_i \circ v \partial_j = v \partial_j(u) \partial_i, \]
\[u \partial_i * v \partial_j = \partial_i(u) v \partial_j.
\]

Denote obtained algebra as $W_n^{sym}(m)$. If $p = 0$, this denotation will be reduced until $W_n^{sym}$. If $n \not\equiv 0(mod p)$, then an element $e = \sum_i x_i \partial_i / n$ is a left unit of $W_n^{sym}(m)$. If $n = 1$, then $a \circ b = a * b$. Thus the algebra $A = W_1^{sym}(m)$ in addition to right-symmetry condition satisfies the following identity

\[a \circ (b \circ c) = b \circ (a \circ c) \quad \forall a, b, c \in A.
\]

Such algebras are called Novikov algebras [7]. Notice that Novikov algebra $W_1(m)$ is unital. If right-symmetric algebra $A$ is Novikov algebra, we will use denotation $A^{nov}$. 


2.2. Right-symmetric modules and comodules. A vector space $M$ is said to be module over right-symmetric algebra $A$, if it is endowed by right action

$$M \times A \rightarrow M, \quad (m, a) \mapsto m \circ a$$

and left action

$$A \times M \rightarrow M, \quad (a, m) \mapsto a \circ m,$$

such that

$$m \circ [a, b] = (m \circ a) \circ b + (m \circ b) \circ a = 0,$$

$$a \circ (m \circ b) - a \circ (m \circ b) - (a \circ b) \circ m + a \circ (b \circ m) = 0,$$

for any $a, b \in A, m \in M$. We will say, that $M$ is antisymmetric $A$-module, if the left action of $A$ is trivial, i.e., $a \circ m = 0$, for any $a \in A, m \in M$. For module $M$ over right-symmetric algebra $A$, denote by $M_{\text{anti}}$ its antisymmetric $A$-module: $M_{\text{anti}} = M, (m, a) \mapsto m \circ a, (a, m) \mapsto 0$, for all $m \in M_{\text{anti}}, a \in A$.

A right-symmetric $A$-module $M$ is said to be special, if the right action satisfies the following condition

$$m \circ (a \circ b) - (m \circ a) \circ b = 0, \quad \forall a, b \in A, \forall m \in M.$$

A special module is antisymmetric, if $a \circ m = 0$, for all $a \in A$.

Example. For right-symmetric algebra $A$ its vector space $A$ can be endowed by a natural structure of $A$-module, $(a, m) \mapsto a \circ m, (m, a) \mapsto m \circ a, a, m \in A$. In such cases we say that $M = A$ is regular $A$-module. If $A$ is associative algebra, then regular module is special.

The functor $A^{\text{lie}}$-module $\rightarrow$ right $A^{\text{lie}}$-module $\rightarrow$ Antisymmetric $A$-module gives us an equivalence of the category of antisymmetric $A$-modules to the category of (right) $A^{\text{lie}}$-modules. Antisymmetric $A$-module corresponding to right $A^{\text{lie}}$-module will be denoted by $M_{\text{anti}}$.

Assume that $A$ is an associative algebra $A$ with multiplication $(a, b) \mapsto a \cdot b$. In the last case of $A^{\text{sym}}$ right-symmetric multiplications can be defined in two ways: by $(a, b) \mapsto a \cdot b$ or by $(a, b) \mapsto b \cdot a$. For definiteness we endow $A^{\text{sym}}$ by multiplication $(a, b) \mapsto a \cdot b$. For associative algebra $A$ the functor

$$A^{\text{ass}}$$

gives us an equivalence of the categories of antisymmetric $A^{\text{ass}}$-modules and right $A^{\text{ass}}$-modules.

Right-symmetric $A$-module $M$ can be endowed by a structure of module over Lie algebra $A^{\text{lie}}$ by action $[a, m] = a \circ m - m \circ a$. The obtained module is denoted by $M^{\text{lie}}$.

So, for defining module structure on a vector space $M$ over a right-symmetric algebra $A$ one should define on $M$ right module structure over the Lie algebra $A^{\text{lie}}$ and endow it by a left action that satisfies condition (AAM). As we mentioned before the last can be done by a trivial way by setting $a \circ m = 0, \forall a \in A, \forall m \in M$.

For a module $M$ over right-symmetric algebra $A$ the subspace

$$M^{\text{ass}} = \{ m \in M : (m, a, b) = 0, \forall a, b \in A \}$$

is called a left associative invariant subspace of $M$, and

$$M^{\text{inv}} = \{ m \in M : m \circ a = 0, \forall a \in A \}$$

is called a left invariant subspace of $M$. If $M = A$ is regular module, then $A^{\text{ass}}$ is called a left associative center. Notice that, $M^{\text{inv}}$ coincides with the left center of $A$. Notice that, $M^{\text{inv}}$ is close under right action of $A$ and

$$M^{\text{inv}} \subseteq M^{\text{ass}}.$$

Module $M$ of (left) unital right-symmetric algebra is called (left) unital, if

$$e \circ m = m, \quad \forall e \in Q_l(A), \forall m \in M.$$
and (left) central, if
\[ z \circ m = 0, \quad \forall z \in Z_r(A), \forall m \in M. \]

Regular module of unital right-symmetric algebra is unital and central.

A vector space \( M \) is called \textit{comodule} over right-symmetric algebra \( A \), if there are given right action
\[ M \times A \to M, \quad (m, a) \mapsto m \circ a, \]
and left action
\[ A \times M \to M, \quad (a, m) \mapsto a \circ m, \]
such that
\[ [a, b] \circ m - a \circ (b \circ m) + b \circ (a \circ m) = 0, \]
\[ -b \circ (m \circ a) + (b \circ m) \circ a - m \circ (a \circ b) + (m \circ a) \circ b = 0, \]
for any \( a, b \in A, m \in M \). A comodule \( M \) is \textit{special}, if it satisfies the identity
\[ (a \circ b) \circ m = a \circ (b \circ m), \quad \forall a, b \in A, \forall m \in M. \]

A (special) comodule \( M \) is called \textit{antisymmetric}, if \( m \circ a = 0 \), for any \( a \in A \).

**Example.** Let \( A \) be right-symmetric algebra, \( M \) be \( A \)-module and \( M' = \{ f : M \to K \} \) be a space of linear functions on \( M \). Set
\[ (a \circ f)(m) = f(m \circ a), \quad (f \circ a)(m) = f(a \circ m). \]
Then \( M' \) under actions \( (a, f) \mapsto a \circ f, \quad (f, a) \mapsto f \circ a \), can be endowed by a structure of \( A \)-comodule. Check it.

The \( A \)-comodule \( A' \) for regular module \( A \) is called \textit{coregular} comodule of \( A \). If \( A \) is associative, then \( A' \) is special comodule.

For \( A \)-comodule \( M \) let
\[ M^{r.ass} = \{ m \in M : (a, b, m) = 0, \forall a, b \in A \} \]
be a \textit{right associative invariant} subspace of \( M \) and
\[ M^{r.inv} = \{ m \in M : a \circ m = 0, \forall a \in A \} \]
be a \textit{right invariant} subspace of \( M \). Notice that \( M^{r.inv} \) is close under left action of \( A \). An inclusion takes place
\[ M^{r.inv} \subseteq M^{r.ass}. \]

2.3. \textbf{Antisymmetric module} \( C_{right}^1(A, M) \).

**Proposition 2.1.** The space of linear maps \( C_{r sym}^1(A, M) := C^1(A, M) = \{ f : A \to M \} \) can be endowed by a structure of antisymmetric \( A \)-module, where the right action is given by
\[ (f \circ a)(b) = f(b \circ a) - f(a \circ b) + b \circ f(a), \quad a, b \in A. \]

**Proof.** For \( f \in C^1(A, M), a, b, c \in A \), we have
\[ (f \circ [b, c])(a) - ((f \circ b) \circ c)(a) + ((f \circ c) \circ b)(a) = d_{r sym}(f, [b, c])(a) - d_{r sym}([f, b])(a, c) + d_{r sym}([f, c])(a, b) = \]
\[ a \circ f([b, c]) - f(a \circ [b, c]) + f(a) \circ [b, c] - a \circ [f, b](c) + [f, b](a \circ c) - [f, b](a) \circ c = \]
\[ a \circ f([b,c]) - f(a \circ [b,c]) + f(a) \circ [b,c] - a \circ d_{rsym}(c(b)) + a \circ (c \circ f(b)) + a \circ f(c \circ b) - a \circ f(b \circ c) + f((a \circ b) \circ c) - f((a \circ b) \circ c) + f(a \circ (f(c) \circ b)) - f(a \circ (f(c) \circ b)) + (a \circ f(c)) \circ b - f(a \circ (f(c) \circ b)) + (a \circ f(c)) \circ b = \]

\[ = 0. \]

So, \( C^1(A,M) \) is a right \( A^{tie} \)-module. •

2.4. **Universal enveloping algebras of right-symmetric algebras.** Consider two copies of \( A \), denoted by \( A^r, A^l \), and the tensor algebra \( T(A^r \oplus A^l) \). Algebras \( A^r, A^l \) supposed to be free as \( K \)-module and the tensor algebra \( T(A^r \oplus A^l) \) is associative and unital. Elements of \( A^r \) and \( A^l \) corresponding to \( a \in A \) denote as \( r_a \) and \( l_a \). Let \( U(A) \) be a factor-algebra of \( T(A^r \oplus A^l) \) over an ideal \( J \) generated by \( r_{[a,b]} = r_a r_b + r_b r_a, \) \( r_{[b,l_a]} = l_b a + l_a o b \). This algebra can be considered as a universal enveloping algebra of right-symmetric algebra \( A \). Denote by \( U(A) \) the factor-algebra of \( T(A^r \oplus A^l) \) over an ideal \( \bar{J} \) generated by \( \{r_{aob} - r_a r_b, \) \( r_{[b,l_a]} = l_b a + l_a o b \}. \) This algebra is called a special universal enveloping algebra of \( A \). Notice that, \( J \subset \bar{J} \), since

\[ r_{[a,b]} - [r_{a},r_{b}] = \{r_{aob} - r_{a} r_{b}\} - \{r_{bca} + r_{b} r_{a}\} \in \bar{J}. \]

So, the following exact sequences of algebras take place

\[ 0 \rightarrow \bar{J} \rightarrow T(A^r \oplus A^l) \rightarrow U(A) \rightarrow 0, \]

\[ 0 \rightarrow \bar{J} \rightarrow T(A^r \oplus A^l) \rightarrow U(A) \rightarrow 0, \]

and

\[ 0 \rightarrow \bar{J}/J \rightarrow U(A) \rightarrow \bar{U}(A) \rightarrow 0. \]

In particular, we can consider \( \bar{U}(A) \) as right \( U(A) \)-module:

\[ \bar{u} \bar{v} = \bar{v} \bar{u}, \]

where \( \bar{u} \) and \( \bar{v} \) are elements of \( \bar{U}(A) \) and \( U(A) \) corresponding to \( u \in T(A^r \oplus A^l) \).

**Theorem 2.2.** Let \( A \) be a right-symmetric algebra.

i) There exists an equivalence of the categories of \( A \)-modules and right \( U(A) \)-modules. The same is true for \( A \)-comodules and left \( U(A) \)-modules.

ii) The category of special \( A \)-modules is equivalent to the category of right \( U(A) \)-modules. The same is true for special \( A \)-comodules and left \( U(A) \)-modules.
Proof. i) Let \((r, l) : A \to \text{End} M\) be a representation of right-symmetric algebra \(A\) corresponding to \(A\)-module \(M\), i.e.,
\[
  r : A \to \text{End} M, \quad a \mapsto r_a, \quad mra = m \circ a,
\]
\[
  l : A \to \text{End} M, \quad a \mapsto l_a, \quad ml_a = a \circ m,
\]
linear operators, such that for any \(a, b \in A\),
\[
  r_{[a,b]} - r_ar_b + r_br_a = 0, \quad (MAA)
\]
\[
  [r_b, l_a] - l_br_a + l_aob = 0. \quad (AAM)
\]
So, any \(A\)-module is a right \(U(A)\)-module and, converse, any right \(U(A)\)-module can be considered as an \(A\)-module.

A corepresentation \((r^\text{co}, l^\text{co}) : A \to \text{End} M\), corresponding to \(A\)-comodule \(M\),
\[
  r^\text{co} : A \to \text{End} M, \quad a \mapsto r^\text{co}_a, \quad r^\text{co}_a m = a \circ m,
\]
\[
  l^\text{co} : A \to \text{End} M, \quad a \mapsto l^\text{co}_a, \quad l^\text{co}_a m = m \circ a,
\]
satisfies conditions (MAA), (AAM) for \(r^\text{co}, l^\text{co}\). So, any \(A\)-comodule is a left \(U(A)\)-module.

Any left \(U(A)\)-module can be considered as a \(A\)-comodule.

ii) Let \(M\) be a special \(A\)-module. Then by the rule
\[
  mr_a = m \circ a, ml_a = m \circ a,
\]
we obtain a right \(\bar{U}(A)\)-module:
\[
  m \circ (a \circ b) - (m \circ a) \circ b = 0 \to r_{aob} = r_ar_b.
\]
Converse, for a right \(\bar{U}(A)\)-module \(N\), one can correspond special \(A\)-module \(N\), by \(n \circ a := nr_a, a \circ n = nl_a\).

For a special \(A\)-comodule \(M\) notice that
\[
  (a \circ b) \circ m - a \circ (b \circ m) = 0 \Rightarrow r^\text{co}_{aob} = r^\text{co}_a r^\text{co}_b,
\]
if \(r^\text{co}_amm = a \circ m, l^\text{co}_amm = m \circ a\). So, any special \(A\)-comodule is a left \(U^{\text{spec}}(A)\)-module. A converse statement is also evident. •

2.5. Right-symmetric cohomologies as a derived functor. Recall that factor-images of the element \(u \in T(A^r \oplus A^t)\) in \(U(A)\) and \(\bar{U}(A)\) are denoted by \(\bar{u}\) and \(\bar{u}\). Consider \(A = A \oplus < 1 >\) as a right \(U(A)\)-module:

\[
  1 \circ \bar{r}_a = r_a \circ 1 = a, \quad 1 \circ \bar{l}_a = l_a \circ 1 = a, \quad a \circ \bar{r}_b = a \circ b, \quad a \circ \bar{l}_b = b \circ a,
\]
for all \(a \in A\). Endow \(\bar{U}(A)\) by a structure of \(U(A)\)-module as in the subsection ???. Consider \(A \otimes \wedge^k(A) \otimes U(A), k \geq 0\), as a right \(U(A)\)-module. Then \(A \otimes \wedge^kA \otimes U(A)\) is a free \(U(A)\)-module. Denote its generators by \(< a_0, a_1, \ldots, a_k >\), where \(a_0 \in A, a_1 \wedge \cdots \wedge a_k \in \wedge^k A\).

Construct homomorphisms
\[
  \partial : A \otimes \wedge^kA \otimes U(A) \to A \otimes \wedge^{k-1}A \otimes U(A), \quad \partial : A \otimes U(A) \to \bar{U}(A)
\]
\[
  \partial : A \otimes U(A) \to \bar{U}(A)
\]
\[
  \epsilon : \bar{U}(a) \to \bar{A},
\]
as below
\[
  \partial < a_0, a_1, \ldots, a_k > = \sum_{i=1}^{k} \{ (-1)^{i+1} < a_i, a_1, \ldots, \hat{a}_i, \ldots, a_k > \bar{l}_{a_0} + (-1)^i < a_0 \circ a_i, a_1, \ldots, \hat{a}_i, \ldots, a_k > + (-1)^{i+1} < a_0, a_1, \ldots, \hat{a}_i, \ldots, a_k > \bar{r}_{a_i} \}
\]
\[
  + \sum_{i<j} (-1)^{i+1} < a_0, a_1, \ldots, \hat{a}_i, \ldots, a_{j-1}, [a_i, a_j], \ldots, a_k >,
\]
\[ \partial(a_0) = < \hat{1} > \bar{I}_{a_0} - < \hat{1} > \bar{r}_{a_0}, \]

Then the following sequence
\[ \cdots \xrightarrow{\partial} A \otimes \wedge^2 A \otimes U(A) \xrightarrow{\partial} A \otimes A \otimes U(A) \xrightarrow{\partial} A \otimes U(A) \xrightarrow{\partial} \hat{U}(A) \xrightarrow{\partial} \hat{A} \to 0 \]
is almost a free resolution of the right \( U(A) \)-module \( \hat{A} \). Here the words ”almost free” mean that all members of the resolution except \( \hat{U}(A) \) is are free right \( U(A) \)-modules.

Notice that

\[ \text{Hom}_{U(A)}(A \otimes \wedge^k A \otimes U(A), M) \cong A \otimes \wedge^k A, \quad k \geq 0, \]
and \( \text{Hom}_{U(A)}(\hat{U}(A), M) \) consists of \( g : \hat{U}(A) \to M \), such that

\[ g(\hat{1})(\bar{r}_a \bar{r}_b - \bar{r}_{a b}) = g((\hat{r}_a \hat{r}_b - \hat{r}_{a b})) = \]

\[ g(\bar{r}_a \bar{r}_b - \bar{r}_{a b}) = g((\bar{r}_a \bar{r}_b - \bar{r}_{a b}) = 0, \]

for any \( a, b \in A \). So,

\[ \text{Hom}_{U(A)}(\hat{U}(A), M) \cong \{ m \in M : (m, a, b) = 0 \}. \]

Therefore, as a right-symmetric cochain complex we can take

\[ C^*_r sym(A, M) = \bigoplus_k C^k_r sym(A, M), \]
\[ C^0_{r sym}(A, M) = \{ m \in M : (m, a, b) = 0, \forall a, b \in A \}, \]
\[ C^k_{r sym}(A, M) = A \otimes \wedge^k A, \quad k \geq 0. \]

These statements will follow from our results on right-symmetric cohomologies in the next sections. Our approach is slightly different from Koszul’s approach. We will argue in cohomological terms and prove that right-symmetric cochain complex has a pre-simplicial structure.

Let us mention these results relating homologies. Let \( M \) be comodule over right-symmetric algebra \( A \). Endow \( M \otimes A \) by a structure of antisymmetric \( A \)-comodule with a left action

\[ b \circ (m \otimes a) = m \otimes a \otimes b - m \otimes a \circ b + b \circ m \otimes a. \]

Set

\[ C^0_{r sym}(A, M) := M^{r, ass} := \{ m \in M : (a, b, m) = 0, \forall a, b \in A \}, \]
\[ C^k_{r sym}(A, M) = M \otimes A \otimes \wedge^k(A), \quad k \geq 0. \]
\[ C^*_s sym(A, M) = \bigoplus_k C^k_{r sym}(A, M). \]

Then \( C^*_{r sym}(A, M) \) is chain complex under the boundary operator

\[ \partial : C^k_{r sym}(A, M) \to C^{k+1}_{r sym}(A, M), \]

\[ \partial(m \otimes a_0 \otimes a_1 \wedge \cdots \wedge a_k) = \]

\[ \sum_{i=1}^k (-1)^{i+1} \{ m \otimes a_0 \otimes a_i \otimes a_1 \wedge \cdots \wedge \hat{a}_i \cdots \wedge a_k \}
\]

\[ - m \otimes a_0 \otimes a_i \otimes a_1 \wedge \cdots \wedge a_k 
\]

\[ + a_i \otimes m \otimes a_0 \otimes a_1 \wedge \cdots \wedge \hat{a}_i \cdots \wedge a_k \}
\]

\[ + \sum_{i<j} (-1)^{i+j} m \otimes a_0 \otimes a_1 \wedge \cdots \wedge a_i \wedge a_j \wedge \cdots \wedge a_k. \]

Moreover, \( C^*_{r sym}(A, M) \) has antisymmetric \( A \)-comodule structure with left action
\[ \rho_{co}^{\text{sym}}(x) : C_{k+1}^{\text{sym}}(A, M) \to C_{k+1}^{\text{sym}}(A, M), \]

\[ \rho_{co}^{\text{sym}}(x)(m \otimes a_0 \otimes a_1 \wedge \cdots \wedge a_k) = \]

\[ \sum_{i=1}^{k} (-1)^{i}(m \otimes a_0 \otimes a_i \otimes a_1 \wedge \cdots \wedge a_i \cdots \wedge a_k \]

\[ - m \otimes a_0 \otimes a_i \otimes a_1 \wedge \cdots \wedge a_i \cdots \wedge a_k \]

\[ + a_i \otimes m \otimes a_0 \otimes a_1 \wedge \cdots \wedge a_i \cdots \wedge a_k \}

\[ + \sum_{i<j} (-1)^{i+j} m \otimes a_0 \otimes a_1 \wedge \cdots \wedge a_{i-1} \wedge [x, a_i] \wedge \cdots \wedge a_k, \]

and an isomorphism of \( A \)-comodules takes place

\[ C_{k+1}^{\text{sym}}(A, M) \cong C_{k}^{\text{lie}}(A, M \otimes A). \]

that induces an isomorphism of homology spaces

\[ H_{k+1}^{\text{sym}}(A, M) \cong H_{k}^{\text{lie}}(A, M \otimes A), \quad k > 0. \]

2.6. Comultiplication of universal enveloping algebra. Let \( U(A) \) be the universal enveloping algebra of a right-symmetric algebra \( A \). As we noticed in section ??, it can be generated by the elements \( r_a, l_a, a \in A, \) such that

\[ r_{[a, b]} - [r_a, r_b] = 0, \]

\[ [l_a, r_b] - l_{aob} + l_{bla} = 0, \quad a, b \in A. \]

Define homomorphism

\[ \Delta : U(A) \to U(A) \otimes U(A), \]

by

\[ \Delta(1) = 1 \otimes 1, \]

\[ \Delta(r_a) = r_a \otimes 1 + 1 \otimes (r_a - l_a), \]

\[ \Delta(l_a) = l_a \otimes 1. \]

Since, according to right-symmetry identities,

\[ \Delta([r_a, r_b]) = \]

\[ (r_a \otimes 1 + 1 \otimes (r_a - l_a))(r_b \otimes 1 + 1 \otimes (r_b - l_b)) = \]

\[ r_{[a, b]} \otimes 1 + 1 \otimes (r_{[a, b]} - l_{[a, b]} = \]

\[ \Delta(r_{[a, b]}), \]

\[ (l_a \otimes 1)(r_b \otimes 1 + 1 \otimes (r_b - l_b) - (r_b \otimes 1 + 1 \otimes (r_b - l_b))(l_a \otimes 1) - l_{aob} \otimes 1 + l_{bla} \otimes 1 = \]

\[ (l_a, r_b - l_{aob} + l_{bla}) \otimes 1 = \]

\[ 0. \]

this definition is correct.

**Theorem 2.3.** For a right-symmetric algebra \( A \) and its universal enveloping algebra \( U(A) \) the following diagram is commutative

\[ \begin{array}{ccc}
U(A) & \Delta & U(A) \otimes U(A) \\
\downarrow \Delta & & \downarrow \Delta \otimes 1 \\
U(A) \otimes U(A) & \cong & U(A) \otimes U(A) \otimes U(A)
\end{array} \]
**Proof.** We must check that

\[(1 \otimes \Delta) \Delta(u) = (\Delta \otimes 1) \Delta(u), \; \forall u \in U(A).\]

We have

\[
(1 \otimes \Delta) \Delta(r_a) = \\
(r_a \otimes 1 \otimes 1 + 1 \otimes r_a \otimes 1 + 1 \otimes (r_a - l_a)) - 1 \otimes l_a \otimes 1 = \\
\Delta(r_a) \otimes 1 + 1 \otimes (r_a - l_a) = \\
(\Delta \otimes 1) \otimes \Delta(r_a),
\]

\[(1 \otimes \Delta) \Delta(l_a) = l_a \otimes 1 \otimes 1 = (\Delta \otimes 1) \Delta(l_a). \bullet
\]

Similarly, homomorphism \(\Delta_1\) defined below is also comultiplication,

\[
\Delta_1 : U(A) \to U(A) \otimes U(A),
\]

\[
\Delta_1(1) = 1 \otimes 1,
\]

\[
\Delta_1(r_a) = (r_a - l_a) \otimes 1 + 1 \otimes r_a,
\]

\[
\Delta_1(l_a) = 1 \otimes l_a
\]

So, we can construct for given \(A\)-modules \(M\) and \(N\) their tensor products \(M \otimes N\) with a module structure induced by comultiplication \(\Delta:\)

\[
(m \otimes n) \circ a = m \circ a \otimes n + m \otimes [n, a],
\]

\[
a \circ (m \otimes n) = a \circ m \otimes n.
\]

Moreover, it is possible for right-symmetric \(A\)-module \(M\) and for \(A^{\text{lie}}\)-module \(N\). These module structures on tensor products are associative: if \(M, N, S\) are modules over right-symmetric algebra \(A\), then

\[
(M \otimes N) \otimes S \cong M \otimes (N \otimes S).
\]

**Definition.** For given modules \(M, N\) over right-symmetric algebra \(A\), a homomorphism of \(A\)-modules \(M \otimes N \to S\) is called a cup product of \(M\) and \(N\).

Denote the image of \(m \otimes n\) in \(S\) by \(m \cup n\). Thus, a bilinear map

\[
M \times N \to S, \; (m, n) \mapsto m \cup n,
\]

is said to be the cup product (pairing) of \(M\) and \(N\) to \(S\), if

\[
(m \cup n) \circ a = m \circ a \cup n + m \cup [n, a],
\]

\[
a \circ (m \cup n) = a \circ m \cup n,
\]

for any \(a \in A, m, n \in M\).

Let

\[
C^1(A, M) = \text{Hom}_k(A, M), \; C^k_{\text{lie}}(A, M) = \text{Hom}_k(\wedge^k A, M), \; k \geq 0,
\]

\[
C^{k+1}_{\text{sym}}(A, M) = \text{Hom}_k(A \otimes \wedge^k A, M), \; k \geq 0.
\]

**Proposition 2.4.** \(C^{k+1}_{\text{sym}}(A, M)\) has an antisymmetric \(A\)-module structure, where the right action

\[
(C^{k+1}_{\text{sym}}(A, M) \times A \to C^{k+1}_{\text{sym}}(A, M), \; (\psi, x) \mapsto \psi \circ x,
\]

is defined by

\[
(\psi \circ x)(a_0, a_1, \ldots, a_k) = \\
a_0 \circ \psi(x, a_1, \ldots, a_k) - \psi(a_0 \circ x, a_1, \ldots, a_k)
\]

\[
\psi(a_0, a_1, \ldots, a_k) \circ x + \sum_{i=1}^{k} \psi(a_0, a_1, \ldots, a_{i-1}, [x, a_i], \ldots, a_k),
\]

for \(\psi \in C^{k+1}_{\text{sym}}(A, M), \; k \geq 0.\)
Proof. Since,
\[ C_{\text{rsym}}^1(A, M) = C^1(A, M), \]
an isomorphism of linear spaces takes place
\[ G : C_{\text{rsym}}^1(A, M) \otimes C^k_{\text{Lie}}(A, k) \rightarrow C_{\text{right}}^k(A, M), \quad k \geq 0, \]
\[ (G(f \otimes \psi))(a_0, a_1, \ldots, a_k) = f(a_0)\psi(a_1, \ldots, a_k). \]

In section ?? we have constructed an antisymmetric right-module structure on \( C_{\text{right}}^1(A, M) \). Lie module structure on \( C^k_{\text{Lie}}(A, k) \) over \( A^k_{\text{Lie}} \) is well known. So, for an antisymmetric \( A^- \) module structure
\[ C_{\text{rsym}}^1(A, M) \otimes C^k_{\text{Lie}}(A, k) = \{ f \otimes \phi : f \in C_{\text{right}}^1(A, M), \phi \in C^k_{\text{Lie}}(A, k) \} \]
we have
\[ ((f \otimes \phi) \circ x)(a_0 \otimes (a_1, \ldots, a_k)) = \]
\[ (f(a_0) \circ x) - f(a_0 \circ x) + a_0 \circ f(x)) \otimes (a_1, \ldots, a_k) \]
\[ + f(a_0) \otimes \sum_{i=1}^{k} \psi(a_1, \ldots, [x, a_i], \ldots, a_k) = \]

We see that
\[ G\{ (f \otimes \psi) \circ x \} = G\{ f \otimes \psi \} \circ x. \]
Therefore, \((\psi, x) \mapsto \psi \circ x \) gives us a right representation. •

2.7. Right-symmetric modules for \( W^{\text{rsym}}_n \). Let
\[ \Gamma_n = \{ \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbb{Z}, i = 1, \ldots, n \} \]
\[ \Gamma^+_n = \{ \alpha \in \Gamma_n : \alpha_i \geq 0, i = 1, \ldots, n \}. \]
and
\[ \Gamma_n(m) = \{ \alpha \in \Gamma^+_n : \alpha_i < p^{m_i}, i = 1, \ldots, n \}. \]
if \( p > 0, m = (m_1, \ldots, m_n) \).

Let
\[ U = K[[x^{\pm 1}, \ldots, x^{\pm 1}]] = \{ x^\alpha : \alpha \in \Gamma_n \}, \]
\[ U^+ = K[[x^1, \ldots, x^n]] = \{ x^\alpha : \alpha \in \Gamma^+_n \}, \]
if \( p = 0, \) and
\[ U = O_n(m) = \{ x^{(\alpha)} : \alpha \in \Gamma_n(m) \}, \]
if \( p > 0. \)
For \( p = 0, \) let \( A = W^{\text{rsym}}_n, \) if \( U = \mathcal{K}[[x^{\pm 1}, \ldots, x^{\pm 1}]], \) and \( A^+ = W^{\text{rsym}}_n, \) if \( U^+ = \mathcal{K}[[x_1, \ldots, x_n]]. \) Let \( A \) be \( W^{\text{rsym}}_n(m), \) if \( U = O_n(m), p > 0. \) Algebras \( A, A^+ \) are right-symmetric and \( U \) is associative commutative.

Notice that \( U \) has a structure of antisymmetric graded \( A^- \) module. The right action is given by \( u \circ a_{\partial_i} = a_{\partial_i}(u). \) The gradings are given by
\[ |x^\alpha| = \sum_i \alpha_i, \quad \alpha \in \Gamma_n \text{ (or } \Gamma_n(m) \text{) if } p > 0, \]
\[ U = \oplus_k U_k, \quad U_k = \{ u \in U : |u| = k \}, \]
\[ A = \oplus_k A_k, \quad A_k = \{ a_{\partial_i} : |a| = k + 1, i = 1, \ldots, n \}, \quad A_k \circ A_l \subseteq A_{k+l}, \]
\[ U \circ U_l \subseteq U_{k+l}, \quad U_k \circ A_l \subseteq U_{k+l}, k, l \in \mathbb{Z}. \]

Notice that \( A_0 \cong gR^{\text{rsym}}_n. \)
Let $A_0 = \oplus_k A_k$, and $A^+ = \oplus_k > 0 A^+_k$, if $p = 0$. Let $M$ be $A_0$—module, if $p > 0$, and $A^+_0$—module if $p = 0$. Define antisymmetric $A$—module structure on $U \otimes M_0$ by (see [?])

$$(u \otimes m) \circ a \partial_t = a \partial_t(u) \otimes m + \sum_{\beta \in T^n} u \partial^\beta(a) \otimes [m, x(\beta) \partial_t], \quad p > 0,$$

$$(u \otimes m) \circ a \partial_t = a \partial_t(u) \otimes m + \sum_{\beta \in T^n} (1/\beta) u \partial^\beta(a) \otimes [m, x^3 \partial_t], \quad p = 0.$$

3. Cohomologies of right-symmetric algebras

3.1. Pre-simplicial structures on $C^{+1}_{rsym}(A, M)$. For a right-symmetric algebra $A$ and its module $M$ we introduce a structure of pre-simplicial cochain complex on $C^{+1}_{rsym}(A, M) = \oplus_{k \geq 0} C^{k+1}_{rsym}(A, M)$, where

$$C^{k+1}_{rsym}(A, M) = \text{Hom}(A \otimes \wedge^k A, M), \quad k \geq 0.$$ 

Define linear operators $D_i : C^{k+1}_{rsym}(A, M) \rightarrow C^{k+1}_{rsym}(A, M), i = 1, 2, \ldots$, by the rules

$$D_i : C^k_{rsym}(A, M) \rightarrow C^{k+1}_{rsym}(A, M),$$

$$D_i \psi(a_0, a_1, \ldots, a_k) = a_0 \circ \psi(a_1, a_1, \ldots, \hat{a}_i, \ldots, a_k) - \psi(a_0 \circ a_i, a_1, \ldots, \hat{a}_i, \ldots, a_k)$$

$$+ \psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_k) \circ a_i + \sum_{i < j} \psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_j, [a_i, a_j], \ldots, a_k),$$

where $a$ means that the element $a$ is omitted.

In the next section we will endow $C^{+1}_{rsym}(A, M) = \oplus_{k \geq 0} C^k_{rsym}(A, M)$ by a structure of cochain complex, where

$$C^k_{rsym}(A, M) = 0, \quad k < 0,$$

$$C^0_{rsym}(A, M) = \{ m \in M : (ma)b = m(ab), \quad \forall a, b \in A \}.$$ 

**Theorem 3.1.** The set of endomorphisms $D_i, i = 1, 2, \ldots$ endows $C^{+1}_{rsym}(A, M) = \oplus_{k \geq 0} C^k_{rsym}(A, M)$ by a pre-simplicial structure:

$$D_j D_i = D_i D_{j-1}, \quad i < j.$$

In particular, $d_{rsym} = -\sum_i (-1)^i D_i$, is a coboundary operator on $C^{+1}_{rsym}(A, M)$:

$$d_{rsym}^2 = 0.$$

**Proof.** For $i < j, 1 < k$, we have

$$D_j D_i \psi(a_0, a_1, \ldots, a_k) = X_1 + X_2 + X_3 + X_4,$$

where

$$X_1 = a_0(D_i \psi(a_j, a_1, \ldots, \hat{a}_i, \ldots, a_k),$$

$$X_2 = -D_i \psi(a_0 \circ a_j, a_1, \ldots, \hat{a}_i, \ldots, a_k),$$

$$X_3 = D_i \psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_k)a_j,$$

$$X_4 = \sum_{j < s} D_i \psi(a_0, \ldots, \hat{a}_j, \ldots, a_s, [a_j, a_s], \ldots, a_k).$$

Direct calculations show that

$$X_1 =$$

$$a_0(a_j \psi(a_i, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_k))$$

$$- a_0 \psi(a_j \circ a_i, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_k)$$

$$+ a_0 \psi(a_j, a_1, \ldots, \hat{a}_i, \ldots, a_j, \ldots, a_k)a_j$$
\[ X_2 = \]
\[ - (a_0 \circ a_j) \psi(a_{\bar{i}}, a_{\bar{j}}, \ldots, a_k) \]
\[ + \psi((a_0 \circ a_j) \circ a_i, a_{\bar{1}}, \ldots, a_{\bar{k}}) \]
\[ - (\psi(a_0 \circ a_j, a_{\bar{i}}, \ldots, a_{\bar{k}})) a_i \]
\[ - \sum_{i < s, s \neq j} \psi(a_0 \circ a_j, a_{\bar{i}}, \ldots, a_{\bar{k}}, [a_{\bar{i}}, a_{\bar{s}}], \ldots, a_k), \]
\[ X_3 = \]
\[ + (a_0 \psi(a_{\bar{i}}, a_{\bar{j}}, \ldots, a_k)) a_j \]
\[ - (\psi(a_0 \circ a_i, a_{\bar{j}}, \ldots, a_k)) a_j \]
\[ + ((\psi(a_0, a_{\bar{i}}, \ldots, a_k)) a_i) a_j \]
\[ + \sum_{i < s} \psi(a_0, a_{\bar{i}}, \ldots, a_{\bar{j}}, \ldots, [a_{\bar{i}}, a_{\bar{s}}], \ldots, a_k) a_j \]
\[ + \sum_{j < s} a_0 \psi(a_{\bar{i}}, \ldots, a_{\bar{j}}, \ldots, a_{\bar{s}-1}, [a_{\bar{j}}, a_{\bar{s}}], \ldots, a_k) \]
\[ X_4 = \]
\[ - \sum_{j < s} \psi(a_0 \circ a_{\bar{i}}, a_{\bar{j}}, \ldots, a_{\bar{s}-1}, [a_{\bar{j}}, a_{\bar{s}}], \ldots, a_k) \]
\[ + \sum_{j < s} \psi(a_0, a_{\bar{i}}, \ldots, a_{\bar{j}}, \ldots, a_{\bar{s}-1}, [a_{\bar{i}}, a_{\bar{s}}], \ldots, a_k) \]
\[ + \sum_{j < s, j < s_1, s < s_1} \psi(a_0, a_{\bar{i}}, \ldots, a_{\bar{j}}, \ldots, a_{\bar{s}-1}, [a_{\bar{i}}, a_{\bar{s}}], \ldots, a_k) \]
\[ + \sum_{j < s, (i < s_1, s \neq s_1)} \psi(a_0, a_{\bar{i}}, \ldots, a_{\bar{j}}, \ldots, a_{\bar{s}-1}, [a_{\bar{i}}, a_{\bar{s}}], \ldots, a_k) \]

Analogously,
\[ D_3 D_3^{-1} \psi(a_0, a_{\bar{1}}, \ldots, a_k) = Y_1 + Y_2 + Y_3 + Y_4, \]
where
\[ Y_1 = a_0(D_3^{-1} \psi(a_{\bar{i}}, a_{\bar{j}}, \ldots, a_k)), \]
\[ Y_2 = -D_3^{-1}(a_0 \circ a_{\bar{i}}, a_{\bar{j}}, \ldots, a_k), \]
\[ Y_3 = D_3^{-1} \psi(a_0, a_{\bar{i}}, \ldots, a_{\bar{j}}, \ldots, a_k), \]
\[ Y_4 = \sum_{i < s} D_3^{-1} \psi(a_0, a_{\bar{i}}, \ldots, a_{\bar{i}}, \ldots, a_{\bar{s}-1}, [a_{\bar{i}}, a_{\bar{s}}], \ldots, a_k). \]

We have
\[ Y_1 = \]
\[ a_0(a_0 \psi(a_{\bar{i}}, a_{\bar{j}}, \ldots, a_k)) \]
\[ - a_0 \psi(a_{\bar{i}} \circ a_{\bar{j}}, a_{\bar{i}}, \ldots, a_k) \]
\[ + a_0 \psi(a_{\bar{i}}, a_{\bar{j}}, \ldots, a_k) \]
\[ + \sum_{j < s} a_0 \psi(a_{\bar{i}}, a_{\bar{j}}, \ldots, a_{\bar{k}}, a_{\bar{s}-1}, [a_{\bar{j}}, a_{\bar{s}}], \ldots, a_k), \]
\[ Y_2 = - (a_0 \circ a_i) \psi(a_j, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_k) \]
\[ + \psi((a_0 \circ a_i) \circ a_j, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_k) \]
\[ - \psi(a_0 \circ a_i, a_1, \ldots, \hat{a}_1, \ldots, \hat{a}_j, \ldots, a_k) a_j \]
\[ - \sum_{j<s} \psi(a_0 \circ a_i, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{s-1}, [a_j, a_s], \ldots, a_k), \]
\[ \psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{s-1}, [a_j, a_s], \ldots, a_k) a_i. \]

Present \( Y_4 \) as a sum
\[ Y_4 = Y_{4,1} + Y_{4,2} + Y_{4,3}, \]
where
\[ Y_{4,1} = \sum_{i<s<j} D_{j-1, i} \psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_{s-1}, [a_i, a_s], \ldots, a_k), \]
\[ Y_{4,2} = D_{j-1} \psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_{j-1}, [a_i, a_j], \ldots, a_k), \]
\[ Y_{4,3} = \sum_{j<s} D_{j-1} \psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_{s-1}, [a_i, a_s], \ldots, a_k). \]

These elements can be expressed in the following ways
\[ Y_{4,1} = \]
\[ \sum_{i<s<j} a_0 \psi(a_j, a_1, \ldots, \hat{a}_i, \ldots, a_{s-1}, [a_i, a_s], \ldots, a_j, \ldots, a_k) \]
\[ - \sum_{i<s<j} \psi(a_0 \circ a_j, a_1, \ldots, \hat{a}_i, \ldots, a_{s-1}, [a_i, a_s], \ldots, a_j, \ldots, a_k) \]
\[ + \sum_{i<s<j} (\psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_{s-1}, [a_i, a_s], \ldots, a_j, \ldots, a_k)) a_j \]
\[ + \sum_{i<s_1<j<s} \psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_{s_1-1}, [a_i, a_{s_1}], \ldots, \hat{a}_j, \ldots, a_{s-1}, [a_j, a_s], \ldots, a_k), \]
\[ Y_{4,2} = \]
\[ + a_0 \psi([a_i, a_j], a_1, \ldots, \hat{a}_i, \ldots, a_j, \ldots, a_k) \]
\[ - \psi(a_0 \circ [a_i, a_j], a_1, \ldots, \hat{a}_i, \ldots, a_j, \ldots, a_k) \]
\[ + (\psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_j, \ldots, a_k)) [a_i, a_j] \]
\[ + \sum_{j<s} \psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_{s-1}, [a_i, a_j], \ldots, a_k), \]
\[ Y_{4,3} = \]
\[ \sum_{j<s} a_0 \psi(a_j, a_1, \ldots, \hat{a}_i, \ldots, a_{s-1}, [a_i, a_s], \ldots, a_k)) \]
Using right-symmetric identity for the expressions underlined, in similar ways we obtain that

\[ D_j D_i = D_i D_{j-1}, \quad i < j. \]

3.2. Cohomologies of right-symmetric algebras and Cartan’s formulas. In the previous section we proved that \( C_{rsym}^{n+1}(A, M) = \bigoplus_{k=0}^{\infty} C_{rsym}^k(A, M) \) is a cochain complex under coboundary operator \( d_{rsym} \), such that

\[
d_{rsym}(a_0, a_1, \ldots, a_k) = -\sum_{i=1}^{k} (-1)^i a_0 \circ (\psi(a_i, a_1, \ldots, a_k)) + \sum_{i=1}^{k} (-1)^i \psi(a_0 \circ a_i, a_1, \ldots, a_k)
\]

\[
\sum_{1 \leq i < j \leq k} (-1)^{i+1} \psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, [a_i, a_j], \ldots, a_k)
\]

\[
- \sum_{i=1}^{k} (-1)^i (\psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_k)) \circ a_i,
\]

\[ \psi \in C^k(A, M), \quad 0 < k. \]

For \( m \in M \), define \( d_{rsym} \in C_{rsym}^1(A, M) \),

\[ d_{rsym}(m) = a \circ m - m \circ a. \]

It is easy to see that

\[
d_{rsym}^2(m, b) = a \circ d_{rsym}(m(b)) - d_{rsym}^2(m(a) \circ b) =
\]

\[
a \circ (b \circ m) - a \circ (m \circ b) - (a \circ b) \circ m + m \circ (a \circ b) + (a \circ m) \circ b - (m \circ a) \circ b =
\]

\[
(a, b, m) - (a, m, b) + m \circ (a \circ b) - (m \circ a) \circ b.
\]

Thus, according to right-symmetric identity,

\[ d_{rsym}^2(m, b) = m \circ (a \circ b) - (m \circ a) \circ b. \]

From this fact two conclusions follow. Firstly, taking a subspace of left associative invariants

\[ M^{a, ass} = \{ m \in M : (m, a, b) = 0, \forall a, b \in A \} \]

as a 0-cochain subspace \( C_{rsym}^0(A, M) \), we obtain cochain complex

\[ C_{rsym}^*(A, M) = \bigoplus_{k \geq 0} C_{rsym}^k(A, M) \]

under coboundary operator \( d_{rsym} \). The second conclusion will be discussed in the next section in the construction of standard 2-cocycles.

Let

\[ Z^k_{rsym}(A, M) = \bigoplus_k Z_{rsym}^k(A, M), \]
be spaces of right-symmetric cocycles,
\[ B^k_{rsym}(A, M) = \oplus_k C^k_{rsym}(A, M), \]
\[ B^k_{rsym}(A, M) = \{ d_{rsym} \psi : \psi \in C_{rsym}^{k-1}(A, M) \}, \]
be spaces of right-symmetric coboundaries, and
\[ H^k_{rsym}(A, M) = C^k_{rsym}(A, M)/B^k_{rsym}(A, M), \]
be right-symmetry cohomology spaces.

**Definitions.** For any \( x \in A \) the interior product endomorphism \( i(x) \) of \( C^*_{rsym}(A, M) \) is defined by
\[ i(x) : C^{k+1}_{rsym}(A, M) \to C^k_{rsym}(A, M), \]
\[ i(x) \psi(a_0, \ldots, a_{k-1}) = \psi(a_0, x, a_1, \ldots, a_{k-1}), \quad k > 0, \]
\[ i(x) \psi = 0, \quad \psi \in C^1_{rsym}(A, M). \]
Let \( \rho_{lie} : A^{lie} \to C^{*+1}_{rsym}(A, M) \) be a representation of Lie algebra \( A^{lie} \) corresponding to anti-symmetric representation
\[ \rho_{rsym} : A \to C^{*+1}_{rsym}(A, M), \quad \rho_{rsym}(x) \psi = 0, \quad \psi \rho_{rsym}(x) = \psi \circ x, \]
constructed in proposition ??,
\[ \rho_{lie}(x) \psi(a_0, a_1, \ldots, a_k) = \]
\[ = -a_0 \circ \psi(x, a_1, \ldots, a_k) + \psi(a_0 \circ x, a_1, \ldots, a_k) \]
\[ -\psi(a_0, a_1, \ldots, a_k) \circ x - \sum_{i=1}^k \psi(a_0, a_1, \ldots, a_{i-1}, [x, a_i], \ldots, a_k). \]
Recall that
\[ \rho_{lie}(x) \psi = -\psi \rho_{lie}(x) = -\psi \circ x = -\psi \rho_{rsym}(x). \]

**Proposition 3.2.** (Cartan’s formulas) Consider \( C^{*+1}_{rsym}(A, M) := \oplus_k C^{k+1}_{rsym}(A, M) \) as a \( A^{lie} \)-module. For linear operators on \( C^{*+1}_{rsym}(A, M) \) the following relations take place
\[ (i) \quad i(x)D_l = D_{l-1}i(x), \quad l > 1, \]
\[ (ii) \quad i(x)D_1 = -\rho_{lie}(x), \]
\[ (iii) \quad \rho_{lie}[x, y] = [\rho_{lie}(x), \rho_{lie}(y)], \]
\[ (iv) \quad [i(x), \rho_{lie}(y)] = -i([x, y]), \]
\[ (v) \quad d_{rsym} i(x) + i(x) d_{rsym} = -\rho_{lie}(x). \]

**Proof.** (i)
\[ i(x)D_l \psi(a_0, \ldots, a_{k-1}) = \]
\[ = \]
\[ = \]
\[ = \]
\[ + \sum_{l-1 < j} \psi(a_0, x, a_1, \ldots, a_{l-1}, a_j, \ldots, a_{k-1}, a_{l-1}, a_{l-1}, \ldots, a_{k-1}) = \]
\[ = a_0 \circ i(x) \psi(a_{l-1}, a_{l-1}, \ldots, a_{k-1}). \]
\[-i(x)\psi(a_0 \circ a_{l-1}, a_1, \ldots, a_{l-1}, \ldots, a_{k-1})
+ i(x)\psi(a_0, a_1, \ldots, a_{l-1}, \ldots, a_{k-1}) \circ a_{l-1}
+ \sum_{l-1 < j} i(x)\psi(a_0, a_1, \ldots, a_{l-1}, \ldots, a_{j-1}, [a_i, a_j], \ldots, a_{k-1}) =
\]
\[D_{l-1}i(x)\psi(a_0, \ldots, a_{k-1}).\]

(ii)
\[i(x)D_1\psi(a_0, \ldots, a_{k-1}) =
\]
\[D_1\psi(a_0, x, a_1, \ldots, a_{k-1}) =
\]
\[a_0 \circ \psi(x, a_1, \ldots, a_{k-1}) - \psi(a_0 \circ x, a_1, \ldots, a_{k-1})
+ \psi(a_0, a_1, \ldots, a_{k-1}) \circ x + \sum_{0 < j} \psi(a_0, a_1, \ldots, a_{j-1}, [a_i, a_j], \ldots, a_{k-1}) =
\]
\[(\psi \rho_{sym}(x))(a_0, \ldots, a_{k-1}).\]

(iii) Proposition ??.

(iv) We obtain
\[-\{(i(x)\psi(y))\psi\}(a_0, a_1, \ldots, a_{k-1}) =
\]
\[(\psi \circ y)(a_0, x, a_1, \ldots, a_{k-1}) =
\]
\[a_0 \circ (\psi(y, x, a_1, \ldots, a_{k-1}) - \psi(a_0 \circ y, x, a_1, \ldots, a_{k-1})
+ \psi(a_0, x, a_1, \ldots, a_{k-1}) \circ y
+ \psi(a_0, [x, y], a_1, \ldots, a_{k-1}) + \sum_{i=1}^{k-1} \psi(a_0, x, a_1, \ldots, a_{i-1}, [a_i, y], \ldots, a_{k-1}),\]

and
\[-\{(\rho_{tie}(y) i(x)) \psi\}(a_0, a_1, \ldots, a_{k-1}) =
\]
\[{(i(x)\psi \circ y)}(a_0, a_1, \ldots, a_{k-1}) =
\]
\[a_0 \circ \{(i(x)\psi)(y, a_1, \ldots, a_{k-1}) - (i(x)\psi)(a_0 \circ y, a_1, \ldots, a_{k-1})
+ \{(i(x)\psi)(a_0, a_1, \ldots, a_{k-1}) \circ y
+ \sum_{i=1}^{k-1} \{(i(x)\psi)(a_0, a_1, \ldots, a_{i-1}, [a_i, y], \ldots, a_{k-1}) =
\]
\[a_0 \circ (\psi(y, x, a_1, \ldots, a_{k-1}) - \psi(a_0 \circ y, x, a_1, \ldots, a_{k-1})
+ \psi(a_0, x, a_1, \ldots, a_{k-1}) \circ y
+ \sum_{i=1}^{k-1} \psi(a_0, x, a_1, \ldots, a_{i-1}, [a_i, y], \ldots, a_{k-1}).\]

Thus
\[\{-i(x)\rho_{tie}(y) + \rho_{tie}(y) i(x)\}\psi(a_0, a_1, \ldots, a_{k-1}) =
\]
\[(i[x, y] \psi)(a_0, a_1, \ldots, a_{k-1}).\]
(v) According (i) and (ii),
\[ i(x)d_{rsym} = i(x)D_1 + \sum_{l>1}(-1)^{l+1}i(x)D_l = \]
\[ -\rho_{iie}(x) + \sum_{l>1}(-1)^{l+1}D_{l-1}i(x) = \]
\[ -\rho_{iie}(x) - \sum_{l>1}(-1)^{l+1}D_l i(x). \]
Thus,
\[ i(x)d_{rsym} + d_{rsym}i(x) = -\rho_{iie}(x). \]

3.3. **Long exact cohomological sequence.** The following theorem follows from standard homological results.

**Theorem 3.3.** Let \( A \) be a right-symmetric algebra and 
\[ 0 \to M \to T \to S \to 0 \]
be a short exact sequence of right-symmetric \( A \)-modules. Then an exact sequence of right-symmetric cohomology spaces take place
\[
\begin{align*}
0 & \to Z_{rsym}^0(A, M) \to Z_{rsym}^0(A, T) \to Z_{rsym}^0(A, S) \xrightarrow{\delta} \\
& \to Z_{rsym}^1(A, M) \to Z_{rsym}^1(A, T) \to Z_{rsym}^1(A, S) \xrightarrow{\delta} H_{rsym}^2(A, M) \to \cdots \\
& \xrightarrow{\delta} H_{rsym}^k(A, M) \to H_{rsym}^k(A, T) \to H_{rsym}^k(A, S) \xrightarrow{\delta} H_{rsym}^{k+1}(A, M) \to \cdots
\end{align*}
\]
Here \( \delta \) is a connected homomorphism:
\[
\delta[\psi] = [d_{rsym}\phi], \quad [\psi] \in H_{rsym}^k(A, S), k > 1,
\]
\[
\delta \psi_1 = [d_{rsym}\phi_1], \quad \psi_1 \in Z_{rsym}^i(A, S), i = 0, 1,
\]
where \( \phi \in Z_{rsym}^0(A, T) \) is a representative of the cohomological class \( [\psi] \) and \( \phi_1 \in Z_{rsym}^i(A, T) \) moves to \( \psi_1 \) under a natural homomorphism \( Z_{rsym}^i(A, T) \to Z_{rsym}^i(A, S), i = 0, 1. \)

Define a homomorphism \( \nabla : S^{lass} \to Z_{rsym}^2(A, M) \), as a composition
\[
\nabla : C_{rsym}^0(A, S) \xrightarrow{d_{rsym}} B_{rsym}^1(A, S) \xrightarrow{\delta} Z_{rsym}^2(A, M), \quad s \mapsto d_{rsym}s \mapsto \delta(d_{rsym}s).
\]
Then
\[
\nabla(m)(a, b) = m \circ (a \circ b) - (m \circ a) \circ b.
\]
In particular, there exist homomorphisms
\[
\delta : S^{lass} \to Z_{rsym}^1(A, M), \quad \delta(s) : a \mapsto [a, s],
\]
\[
\delta : S^{lass} \to Z_{rsym}^1(A, M), \quad -\delta(s) : a \mapsto a \circ s.
\]
3.4. Connections between right-symmetric cohomologies and Chevalley-Eilenberg cohomologies. Recall that for any $A^\text{lie}$—module $Q$ a standard representation $\rho : A^\text{lie} \to C^\text{lie}_i(A, Q)$ is given by

$$\rho(x)\psi(a_1, \ldots, a_k) = [x, \psi(a_1, \ldots, a_k)] - \sum_{i=1}^{k} \psi(a_1, \ldots, a_{i-1}, [x, a_i], \ldots, a_k),$$

where $(x, q) \mapsto [x, q]$ is representation corresponding to the Lie module $Q$ and $\psi \in C^k_{\text{lie}}(A, Q)$.

**Theorem 3.4.** Let $A$ be a right-symmetric algebra and $M$ be an $A$—module. An operator

$$F : C^k_{\text{lie}}(A, C^1(A, M)) \to C^{k+1}_{\text{sym}}(A, M), \ k > 0,$$

defined by the rule

$$F\psi(a_0, a_1, \ldots, a_k) = -\psi(a_1, \ldots, a_{k-1})(a_0)$$

induces an isomorphism of $A^\text{lie}$—modules. Moreover, $F$ induces an isomorphism of cochain complexes $C^{\text{sym}}_{k+1}(A, M)$ and $C^k_{\text{lie}}(A, C^1(A, M))$. In particular,

$$H^{k+1}_{\text{sym}}(A, M) \cong H^k_{\text{lie}}(A, C^1(A, M)), \ k > 0.$$  

The following sequence is exact

$$0 \to Z^0_{\text{sym}}(A, M) \to C^0_{\text{sym}}(A, M) \to H^0_{\text{lie}}(A, C^1(A, M)) \to H^1_{\text{sym}}(A, M) \to 0.$$  

**Proof.** Prove that for any $x \in A$, $k > 0$, the following diagram is commutative

$$\begin{array}{ccc}
C^k_{\text{lie}}(A, C^1(A, M)) & \xrightarrow{\partial(x)} & C^{k+1}_{\text{lie}}(A, C^1(A, M)) \\
\downarrow F & & \downarrow F \\
C^{k+1}_{\text{sym}}(A, C^1(A, M)) & \xrightarrow{\rho_{\text{sym}}(x)} & C^{k+1}_{\text{sym}}(A, M)
\end{array}$$

For $\psi \in C^k_{\text{lie}}(A, C^1(A, M))$ we have

$$F(\rho(x)\psi)(a_0, a_1, \ldots, a_{k+1}) = -\rho(x)\psi(a_1, \ldots, a_{k+1})(a_0) =$$

$$-\{x \circ (\psi(a_1, \ldots, a_k))(a_0) + \sum_{i=1}^{k} \psi(a_1, \ldots, a_{i-1}, [x, a_i], \ldots, a_k)(a_0) =$$

$$+\{(\psi(a_1, \ldots, a_k)) \circ x\}(a_0) + \sum_{i=1}^{k} \psi(a_1, \ldots, a_{i-1}, [x, a_i], \ldots, a_k)(a_0) =$$

$$-a_0 \circ \psi(x, a_1, \ldots, a_k) + \psi(a_0 \circ x, a_1, \ldots, a_k) - \psi(a_0, a_1, \ldots, a_k) \circ x =$$

$$-\sum_{i=1}^{k} \psi(a_0, a_1, \ldots, [x, a_i], \ldots, a_k) =$$

$$-(\rho_{\text{sym}}(x))(a_0, a_1, \ldots, a_k).$$

Thus, $F : C^k_{\text{lie}}(A, C^1(A, M)) \to C^{k+1}_{\text{sym}}(A, M)$ is a homomorphism of $A^\text{lie}$—modules. It is evident that, $F$ has no kernel and $F$ is epimorphism.

Now, prove that for any $k \geq 0$ the following diagram is commutative

$$\begin{array}{ccc}
C^k_{\text{lie}}(A, C^1(A, M)) & \xrightarrow{d_{\text{lie}}} & C^{k+1}_{\text{lie}}(A, C^1(A, M)) \\
\downarrow F & & \downarrow F \\
C^{k+1}_{\text{sym}}(A, C^1(A, M)) & \xrightarrow{d_{\text{sym}}} & C^{k+2}_{\text{sym}}(A, M)
\end{array}$$
For \( \psi \in C^k_{\text{tie}}(A, C^1(A, M)) \),

\[
F(d_{\text{tie}} \psi)(a_0, a_1, \ldots, a_{k+1}) = d_{\text{tie}} \psi(a_1, \ldots, a_{k+1})(a_0) = \\
\sum_{1 \leq i < j \leq k+1} (-1)^i \psi(a_1, \ldots, a_i, \ldots, [a_i, a_j], \ldots, a_{k+1})(a_0) \\
- \sum_{i=1}^{k+1} (-1)^i [a_i, \psi(a_1, \ldots, a_{i-1}, a_{i+1})](a_0) = \\
\sum_{1 \leq i < j \leq k+1} (-1)^{i+j} F\psi(a_0, a_1, \ldots, a_i, \ldots, [a_i, a_j], \ldots, a_{k+1}) \\
+ \sum_{i=1}^{k+1} (-1)^i d_{\text{rsym}}(\psi(a_1, \ldots, a_{i-1}, a_{i+1}, a_i))(a_{k+1})(a_0, a_i) = \\
\sum_{1 \leq i < j \leq k+1} (-1)^{i+j} F\psi(a_0, a_1, \ldots, a_i, \ldots, [a_i, a_j], \ldots, a_{k+1}) \\
- \sum_{i=1}^{k+1} (-1)^i (F\psi(a_0, a_1, \ldots, a_{i-1}, a_{i+1}, a_i))(a_0, a_i) = \\
\sum_{1 \leq i < j \leq k+1} (-1)^{i+j} F\psi(a_0, a_1, \ldots, a_i, \ldots, [a_i, a_j], \ldots, a_{k+1}) \\
- \sum_{i=1}^{k+1} (-1)^i (F\psi(a_0, a_1, \ldots, a_{i-1}, a_{i+1}, a_i))(a_0, a_i).
\]

Thus we obtain the equivalence of cochain complexes \( \oplus_{k>0} C^k_{\text{tie}}(A, C^1(A, M)) \) and \( \oplus_{k>1} C^k_{\text{rsym}}(A, M) \).

In particular, an isomorphism \((??)\) takes place.

Since, \( H^1_{\text{rsym}}(A, M) = Z^1_{\text{rsym}}(A, M)/B^1_{\text{rsym}}(A, M) \), and

\[
Z^1_{\text{rsym}}(A, M) = \{ \psi \in C^1(A, M) : d_{\text{rsym}} \psi = 0 \} = Z^0_{\text{tie}}(A, C^1(A, M)),
\]

\[
B^1_{\text{rsym}}(A, M) = \{ d_{\text{sym}} m : m \in M, (m, a, b) = 0, \forall a, b \in A \} = M^{\text{ass}}/M^{\text{ass}} \cap M^{\text{inv}},
\]

we have the exactness of \((??)\).

**Definition.** Let \( f : C^*_{\text{rsym}}(A, M) \to C^*_{\text{tie}}(A, M) \) be a linear operator, such that

\[
f : C^k_{\text{rsym}}(A, M) \to C^k_{\text{tie}}(A, M),
\]

\[
f\psi(a_1, \ldots, a_k) = \sum_{i=1}^k (-1)^{i+k+1} \psi(a_i, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k).
\]

Introduce subspaces

\[
C^r_{\text{rsym}}(A, M) = \oplus_k C^k_{\text{rsym}}(A, M),
\]
$C^k_{rsym}(A, M) = \{ \psi \in C^{k+2}_{rsym}(A, M) : f\psi = 0 \}, \ k \geq 0$.

**Theorem 3.5.** Let $A$ be a right-symmetric algebra and $M$ be an $A$-module. Then the operator $f : C^{k}_{rsym}(A, M) \rightarrow C^{k+1}_{lie}(A, M)$ is the homomorphism of cochain complexes and the following cohomological sequence:

$$0 \rightarrow Z^1_{rsym}(A, M) \rightarrow Z^1_{lie}(A, M) \xrightarrow{\delta} \tilde{H}^0_{rsym}(A, M) \rightarrow H^1_{rsym}(A, M) \xrightarrow{\delta} \tilde{H}^1_{rsym}(A, M) \rightarrow \cdots$$

is exact. A connected homomorphism

$$\delta : \tilde{H}^k_{lie}(A, M) \rightarrow H^{k-1}_{rsym}(A, M)$$

is induced by homomorphism

$$\delta : Z^k_{lie}(A, M) \rightarrow Z^{k-1}_{rsym}(A, M), \ \psi \mapsto d_{rsym}\psi.$$

**Proof.** We will check that for $k > 0$ the following diagram is commutative

$$
\begin{array}{ccc}
C^k_{rsym}(A, M) & \xrightarrow{d_{rsym}} & C^{k+1}_{rsym}(A, M) \\
\downarrow f & & \downarrow f \\
C^k_{lie}(A, M) & \xrightarrow{d_{lie}} & C^{k+1}_{lie}(A, M)
\end{array}
$$

For $\psi \in C^k_{rsym}(A, M)$, we have

$$fd_{rsym}\psi(a_1, \ldots, a_{k+1}) = \sum_s (-1)^{s+k} d_{rsym}\psi(a_s, a_1, \ldots, a_{s-1}, a_s, \ldots, a_{k+1}) =$$

$$\sum_{i<s} (-1)^{i+s+k+1} a_s \circ (\psi(a_s, a_1, \ldots, a_i, \ldots, a_s, a_{k})) + \sum_{s<i} (-1)^{i+s+k} a_s \circ (\psi(a_i, a_1, \ldots, a_k))$$

$$- \sum_{i<s} (-1)^{i+s+k+1} \psi(a_s \circ a_i, a_1, \ldots, a_i, \ldots, a_s, a_{k+1})$$

$$- \sum_{s<i} (-1)^{i+s+k} \psi(a_s \circ a_i, a_1, \ldots, \hat{a}_i, \ldots, a_s, \ldots, a_{k+1})$$

$$+ \sum_{i<j<s} (-1)^{s+i+k} \psi(a_s, a_1, \ldots, \hat{a}_i, \ldots, a_{j-1}, a_i, a_j, \ldots, a_{k+1})$$

$$+ \sum_{i<s<j} (-1)^{s+i+k} \psi(a_s, a_1, \ldots, a_i, \ldots, a_s, \ldots, a_{j-1}, a_i, a_j, \ldots, a_{k+1})$$

$$+ \sum_{s<i<j} (-1)^{s+i+k+1} \psi(a_s, a_1, \ldots, \hat{a}_i, \ldots, a_{j-1}, a_i, a_j, \ldots, a_{k+1})$$

and

$$d_{lie}f\psi(a_1, \ldots, a_{k+1}).$$
Thus, according to right-symmetric identity,
\[ f \circ d_{\text{rsym}} \psi = d_{\text{lie}} f \psi, \quad \forall \psi \in C^k_{\text{rsym}}(A, M), \quad \forall k > 0. \]

So, a short exact sequence of cochain complexes takes place
\[ 0 \rightarrow \oplus_{k>0} C^k_{\text{rsym}}(A, M) \rightarrow \oplus_{k>0} C^k_{\text{lie}}(A, M) \rightarrow \oplus_{k>0} C^k_{\text{lie}}(A, M) \rightarrow 0. \]

In particular, a long cohomological sequence
\[ H^0_{\text{rsym}}(A, M) \rightarrow H^2_{\text{rsym}}(A, M) \rightarrow H^2_{\text{lie}}(A, M) \rightarrow \cdots \]
\[ \rightarrow H^k_{\text{rsym}}(A, M) \rightarrow H^k_{\text{rsym}}(A, M) \rightarrow H^k_{\text{lie}}(A, M) \rightarrow \cdots \]
is exact. The exactness of the beginning part
\[ 0 \rightarrow Z^1_{\text{rsym}}(A, M) \rightarrow Z^1_{\text{lie}}(A, M) \rightarrow H^0_{\text{rsym}}(A, M) \rightarrow H^2_{\text{rsym}}(A, M) \]
we check directly. It is clear that \( d_{\text{lie}} \psi = 0 \), if \( d_{\text{rsym}} \psi = 0 \), \( \psi \in C^1(A, M) \). So, the natural homomorphism \( Z^1_{\text{rsym}}(A, M) \rightarrow Z^1_{\text{lie}}(A, M) \) is a monomorphism. Let \( \delta \phi, \phi \in Z^1_{\text{lie}}(A, M) \), gives us a trivial class in \( H^0(A, M) = Z^0_{\text{rsym}}(A, M) \). Then \( \phi \in Z^0_{\text{rsym}}(A, M) \), since \( \delta \phi = d_{\text{rsym}} \phi \).

Suppose that \( \sigma \in Z^0_{\text{rsym}}(A, M) \) is a coboundary in \( Z^2_{\text{rsym}}(A, M) \), say \( \sigma = d_{\text{rsym}} \omega \), for some \( \omega \in C^1_{\text{rsym}}(A, M) \). Then \( d_{\text{rsym}} \omega(a, b) = \sigma(a, b) = \sigma(b, a) = d_{\text{rsym}} \omega(b, a) \), for any \( a, b \in A \). This means that \( d_{\text{lie}} \omega = 0 \).

The theorem is proved completely. •

3.5. Cup product in right-symmetric cohomologies.

**Theorem 3.6.** Assume that a cup product of \( A \)-modules \( \cup : M \times N \rightarrow S \) is given. Then a bilinear map
\[ C^*_{\text{rsym}}(A, M) \times C^*_{\text{lie}}(A, N) \rightarrow C^{*+1}_{\text{rsym}}(A, S), \quad (\psi, \phi) \mapsto \psi \cup \phi, \]
defined by
\[ C^k_{\text{rsym}}(A, M) \times C^l_{\text{lie}}(A, N) \rightarrow C^{k+l+1}_{\text{rsym}}(A, S), \quad (\psi, \phi) \mapsto \psi \cup \phi, \]
\[ \psi \cup \phi(a_0, a_1, \ldots, a_{k+l}) = \]
\[
\sum_{\sigma \in \text{Sym}_{k+l}, \atop \sigma(1) < \cdots < \sigma(k), \atop \sigma(k+1) < \cdots < \sigma(k+l)} \text{sgn } \sigma \psi(a_0, a_{\sigma(1)}, \ldots, a_{\sigma(k)}) \cup \phi(a_{\sigma(k+1)}, \ldots, a_{\sigma(k+l)}).
\]

is also a cup product:
\[
(\alpha \cup \beta)\rho_{\text{rsym}}(x) = \alpha \rho_{\text{rsym}}(x) \cup \beta + \alpha \cup \beta \rho_{\text{lie}}(x), \quad \forall \alpha \in C_{r\text{sym}}^{k+1}(A, M), \ \forall \beta \in C_{\text{lie}}^{k}(A, N).
\]
Moreover,
\[
d_{\text{rsym}}(\psi \cup \phi) = d_{\text{rsym}}\psi \cup \phi - (-1)^k\phi \cup d_{\text{lie}}\phi,
\]
for any \( \psi \in C_{r\text{sym}}^{k+1}(A, M), \ \phi \in C_{\text{lie}}^{l}(A, N), \ k, l \geq 0.

**Proof.** By theorem (??),
\[
F(\eta \cup \phi) = F(\eta) \cup \phi,
\]
\[
F(\alpha \rho_{\text{lie}}(x)) = (F\alpha)\rho_{\text{rsym}}(x),
\]
for any \( \eta \in C_{\text{lie}}^{k}(A, C_{\text{Lie}}^{1}(A, M)), \ \alpha \in C_{r\text{sym}}^{k+1}(A, M), \ \phi \in C_{\text{lie}}^{l}(A, N), \ x \in A.

Prolongate the cup product \( M \times N \to S, (m, n) \mapsto m \cup n, \) of \( A-\) modules to a cup product of \( A_{\text{lie}}-\)modules
\[
C_{\text{lie}}^{1}(A, M)_{\text{lie}} \times N_{\text{lie}} \to C_{\text{lie}}^{1}(A, S)_{\text{lie}},
\]
\[
(f, n) \mapsto f \cup n, \ (f \cup n)(a) = f(a) \cup n.
\]
Check the correctness of this definition:
\[
((f \cup n) \circ (a))(b) =
\]
\[
d_{\text{rsym}}(f \cup n)(b, a) =
\]
\[
b \circ ((f \cup n)(a)) - (f \cup n)(b \circ a) + ((f \cup n)(b)) \circ a =
\]
\[
b \circ (f(a) \cup n) - f(b \circ a) \cup n + (f(b) \cup n) \circ a =
\]
\[
(b \circ f(a)) \cup n - f(b \circ a) \cup n + (f(b) \circ a) \cup n + f(b) \cup [n, a] =
\]
\[
(d_{\text{rsym}}f(b, a)) \cup n + f(b) \cup [n, a] =
\]
\[
(f \circ a)(b) \cup n + f(b) \cup [n, a] =
\]
\[
((f \circ a) \cup n + (f \cup [n, a]))(b).
\]
Thus, we have a cup product of Chevalley-Eilenberg cochain complexes (??)
\[
C_{\text{lie}}^{k}(A, C_{\text{lie}}^{1}(A, M)) \times C_{\text{lie}}^{l}(A, N) \to C_{r\text{sym}}^{k+l}(A, C_{\text{lie}}^{1}(A, S))
\]
\[
\{(\eta \cup \phi)(a_1, \ldots, a_{k+l})\} \{a_0\} =
\]
\[
\sum_{\sigma \in \text{Sym}_{k+l}, \atop \sigma(1) < \cdots < \sigma(k), \atop \sigma(k+1) < \cdots < \sigma(k+l)} \text{sgn } \sigma \{\eta(a_{\sigma(1)}, \ldots, a_{\sigma(k)}) \cup \phi(a_{\sigma(k+1)}, \ldots, a_{\sigma(k+l)})\}(a_0).
\]
We see that cup products for Chevalley-Eilenberg complexes and right-symmetric complexes are compatible. Namely,
\[
(F\eta) \cup \phi = F(\eta \cup \phi),
\]
for any \( \eta \in C_{\text{lie}}^{k}(A, C_{\text{lie}}^{1}(A, M)), \ \phi \in C_{\text{lie}}^{l}(A, N) \) (definition of isomorphism \( F : C_{\text{lie}}^{k}(A, C_{\text{lie}}^{1}(A, M)) \to C_{r\text{sym}}^{k+l}(A, M) \) see (??)). Since, (??)
\[
d_{\text{lie}}(\eta \cup \phi) = d_{\text{lie}}\eta \cup \phi + (-1)^k\eta \cup d_{\text{lie}}\phi,
\]
accordingly (??),
\[
d_{\text{rsym}}((F\eta) \cup \phi) = d_{\text{rsym}}F(\eta \cup \phi) =
\]
\[
Fd_{\text{lie}}(\eta \cup \phi) = F(d_{\text{lie}}\eta \cup \phi + (-1)^k\eta \cup d_{\text{lie}}\phi) =
\]
26

By theorem ?? for any $\psi \in C^{k+1}_{rsym}(A, M), k \geq 0$, there exists $\eta \in C^k(A, C^1(A, M))$, such that $\psi = F\eta$. Hence, (??) is true. •

Corollary 3.7. The cup product

$C^{*+1}_{rsym}(A, M) \times C^*_\text{lie}(A, N) \rightarrow C^{*+1}_{rsym}(A, S)$, $(\psi, \phi) \mapsto \psi \cup \phi$,

induces a cup product of cohomology spaces

$H^{k+1}_{rsym}(A, M) \times H^l_{\text{lie}}(A, N) \rightarrow H^{k+1}_{rsym}(A, S)$, $([\psi], [\phi]) \mapsto [\psi \cup \phi]$, $k > 0, l \geq 0$.

$Z^1_{rsym}(A, M) \times H^1_{\text{lie}}(A, N) \rightarrow H^{1+1}_{rsym}(A, S)$, $(\psi, [\phi]) \mapsto (\psi \cup \phi)$, $l \geq 0$.

Proof.

$Z^{k+1}_{rsym}(A, M) \cup Z^l_{\text{lie}}(A, N) \subseteq Z^{k+1}_{rsym}(A, S), k, l \geq 0$,

$B^{k+1}_{rsym}(A, M) \cup Z^l_{\text{lie}}(A, N) \subseteq B^{k+1}_{rsym}(A, S), k > 0, l \geq 0$,

$Z^{k+1}_{rsym}(A, N) \cup B^l_{\text{lie}}(A, N) \subseteq B^{k+1}_{rsym}(A, S), k, l \geq 0$.

Notice that for any module $M$ of right-symmetric algebra $A$ and trivial $A$-module $K$ there exists a natural cup product

$M \times K \rightarrow M$, $(m, \lambda) \mapsto m\lambda$.

So, we have a pairing of cohomology spaces

$H^*_\text{sym}(A, M) \times H^*_\text{lie}(A, K) \rightarrow H^*_\text{sym}(A, M)$.

In particular, $H^*_\text{sym}(A, M)$ has a natural structure of module over $H^*_\text{lie}(A, K)$. As it turned out in some cases $H^*_\text{sym}(A, M)$ is a free $H^*_\text{lie}(A, K)$-module. In section ?? we will see that this is the case, if $A = gl_{rsym}$.

Denote by $M$ antisymmetric $A$-module obtained from $M$ by $a \leftrightarrow -a$, $a \in A$. One can construct another cup product

$\mathcal{K} \times M \rightarrow M$, $\lambda \cup m = \lambda m$.

We use this cup product in consideration of right-symmetric cohomologies for $A = W^n_{rsym}$, section ??.

4. Deformations of right-symmetric algebras.

4.1. Deformation equations. We will follow the Gerstenhaber theory of deformations of algebras [?]. Let $A$ be a right-symmetric algebra over a field $K$ of any characteristic $p$. Let $K((t))$ be a fraction field for formal power series algebra $K[[x]]$. Extend the main field $K$ until $K((t))$ and construct on the vector space $A \otimes K((t))$ a new right-symmetric multiplication

$\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \ldots$,

where

$\mu_i \in \mathcal{C}^i_{rsym}(A, M), i = 0, 1, 2, \ldots$, and $\mu_0(a, b) = a \circ b$.

The right-symmetric condition for $\mu_t$ in terms of $\mu_k$ can be regarded as the following deformation equations

$\mu_1 \in \mathcal{Z}^1_{rsym}(A, A)$, (DFR.1)

$\sum_{l=1}^{k-1} \mu_l \ast \mu_{k-1} = -d_{rsym}\mu_k$, (DFR.k)

$k = 2, 3, \ldots$,

where

$(\psi \ast \phi)(a, b, c) = \psi(a, \phi(b, c)) - \psi(\phi(a, b), c) - \psi(a, \phi(c, b)) + \psi(\phi(a, c), b)$,
Right-symmetric deformations $\mu_t$, and $\nu_t$ are said to be equivalent, if there exists a map

$$g_t = g_0 + t g_1 + t g_2 + \cdots , \quad g_k \in C^1_{rsym}(A, A), \quad k = 0, 1, 2, \ldots ,$$

with an identity map $g_0$, such that

$$g_t^{-1}(\mu_t(a, b)) = \nu_t(a, b), \quad \forall a, b \in A.$$

In particular, for equivalent deformations $\mu_t, \nu_t$, should be

$$\nu_1 = \mu_1 + d_{rsym} g_1.$$

In other words the first deformation terms, so called local deformations will define equivalent 2-right-symmetry cohomology classes $[\mu_1] = [\nu_1]$.

Converse, suppose that there is given a 2-cocycle of right-symmetric algebra with coefficients in the regular module, $\psi \in Z^2_{rsym}(A, A)$, with a cohomology class $[\psi] \in H^2_{rsym}(A, A)$. One can take $\mu_1 := \psi$, and try to construct $\mu_k$ that will satisfy deformation equations. Evidently, (DFR.1) is true. We will say that local deformation $\mu_1 = \psi$ can be prolonged to a global deformation until $k$-th term, if there exist $\mu_2, \ldots , \mu_k$, such that equations (DFR.k) are true. If this is the case for any $k > 0$, we will say that local deformation $\mu_1$ can be prolonged until global deformation $\mu_t$ or, equivalently, that $\mu_t$ is global deformation or prolongation of $\mu_1$. Set,

$$\text{Obs}_k(\psi) = \sum_{l=1}^{k-1} \mu_l \star \mu_{k-l}. $$

Notice that the definition of $\text{Obs}_k(\psi)$ depends not only from $\psi$ but, also from the first $k - 1$ terms of deformation.

4.2. Third cohomologies as obstruction.

**Proposition 4.1.** Suppose that a local deformation $\mu_1 = \psi$ can be prolonged to a global deformation until $(k - 1)$-th term. Then, $\text{Obs}_k(\psi) \in Z^3_{rsym}(A, A)$ and the prolongation of $\mu_1$ until $k$-th term is possible, if and only if $[\text{Obs}_k(\psi)] = 0$.

**Proof.** For $\alpha \in C^{k+1}(A, A), \beta \in C^{l+1}(A, A)$ define multiplications $\alpha \star \beta \in C^{k+l+2}(A, A)$ by

$$\alpha \star \beta(a_1, \ldots, a_{k+l+1}) = \sum_{s=1}^{k+1} (-1)^{(s+1)l} \alpha(a_1, \ldots, a_{s-1}, \beta(a_{s+1}, \ldots, a_{s+l}), a_{s+l+1}, \ldots, a_{k+l+1}).$$

Then,

$$\psi \star \phi(a, b, c) = \psi \star \phi(a, c, b) - \psi \star \phi(a, b, c), \quad \psi, \phi \in C^2(A, A),$$

and

$$d_{rsym} \text{Obs}_k(\psi)(a_0, a_1, a_2, a_3) = \sum_{l+s=k, l>0, s>0} \sum_{\sigma \in \text{Sym}_3} \text{sgn} \sigma d_{\text{ass}}(\alpha \star \beta)(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$$

where $d_{\text{ass}}$ means Hochshild coboundary operator as in associative algebras. By [?], §7, Th.3],

$$d_{\text{ass}} \alpha \star \beta = \alpha \star d_{\text{ass}} \beta - d_{\text{ass}} \alpha \star \beta - \alpha \star \beta - \alpha \star \beta, \quad \alpha, \beta \in C^2(A, A).$$

Notice that

$$\sum_{l+s=k, l>0, s>0} \mu_l \star \mu_s = \sum_{l+s=k, l>0, s>0} \mu_l - \mu_s = \sum_{l+s=k, l>0, s>0} \mu_l - \mu_s = 0.$$

Hence, according to conditions (DFR.1), $l < k,$
$$d_{\text{rsym}} \text{Obs}_k(\psi)(a_0, a_1, a_2, a_3) =$$

$$\sum_{l+s=k, l>0, s>0} \sum_{\sigma \in \text{Sym}_3} \sgn \sigma d_{\text{ass}}(\mu_l \sim \mu_s)(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) =$$

$$\sum_{l+s=k, l>0, s>0} \sum_{\sigma \in \text{Sym}_3} \sgn \sigma \mu_l * d_{\text{ass}} \mu_s(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$$

$$- \sgn \sigma d_{\text{ass}} \mu_l * \mu_s(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) =$$

$$\sum_{l+s=k, l>0, s>0} \sum_{\sigma \in \text{Sym}_3} \sgn \sigma \mu_l(d_{\text{ass}} \mu_s(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}))$$

where

$$S_1 = \sum_{l+s=k, l>0, s>0} \sum_{l_1+s_1=\delta_{l,s_1}, s_1+s_2>0} \{ -\mu_l(\mu_{s_1} * \mu_{s_2}(a_0, a_1, a_2), a_3) - \mu_l(\mu_{s_1} * \mu_{s_2}(a_0, a_1, a_3), a_2) + \mu_l(\mu_{s_1} * \mu_{s_2}(a_0, a_2, a_3), a_1) - \mu_l(\mu_{s_1} * \mu_{s_2}(a_0, a_3, a_1), a_2) + \mu_l(\mu_{s_1} * \mu_{s_2}(a_0, a_3, a_2), a_1) \}$$

$$S_2 = \sum_{l+s=k, l>0, s>0} \sum_{l_1+s_1=\delta_{l,s_1}, l_1+l_2>0} \{ -\mu_l * \mu_{l_2}(\mu_{s_1} * \mu_{s_2}(a_0, a_1, a_2), a_3) - \mu_l * \mu_{l_2}(\mu_{s_1} * \mu_{s_2}(a_0, a_1, a_3), a_2) + \mu_l * \mu_{l_2}(\mu_{s_1} * \mu_{s_2}(a_0, a_2, a_3), a_1) - \mu_l * \mu_{l_2}(\mu_{s_1} * \mu_{s_2}(a_0, a_3, a_2), a_1) + \mu_l * \mu_{l_2}(\mu_{s_1} * \mu_{s_2}(a_0, a_3, a_3), a_1) \}.$$
\[-\mu_1(a_0, \mu_{s_1} * \mu_{s_2}(a_1, a_3, a_2)) + \mu_1(a_0, \mu_{s_1} * \mu_{s_2}(a_2, a_3, a_1)) - \mu_1(a_0, \mu_{s_1} * \mu_{s_2}(a_3, a_2, a_1))\] 

\[= \sum_{t+s_1+s_2=k,l>0,s_1>0,s_2>0} \sum_{\sigma \in \text{Sym}_3} \text{sgn } \sigma \mu_1 * (\mu_{s_1} * \mu_{s_2})(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}),\]

and

\[S_2 = \sum_{t+s=k,l>0,s>0} \sum_{l_1+l_2=l_1>0,l_2>0} \{ \mu_1 * \mu_{s_2}(\mu_s(a_0, a_1), a_2, a_3) - \mu_1 * \mu_{s_2}(\mu_s(a_0, a_2), a_1, a_3) + \mu_1 * \mu_{s_2}(\mu_s(a_0, a_3), a_1, a_2) \]

\[\rightarrow \mu_1 * \mu_{s_2}(a_0, a_1, a_2) - \mu_1 * \mu_{s_2}(a_0, a_2, a_3) + \mu_1 * \mu_{s_2}(a_0, a_3, a_2) + \mu_1 * \mu_{s_2}(a_1, a_2, a_3) - \mu_1 * \mu_{s_2}(a_1, a_3, a_2) + \mu_1 * \mu_{s_2}(a_2, a_3, a_1) \]

\[= \sum_{l_1+l_2+s=k,l_1>0,l_2>0,s>0} \text{sgn } \sigma \mu_1 * \mu_{s_2}(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}),\]

Let \(\alpha, \beta, \gamma \in C^2(A, A)\). Then,

\[\{\alpha * (\beta * \gamma) - (\alpha * \beta) * \gamma\}(a, b, c, d) = \]

\[\alpha(\beta * \gamma(a, b, c), d) + \alpha(a, \beta * \gamma(b, c, d)) - \alpha * \beta(\gamma(a, b, c), d) - \alpha * \beta(a, \gamma(b, c), d) = \]

\[\alpha(\beta(a, \gamma(b, c), d)) - \alpha(\beta(a, \gamma(b, c), d)) + \alpha(a, \beta(b, \gamma(c, d))) - \alpha(a, \beta(b, \gamma(c, d))) \]

\[-\alpha(\beta(a, \gamma(b, c), d)) + \alpha(a, \beta(b, \gamma(c, d))) \]

\[= -\alpha(\beta(a, b, \gamma(c, d))) + \alpha(a, \beta(b, \gamma(c, d))) \]

So, for any \(\alpha, \beta, \gamma \in C^2(A, A)\),

\[\alpha * (\beta * \gamma + \gamma * \beta) - (\alpha * \beta) * \gamma - (\alpha * \gamma) * \beta = 0\]

For these reasons,

\[S_1 = \sum_{s_1+s_2+s_3=k,s_1>0,s_2>0,s_3>0} \sum_{\sigma \in \text{Sym}_3} \text{sgn } \sigma \mu_{s_1} * (\mu_{s_2} * \mu_{s_3})(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = \]
So, we prove that \( d_{rsym} \text{Ob}_k(\psi) = 0 \), if \( d_{rsym} \text{Ob}_l(\psi) = 0 \), for any \( 0 < l < k \). 

**Corollary 4.2.** If \( H^2_{rsym}(A, A) = 0 \), then any local deformation can be prolonged.

### 4.3. Steenrod squares

Let \( \text{char } k = p > 0 \). In this subsection we recall Gerstenhaber’s construction \[?\] of the homomorphism

\[
Sq : Z^1_{rsym}(A, A) \rightarrow Z^2_{rsym}(A, A), \quad D \mapsto SqD
\]

regarding the right-symmetric algebras. For any derivation \( D \in Z^1_{rsym}(A, A) \) its \( p \)-th power \( D^p \) is also a derivation, \( D^p \in Z^1_{rsym}(A, A) \). The proof is based on the following property of binomial coefficients: an integer \( \binom{p}{i} \) can be divided into \( p \), if \( 0 < a < p \).

\[
D^p(a \circ b) - D^p(a) \circ b - a \circ D^p(b) = \sum_{i=1}^{p-1} \binom{p}{i} D^i(a) \circ D^{p-i}(b) \equiv 0(\text{mod } p).
\]

In particular, we can consider integers \( \binom{p}{i}/p \), \( 0 < i < p \) by modulus \( p \) and introduce 2-cocycle \( SqD \) with coefficients in the regular module

\[
Sq D(a, b) = \sum_{i=1}^{p-1} D^i(a) \circ D^{p-i}(b)/i!(p - i)!
\]

This cocycle is called the Steenrod Square of derivation \( D \) and can be interpreted as an obstruction to prolongation of derivation to automorphism.

### 5. Calculations

#### 5.1. Standard 2-cocycles of right-symmetric algebras

In this subsection we will give the second interpretation of the identity \( d^2_{rsym}m = 0 \), \( m \in M \), mentioned in section \[?\].

**Proposition 5.1.** i) Let \( \tilde{M} \) be a module over right-symmetric algebra \( A \) and \( M \) is its submodule. Suppose that for \( \tilde{m} \in M \),

\[
\tilde{m}, a, b) \in M, \quad \forall a, b \in A.
\]

Then 2-cochain \( \psi_{\tilde{m}} \in C^2(A, M) \) defined by

\[
\psi_{\tilde{m}}(a, b) = \tilde{m} \circ (a \circ b) - (\tilde{m} \circ a) \circ b,
\]

is symmetric 2-cocycle, \( \psi_{\tilde{m}} \in \tilde{Z}^2(A, M) \).

If \( \tilde{m} \circ a \in M, \forall a \in A, \) then \( [\psi_{\tilde{m}}] = [\phi_{\tilde{m}}] \), where

\[
\phi_{\tilde{m}}(a, b) = a \circ (\tilde{m} \circ b).
\]

Notice that in the denotations of section \[?\], \( \psi_{\tilde{m}} = \nabla(\tilde{m}) \).

**Proof.**

\[
\psi_{\tilde{m}}(a, b) = (m, a, b) = (m, b, a) = \psi_{\tilde{m}}(b, a),
\]

\[
\begin{align*}
d_{rsym} \psi_{\tilde{m}}(a, b, c) = & \quad a \circ (\tilde{m}, b, c) - a \circ (\tilde{m}, c, b) - (\tilde{m}, a \circ b, c) + (\tilde{m}, a, b \circ c) + (\tilde{m}, a, [b, c]) - (\tilde{m}, a, b) \circ c + (\tilde{m}, a, c) \circ b = \\
= & -(\tilde{m}, a \circ b, c) + (\tilde{m}, a, b \circ c) - (\tilde{m}, a, b) \circ c \\
+ & (\tilde{m}, a \circ c, b) - (\tilde{m}, a, c \circ b) + (\tilde{m}, a, c) \circ b = \\
= & \tilde{m} \circ (a, b, c) - (\tilde{m} \circ a, b, c) \\
- & \tilde{m} \circ (a, c, b) + (\tilde{m} \circ a, c, b) = 0.
\end{align*}
\]
If $\bar{m} \circ a \in M$, then we can introduce a linear map $\omega : A \to M$, $a \mapsto \bar{m} \circ a$. We obtain
\[
\begin{align*}
\psi_{\bar{m}}(a, b) + d_{rsym}(a, b) &= \\
\psi_{\bar{m}}(a, b) - \omega(a \circ b) + \omega(a) \circ b + a \circ \omega(b) &= \\
\psi_{\bar{m}}(a, b) - \bar{m} \circ (a \circ b) + (\bar{m} \circ a) \circ b + a \circ (\bar{m} \circ b) &= a \circ (\bar{m} \circ b).
\end{align*}
\]
In other words, $\phi_{\bar{m}} = \psi_{\bar{m}} + d_{rsym}\omega$.

5.2. Semi-center and derivations of $W_n^{rsym}$. Suppose that $A$ is an algebra with multiplications $(a, b) \mapsto a \circ b$, and $(a, b) \mapsto a \ast b$, such that the following conditions hold
\[
\begin{align*}
a \circ (b \circ c) - (a \circ b) \circ c - a \circ (c \circ b) + (a \circ c) \circ b &= 0, \\
a \ast (b \ast c) - b \ast (a \ast c) &= 0, \\
a \circ (b \ast c) - b \ast (a \circ c) &= 0, \\
(a \ast b - b \ast a - a \circ b + b \circ a) \ast c &= 0,
\end{align*}
\]
In particular, $A$ is a right-symmetric algebra.

Let $Z(A)$ be the left center of $A$, $Q(A)$ is the left units space and $N(A) = Z(A) \oplus Q(A)$ is the left semi-center.

**Theorem 5.2.** For $A = W_n^{rsym}$, if $p = 0$, or $A = W_n(m)$, if $p > 2$, $Z(A)$ is semi-center and derivations of $W_n^{rsym}$. Suppose that $A$ is an algebra with multipli-
cations $(a, b) \mapsto a \circ b$, and $(a, b) \mapsto a \ast b$, such that the following conditions hold
\[
\begin{align*}
a \circ (b \circ c) - (a \circ b) \circ c - a \circ (c \circ b) + (a \circ c) \circ b &= 0, \\
a \ast (b \ast c) - b \ast (a \ast c) &= 0, \\
a \circ (b \ast c) - b \ast (a \circ c) &= 0, \\
(a \ast b - b \ast a - a \circ b + b \circ a) \ast c &= 0,
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\[
\begin{align*}
a \circ (b \circ c) - (a \circ b) \circ c - a \circ (c \circ b) + (a \circ c) \circ b &= 0, \\
a \ast (b \ast c) - b \ast (a \ast c) &= 0, \\
a \circ (b \ast c) - b \ast (a \circ c) &= 0, \\
(a \ast b - b \ast a - a \circ b + b \circ a) \ast c &= 0,
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\[
\begin{align*}
a \circ (b \circ c) - (a \circ b) \circ c - a \circ (c \circ b) + (a \circ c) \circ b &= 0, \\
a \ast (b \ast c) - b \ast (a \ast c) &= 0, \\
a \circ (b \ast c) - b \ast (a \circ c) &= 0, \\
(a \ast b - b \ast a - a \circ b + b \circ a) \ast c &= 0,
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\[
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a \circ (b \circ c) - (a \circ b) \circ c - a \circ (c \circ b) + (a \circ c) \circ b &= 0, \\
a \ast (b \ast c) - b \ast (a \ast c) &= 0, \\
a \circ (b \ast c) - b \ast (a \circ c) &= 0, \\
(a \ast b - b \ast a - a \circ b + b \circ a) \ast c &= 0,
\end{align*}
\]
In particular, $A$ is a right-symmetric algebra.

Let $Z(A)$ be the left center of $A$, $Q(A)$ is the left units space and $N(A) = Z(A) \oplus Q(A)$ is the left semi-center.
5.3. Pairing of $W^\text{sym}_n$—modules and cocycle constructions.

**Theorem 5.3.** Let $A = W_n, p = 0$, or $A = W_n(m), p > 3$. The space $H^2_{\text{sym}}(A, A), p = 0$, has a basis consisting of cocycle classes of four types $\psi_{s,l,r}^1, \psi_{l,r}^2, \psi_{s,l}^3, \psi_l^4$, $s, l, r = 1, \ldots, n$, such that
\[
\begin{align*}
\psi_{s,l,r}^1(u \partial_i, v \partial_j) &= \delta_{j,r}x^{-1}_r(\delta_{l,s}uv\partial_i - x_s\partial_i(u)v\partial_i), \\
\psi_{l,r}^2(u \partial_i, v \partial_j) &= \delta_{j,r}x^{-1}_r\partial_i(u)v\partial_i, \\
\psi_{s,l}^3(u \partial_i, v \partial_j) &= (\delta_{l,s}u\partial_j(v)\partial_i - x_s\partial_i(u)\partial_j(v)\partial_i), \\
\psi_l^4(u \partial_i, v \partial_j) &= \partial_i(u)\partial_j(v)\partial_i.
\end{align*}
\]

In the case of $p > 3$, the space $H^2_{\text{sym}}(A, A)$ has a basis with cohomological classes of the following cocycles of five types.
\[
\begin{align*}
\psi_{s,l,r}^1(u \partial_i, v \partial_j) &= \delta_{j,r}x^{-1}_r(\delta_{l,s}uv\partial_i - x_s\partial_i(u)v\partial_i), \\
\psi_{l,k_i,r}^2(u \partial_i, v \partial_j) &= \delta_{j,r}x^{-1}_r(\delta_{l,s}uv\partial_i - x_s\partial_i(u)v\partial_i), \\
\psi_{s,l}^3(u \partial_i, v \partial_j) &= (\delta_{l,s}u\partial_j(v)\partial_i - x_s\partial_i(u)\partial_j(v)\partial_i), \\
\psi_{l,k_i}^4(u \partial_i, v \partial_j) &= \partial_i(u)\partial_j(v)\partial_i, \\
\psi_{l,k_i}^5 &= Sq\partial_i^{k_i},
\end{align*}
\]
where $s, l, r = 1, \ldots, n$, $0 \leq k_i < m_i$.

The proof is based on the following observations.

Suppose that $A$—module $M$ preserve the action of $N_l(A) :$ $z \circ m = 0, \forall z \in Z_l(A), \ e \circ m = m, \forall e \in Q_l(A),$ for any $m \in M$. Define an operator
\[
C^k_{\text{sym}}(A, M) \rightarrow C^{k+1}_{\text{sym}}(A, M),
\]
and an operator
\[
T : C^k_{\text{sym}}(A, A) \rightarrow C^{k+1}_{\text{sym}}(A, A),
\]
\[
T\psi(a_0, a_1, \ldots, a_k) = \sum_{i=1}^k (-1)^{i+k}a_i \ast \psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_k).
\]
Then
\[
Td_{\text{sym}} = d_{\text{sym}}T,
\]
and for any $a \in N_l(A), \psi \in Z^{k+1}_{\text{sym}}(A, A),$
\[
i_0(a)\psi \in Z^k_{\text{lie}}(A, A_{\text{anti}}).
\]
Define a pairing of regular $A$—module $A$ and antisymmetric $A$—module $U :$
\[
A \times U \rightarrow A, \ u\partial_i \cup v = uv\partial_i.
\]
Notice that,
\[
A_{\text{anti}} \cong U \otimes Z_l(A).
\]
In particular, we have pairing
\[
A \times A_{\text{anti}} \rightarrow A, \ u\partial_i \cup v\partial_j = uv\partial_i.
\]
Therefore we have imbedding
\[
Z^1_{\text{sym}}(A, A) \times H^k_{\text{lie}}(A, U) \rightarrow H^{k+1}_{\text{sym}}(A, A).
Notice that the four types of cocycles mentioned above can be obtained from $Z^1_{rsym}(A, A)$ (see section ??) and $H^1_{lie}(A, U)$, by pairing $\psi \cup \phi$, $\psi \in Z^1_{rsym}(A, A)$, $\phi \in H^1_{lie}(A, U)$. Recall that $H^1_{lie}(A, U)$ can be generated by the classes of cocycles $u\partial_i \mapsto ux^{-1}_{\partial_i}$, and $u\partial_i \mapsto \partial_i(u)$.

Another interpretation of cocycles of types 1 and 2 can be given in terms of standard cocycles (see section ??). For simplicity consider only the case of $p = 0$. We have

$$\psi^1_{s,t,r} = d\omega_{s,t,r}, \text{for } \omega_{s,t,r}(u\partial_i) = \ln x_r(x_s\partial_i, u\partial_i),$$

$$\psi^2_{t,r} = d\omega_{t,r}, \text{for } \omega_{t,r}(u\partial_i) = \ln x_r(\partial_i(u)\partial_i),$$

and

$$\nabla(x_s \ln x_r(\partial_i(u\partial_i, v\partial_j)) = \partial_i \partial_j(x_s \ln x_r)uv,$$

$$\nabla(\ln x_r(\partial_i(u\partial_i, v\partial_j)) = -\delta_i \delta_j, uv\partial_i.$$

Therefore,

$$[\psi^1_{s,t,r}] = [\nabla(x_s \ln x_r(\partial_i))],$$

$$[\psi^2_{t,r}] = [\nabla(\ln x_r(\partial_i))],$$

because of

$$\psi^1_{s,t,r} = \nabla(x_s \ln x_r(\partial_i) - d^{rsym}\omega_{s,t,r}^1, \text{for } \omega_{s,t,r}^1 \in C^1_{rsym}(W_n, W_n), \omega_{s,t,r}^1(u\partial_i) = \delta_i x_r^{-1} x_s u\partial_i,$$

$$\psi^2_{t,r} = \nabla(\ln x_r(\partial_i) - d^{rsym}\omega_{t,r}^2, \text{for } \omega_{t,r}^2 \in C^1_{rsym}(W_n, W_n), \omega_{t,r}^2(u\partial_i) = \delta_i x_r^{-1} u\partial_i.$$

**Theorem 5.4.** Let $A = W^1_{sym}$, if $p = 0$, and $A = W_1(m)$, if $p > 3$. Then $H^2_{rsym}(A, A)$ has dimension $4$, if $p = 0$, and cohomological classes of the following cocycles generate a basic

$$\psi^1(u\partial_i, v\partial) = x^{-1} uv\partial,$$

$$\psi^2(u\partial, v\partial) = x^{-1} \partial(u)v\partial,$$

$$\psi^3(u\partial, v\partial) = (u - x\partial(u))\partial(v)\partial,$$

$$\psi^4(u\partial, v\partial) = \partial(u)\partial(v)\partial.$$

(Recall that, $\partial = \partial_z$, for $n = 1$.)

For $A = W_1(m), p > 3$, the group $H^2_{rsym}(A, A)$ is $(3m + 2)$–dimensional and the classes of the following cocycles generate its basis

$$\psi^1(u\partial, v\partial) = x^{m-1} uv\partial,$$

$$\psi^2_k(u\partial, v\partial) = x^{m-1} \partial^k(u)v\partial, \quad 0 \leq k < m,$$

$$\psi^3(u\partial, v\partial) = (u - x\partial(u))\partial(v)\partial,$$

$$\psi^4_k(u\partial, v\partial) = \partial^k(u)\partial(v)\partial, \quad 0 \leq k < m.$$

$$Sq D : (a, b) \mapsto \sum_{i=1}^{p-1} D^i(a) \circ D^{p-i}(b)/(i!(p - i)!), \quad D = \partial^k, \quad 0 \leq k < m.$$

Cocycles of types 1 and 2 are also Novikov cocycles. Local deformation $\psi = \sum_{i=1}^4 t_i \psi^i, p = 0$, can be prolonged if and only if $t_1 t_3 = 0, t_2 t_4 = 0$. 
5.4. **Right-symmetric central extensions of Novikov algebras.** Let \( A \) be a Novikov algebra, \( R \in \text{Der}_0 A := \{D \in \text{Der} A : a \circ R(b) = b \circ R(a), \forall a, b \in A\}, \) and \( \pi : A \rightarrow K, \) a linear map, such that

\[
\pi(R(a)) = 0, \quad \forall a \in A.
\]

Define \( \psi \in \Omega^2_{rsym}(A, K), \) by

\[
\vartheta(a, b) = \pi(a \circ R(b)).
\]

**Lemma 5.5.** \( \vartheta \in Z^2_{rsym}(A, K). \)

**Proof.** Since, \( a \circ R(b) = b \circ R(a), \) then

\[
\vartheta(a, b) = \vartheta(b, a).
\]

We have

\[
d_{rsym}(a, b, c) = \\
\pi(- (a \circ b) \circ R(c) + (a \circ c) \circ R(b) + a \circ R[b, c]) = \\
-(a \circ R(c)) \circ b - a \circ [b, R(c)] \\
+(a \circ R(b)) \circ c + a[c, R(b)] + a \circ R[b, c] = \\
\pi(- (a \circ R(c)) \circ b + (a \circ R(b)) \circ c) = \\
\pi(-R((a \circ c) - b) + (R(a) \circ c) \circ b + (a \circ c) \circ R(b) \\
+a \circ R(b)) \circ c = \\
\pi((R(a) \circ c) \circ b + (R(a) \circ b) \circ c + (a \circ R(b)) \circ c) = \\
\pi((R(a) \circ c) \circ b + b \circ (R(a) \circ c) + b \circ (a \circ R(c)) + (a \circ R(b)) \circ c) = \\
\pi((R(a) \circ c) \circ b + R(a) \circ (b \circ c) + b \circ (a \circ R(c)) + (a \circ R(b)) \circ c) = \\
\pi((R(a) \circ c) \circ b + R(a \circ (b \circ c)) - a \circ (R(b) \circ c) + (a \circ R(b)) \circ c) = \\
\pi((-a \circ (c \circ R(b)) + (a \circ c) \circ R(b)) = \\
\pi(-a \circ (c \circ R(b)) - (a \circ R(c)) \circ b) = \\
\pi(-R(a \circ (c \circ b)) + R(a) \circ (c \circ b) + a \circ (R(c) \circ b) - (a \circ R(c)) \circ b) = \\
\pi(R(a) \circ (c \circ b) + a \circ (b \circ R(c)) - (a \circ b) \circ R(c)) = \\
\pi(c \circ (R(a) \circ b) + a \circ (c \circ R(b)) - (a \circ b) \circ R(c)) = \\
\pi(c \circ (R(a) \circ b) + c \circ (a \circ R(b)) - (a \circ b) \circ R(c)) = \\
\pi(c \circ R(a \circ b) - (a \circ b) \circ R(c)) = 0.
So, \( \vartheta \in Z^2_{\operatorname{rsym}}(A,K) \).

In particular, algebras

\[
A = W^\operatorname{rsym}_1 = \{e_i : e_i \circ e_j = (i+1)e_{i+j}, \ i, j \in \mathbb{Z}\}, \ p = 0,
\]

\[
A = W^\operatorname{rsym}_1(m) = \{e_i : e_i \circ e_j = \binom{i + j + 1}{i} e_{i+j}, -1 \leq i, j \leq p^m - 1\}, \ p > 0,
\]

have right-symmetric 2-cocycles with coefficients in the trivial \( A \)-module \( K \). Prove that, the cohomological class of the cocycle \( \vartheta \) in both cases is not trivial. If \( \vartheta = d_{\operatorname{rsym}}\omega, \ \omega \in C^1(N,K) \), then

\[
\vartheta(e_i, e_j) = -(j + 1)\delta_{i,j,-1}, \ \text{if} \ p = 0,
\]

\[
\vartheta(e_i, e_j) = -(-1)^{i}j\delta_{i,j,p^m,-1}, \ \text{if} \ p > 0,
\]

\[
d_{\operatorname{rsym}}\omega(e_i, e_j) = -(i + 1)\omega(e_{i+j}), \ \text{if} \ p = 0,
\]

\[
d_{\operatorname{rsym}}\omega(e_i, e_j) = -\binom{i + j + 1}{i}\omega(e_{i+j}), \ \text{if} \ p > 0.
\]

In the case of \( p = 0 \) we have a contradiction:

\[
-2 = \vartheta(e_{-2}, e_1) = d_{\operatorname{rsym}}(e_{-2}, e_1) = -\omega(e_{-1}),
\]

\[
-6 = \vartheta(e_{-3}, e_2) = d_{\operatorname{rsym}}(e_{-3}, e_2) = -2\omega(e_{-1}).
\]

Since, \( d_{\operatorname{rsym}}\omega(e_i, e_{p^m-i-1}) = 0 \), we also obtain a contradiction, if \( p > 0 \).

**Theorem 5.6.** Let \( A = W^\operatorname{rsym}_1 \), if \( p = 0 \) and \( A = W^\operatorname{sym}_1(m) \), if \( p > 0 \). Then the second right-symmetric cohomology space \( H^2_{\operatorname{rsym}}(A,K) \) has dimension 1 and generates by a class of cocycles

\[
\vartheta(e_i, e_j) = -(j + 1)\delta_{i,j,-1}, \ \text{if} \ p = 0,
\]

\[
\vartheta(e_i, e_j) = -(-1)^{i}j\delta_{i,j,p^m,-1}, \ \text{if} \ p > 0.
\]

If \( A \) is considered as a Novikov algebra, then any Novikov central extension is split: \( H^2_{\operatorname{Nov}}(W^1_1, K) = 0 \).

Recall that Novikov cohomologies are defined in [?]. Central extensions of Cartan Type Lie algebras are described in [?].

**Proof.** For \( u \in U = K[[x_{\pm 1}]] \) let \( \pi(u) \) be its coefficient at \( x^{-1} \). Then \( \pi(\partial(u)) = 0, \ \forall u \in U \).

Recall that \( e_i = x^{i+1}, \ i \in \mathbb{Z} \), and the multiplication in \( A \) is given by \( a \circ b = \partial(a)b, \ a, b \in U \).

Prove that an isomorphism of \( A_{\operatorname{lie}} \)-modules takes place

\[
C^1(A,K) \cong U_1.
\]

A bilinear map

\[
(,): U_0 \times U_1 \to K, \ (u,v) \mapsto \pi(u \cdot v),
\]

is compatible with the action of \( A_{\operatorname{lie}} \):

\[
((a)_0(u), v) + (u, (a)_1(v)) = \pi(-u \circ a \cdot v - u \cdot (v \circ a) - u \cdot (a \circ v)) = \pi(-\partial(a \cdot (u \cdot v))) = 0,
\]

for all \( a \in A, u \in U_0, v \in U_1 \). So, we have a pairing of \( A_{\operatorname{lie}} \)-modules \((,): U_0 \times U_1 \to K \). This pairing is nondegenerate. Thus, a dual \( A_{\operatorname{lie}} \)-module to \( U_0 \) is \( U_1 \). Since,

\[
(f \circ a)(b) = d_{\operatorname{sym}}f(b, a) = -f(b \circ a), \ f \in C^1(N,K), \ a,b \in N,
\]

we see that the \( A_{\operatorname{lie}} \)-module \( C^1(A,K) \) is isomorphic to the dual of \( U_0 \). This ends the proving of (??).

By theorem ??

\[
H^2_{\operatorname{rsym}}(A,K) \cong H^1_{\operatorname{lie}}(A, C^1(A,K)) \cong H^1_{\operatorname{lie}}(W^1_1, U_1).
\]
By the results of Gelfand and Fuchs [?] the space $H_1^{1e}(W_1, U_1)$ is 1-dimensional and generates by a class of cocycle $a \mapsto \partial^p(a)$. An analogous statement is also true in the case of $p > 0$ [?]. The corresponding right-symmetric cocycle is the cocycle $\psi$.

If $\psi: A \times A \to K$ is a cocycle for central extension in the category of Novikov algebras, then

$$\psi(a, b \circ c) - \psi(b, a \circ c) = 0, \forall a, b, c \in A.$$ 

The algebra $A = W_1^{nov}$ has an element $e_0$ that has the property $e_0 \circ c = c$, for any $c \in A$. Take $a := e_0$. We have

$$\psi(e_0, b \circ c) = \psi(b, e_0 \circ c) = \psi(b, c), \forall b, c \in A.$$ 

Therefore, for $\omega \in C^1_{nov}(A, K) = C^1(A, K)$, such that $\omega(a) = -\psi(e_0, a)$, we have

$$\psi(b, c) = -\omega(b \circ c) = d_{nov}\omega(b, c).$$

Recall that, $d_{nov}\phi = d_{right}\phi$, for any $\phi \in C^1(A, K)$. So, any 2-cocycle of $W_1^{nov}$ (in sense of Novikov) with coefficients in the trivial module, is a coboundary. $
$

### 5.5. Cohomologies of $W_n^{rsym}$ in an antisymmetric module.

Recall that a multiplication in $W_n^{rsym}$ is given by $a \partial_i \circ b \partial_j = b \partial_j(a) \partial_i$, where $a, b \in U = K[[x^\pm_1, \ldots, x^\pm_n]]$. Endow $U$ by a structure of antisymmetric $W_n^{rsym}$-module: $u \circ a \partial_i = a \partial_i(u)$. Let $\Omega^n_n = \{u dx_1 \wedge \cdots \wedge dx_n : u \in U\}$ be an antisymmetric $W_n^{rsym}$-module of $n$-dimensional differential forms:

$$(u dx_1 \wedge \cdots \wedge dx_n) \circ a \partial_i = a \partial_i(u) dx_1 \wedge \cdots \wedge dx_n.$$ 

Let $M$ be $W_n^{rsym}$-module. Construct a cup product of $W_n^{rsym}$-modules

$$(u \circ a \partial_i) \cup (v dx_1 \wedge \cdots \wedge dx_n \otimes m) + u \cup [v dx_1 \wedge \cdots \wedge dx_n \otimes m, a \partial_i] =$$

$$(a \partial_i(u) \cup (v dx_1 \wedge \cdots \wedge dx_n \otimes m) + u \cup (a \partial_i(au) dx_1 \wedge \cdots \wedge dx_n \otimes m)$$

$$+ u \cup v dx_1 \wedge \cdots \wedge dx_n \otimes [m, a \partial_i] =$$

$$\pi(a \partial_i(u)v + a \partial_i(au)v) + \pi(auv)[m, a \partial_i] =$$

$$\pi(a \partial_i(auv)) + \pi(auv)[m, a \partial_i] =$$

$$(u \cup v dx_1 \wedge \cdots \wedge dx_n) \otimes [m, a \partial_i] =$$

$$(u \cup v dx_1 \wedge \cdots \wedge dx_n \otimes m) \circ a \partial_i.$$ 

and the definition of the cup product (??) is correct.

**Theorem 5.7.** Let $M$ be an antisymmetric $W_n^{rsym}$-module. The cup product of $W_n^{rsym}$-modules (??) induces an isomorphism

$$H_{rsym}^{k+1}(W_n, M) \cong Z_{rsym}^1(W_n, U) \otimes H_{1e}^k(W_n, \Omega^n \otimes M), \ k > 0.$$ 

An isomorphism takes place

$$Z_{rsym}^i(W_n, U) \cong B_{rsym}^i(W_n, U) = \{dx_i : i = 1, \ldots, n\} \cong \wedge^1.$$
Proof. We will argue as in the previous subsection. For a $W_n$-module $M$ its dual module is denoted by $M'$. Consider $\Lambda^1 = \{dx_1, \ldots, dx_n\}$ as a trivial module over Lie algebra $W_n$. Endow $U \otimes \Lambda^1$ by a structure of $W_n$-module using a natural $W_n$-module structure on $U = \mathbb{K}[x_1, \ldots, x_n]$ and a trivial $W_n$-module structure on $\Lambda^1$:

$$(u \otimes dx_i)b\partial_j = b\partial_j(u) \otimes dx_i.$$ 

It is easy to see that, $C^1(W_n, \mathbb{K}) \cong \Lambda^1 \otimes U'$, since for any $f \in C^1(W_n, \mathbb{K})$,

$$[f, ad_b](b\partial_j) = (f \circ ad_b)(b\partial_j) = -f(ad_b(b)\partial_j).$$

The bilinear map (**) is nondegenerate and gives a pairing of modules over Lie algebra $W_n$. So,

$$U' \cong \Omega_n.$$ 

Therefore, an isomorphism of $W_n$-modules takes place:

$$C^1(W_n, \mathbb{K}) \cong \Lambda^1 \otimes \Omega_n, \quad C^1(W_n, M) \cong \Lambda^1 \otimes \Omega_n \otimes M,$$

and by theorem **,

$$H^k_{rsym}(W_n, M) \cong \Lambda^1 \otimes H^k_{\text{Lie}}(W_n, \Omega^n \otimes M), \quad k > 0.$$ 

Corollary 5.8. $H^k_{rsym}(W_n, \mathbb{K}) \cong \Lambda^1 \otimes H^k_{\text{Lie}}(W_n, \Omega^n), \quad k > 0.$

Recall that $H^k_{\text{Lie}}(W_n, \Omega^n)$ is calculated by Gelfand and Fuchs [?].

5.6. Cohomologies of $gl^\text{sym}_n$ in an antisymmetric module.

Theorem 5.9. Let $A = gl^\text{sym}_n$, char $k = 0$. Let $M$ be a finite-dimensional $A$-module, such that a $A^{\text{Lie}}$-module $M$ is a tensor module. Then the cap product $M \times \mathbb{K} \rightarrow M$, $m \cup \lambda = \lambda m$, induces an isomorphism

$$H^k_{rsym}(A, M) \cong Z^1_{rsym}(A, M) \otimes H^k_{\text{Lie}}(A, \mathbb{K}), \quad k > 0.$$ 

Proof. By theorem **,

$$H^k_{rsym}(A, M) \cong H^k_{\text{Lie}}(A, C^1(A, M)), \quad k > 0.$$ 

By theorem 2.1.2 of [?],

$$H^k_{\text{Lie}}(A, C^1(A, M)) \cong H^k_{\text{Lie}}(A, \mathbb{K}) \otimes C^1(A, M)^{A^{\text{Lie}}}. $$

It remains to notice that,

$$C^1(A, M)^{A^{\text{Lie}}} := \{f \in C^1(A, M) : [f, a] = 0, \forall a \in A\}$$

is exactly $Z^1_{rsym}(A, M)$. It is evident:

$$[f, a](b) = (f \circ a)(b) = d_{rsym}f(b, a), \quad \forall a, b \in A.$$ 

Corollary 5.10. Let $A = gl^\text{sym}_n$ and $M$ be a irreducible antisymmetric $A$-module. Then $H^k_{rsym}(A, M) \neq 0, k \geq 0$, if and only if $M = A$, and

$$H^k_{rsym}(gl_n, (gl_n)^\text{anti}) \cong H^k_{\text{Lie}}(gl_n, \mathbb{K}), k \geq 0.$$

In particular, $H^k_{\text{Lie}}(gl_n, \mathbb{K}) = 0, k > 0.$

Proof. Since $M$ is antisymmetric,

$$Z^1_{\text{rsym}}(A, M) = \{f : A \rightarrow M : f(b \circ a) = f(b) \circ a\}.$$ 

Thus, any $f \in Z^1_{\text{rsym}}(A, M)$ gives us a homomorphism of right modules $f : A \rightarrow M$. Since right modules $M$ and $A$ are irreducible, by the Lemma of Shur, $Z^1_{\text{rsym}}(A, M) \cong \mathbb{K}$, if $M \cong A$, and $Z^1_{\text{rsym}}(A, M) = 0$, if $M \not\cong A.$
5.7. Cohomologies of $gl^n_{rsym}$ in a regular module.

Theorem 5.11. Let $A = gl^n_{rsym}$ over a field $K$ of characteristic 0 and $M = A$ be its regular module. Then the cup product $M \times \mathcal{C} \rightarrow M, m \cup \lambda = \lambda m$, induces an isomorphism

$$H_{rsym}^{k+1}(A, M) \cong H_{rsym}^1(A, A) \otimes H_{lie}^k(A, K), Z_{rsym}^1(A, A) \cong \mathfrak{sl}_n, \ k \geq 0.$$  

In particular, any cocycle class in $H_{rsym}^{k+1}(gl^n, gl^n)$ has a representative that can be presented as $ad X \cup \psi$, where $\psi \in Z_{lie}^k(gl^n, K)$.

**Proof.** Any right-symmetric 1-cocycle of an associative algebra $A$ is also an associative 1-cocycle and converse, any associative 1-cocycle is a right-symmetric 1-cocycle. So, $Z_{rsym}^1(gl^n, gl^n) = Z_{ass}^1(gl^n, gl^n)$. Any derivation of the associative algebra $gl_n$ is a derivation of the Lie algebra of $gl_n$. Any derivation of $gl^n_{lie}$, except $a \mapsto tr a$, is inner. So, the following sequence is exact

$$0 \rightarrow Z_{rsym}^1(gl^n, gl^n) \rightarrow Z_{lie}^1(gl^n, gl^n) \rightarrow K \rightarrow 0.$$  

In particular,

$$Z_{rsym}^1(gl^n, gl^n) = \{ ad X : X \in \mathfrak{sl}_n \} \cong Z_{lie}^1(sl_n, sl_n) \cong \mathfrak{sl}_n.$$  

It remains to use theorem ??.

Corollary 5.12. An algebra $gl_n$ as a right-symmetric algebra has nontrivial deformations. Any right-symmetric local deformation (2-cocycle of the regular module) is equivalent to a 2-cocycle $\eta_X$ of the form

$$\eta_X(a, b) = (tr b)[X, a],$$

for some $X \in \mathfrak{sl}_n$. Any local right-symmetric deformation can be prolonged. Any formal right-symmetric deformation of $gl_n$ is equivalent to a deformation of the form

$$\mu_t(a, b) = a \circ b + t (tr b)[X, a], \quad X \in \mathfrak{sl}_n,$$

where $(a, b) \mapsto a \circ b$ is a usual associative multiplication of matrices.

**Proof.** These statements can be obtained from the following cohomological facts

$$H_{rsym}^2(gl^n, gl^n) \cong Z_{ass}^1(gl^n, gl^n) \otimes H_{lie}^1(gl^n, K) \cong \{ ad X \cup tr : X \in \mathfrak{sl}_n \},$$

$$H_{rsym}^3(gl^n, gl^n) = 0,$$

and corollary ??.

**Remark.** For $\omega \in C^1(gl^n, gl^n)$, $\omega(a) = (tr a)X$, we have

$$(\eta + d_{rsym}\omega)(a, b) = (tr b)[X, a] + (tr b)a \circ X - (tr a \circ b)X + (tr a)X \circ b = \eta_X(a, b),$$

where

$$\eta_X(a, b) = (tr b)X \circ a - (tr a \circ b)X + (tr a)X \circ b.$$  

Therefore, $[\eta_X] \sim [\bar{\eta}_X]$. Notice that $\bar{\eta}_X$ is a symmetric cocycle:

$$\bar{\eta}(a, b) = \bar{\eta}(b, a).$$

The prolongation formula for $\bar{\eta}_X$ is a little bit complicated. It can be given by

$$\mu_t(a, b) = \Phi_t^{-1}(\mu_t(\Phi_t(a), \Phi_t(b))),$$

where

$$\Phi_t = id + t \omega.$$  

In particular,

$$\Phi_t^{-1}(a) = a - t (tr a)X + t^2 (tr a)^2 X^2 - \cdots,$$

and some of the beginning terms of $\mu_t$ look like

$$\mu_t(a, b) = a \circ b + t X \circ ( (tr a)b - (tr a \circ b) + (tr b)a)$$

$$+ t^2 (tr a tr b - (tr a \circ b)^2 X^2 - (tr a tr (X \circ b))X - (tr (a \circ X) tr b)X + \cdots,$$
So, right-symmetric prolongation can be constructed in a such way, that corresponding Lie multiplication will not be changed:

$$\mu_t(a,b) - \mu_t(b,a) = [a,b].$$

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**REFERENCES**


