We present five dimensional extreme black hole solutions of M-theory compactified on Calabi-Yau threefolds and study these solutions in the context of flop transitions in the extended Kähler cone. In particular, we consider a specific model and present black hole solutions, breaking half of $N=2$ supersymmetry, in two regions of the extended Kähler cone, which are connected by a flop transition. The conditions necessary to match both solutions at the flop transition are analysed. Finally, we also discuss the conditions to obtain massless black holes at the flop transition.
1 Introduction

In recent times there has been a lot of interest in the study of various dualities in string theory. The central observation is that the five distinct ten-dimensional string theories can be thought of as the weak coupling limits of various compactifications of eleven dimensional M-theory, whose low-energy limit is given by eleven dimensional supergravity [7]. This can be used to lift known dualities by one dimension. In four dimensions $N = 2$ heterotic string theory on $K_3 \times T^3$ and type IIA string theory on Calabi-Yau threefolds are dual to each other [8]. Using M-theory this duality can be lifted to five dimensions so that heterotic string theory on $K_3 \times S^1$ is related to M-theory on Calabi-Yau spaces with $N = 2$ supersymmetry [9]. This is equivalent to taking the large volume limit of type II string theory compactified on the same Calabi-Yau manifold [9].

When compactifying M-theory on a Calabi-Yau manifold, the two-brane and the five-brane can be wrapped around the two- and four-cycles of the Calabi-Yau space to give rise to BPS states of the $D = 5$ theory which has $N = 2$ supersymmetry [9]. The study of BPS black hole solutions in the compactified theories has become interesting in the context of a microscopic derivation of the macroscopic Bekenstein-Hawking entropy [7] through the counting of microscopic degrees of freedom of D-branes [7]. Moreover BPS solutions account for the additional light states that have to be present in certain regions of moduli space in order to cure apparent singularities of the low energy effective action and to describe phase transitions between different branches of the moduli space [7].

BPS saturated solutions in toroidal compactifications have $N = 4$ and $N = 8$ supersymmetry and they do not receive quantum corrections. On the other hand, BPS saturated solutions in $N = 2$ supergravity can receive quantum corrections at one-loop level. These $N = 2$ models with vector and hyper multiplets in four and five dimensions can arise as compactifications of type-II or M-theory on a Calabi-Yau threefold, respectively. The Bekenstein-Hawking entropy of these $N = 2$ black holes or more generally solitons with non-singular horizons can be obtained by extremizing the underlying central charge [7] and the scalar fields take fixpoint values at the horizon, which are independent of their values at infinity. These fixed values are determined in terms of conserved charges and topological data of the compactified space. The most simple example is the double-extreme black hole, where the scalar fields (moduli) are constant throughout the entire space-time [7].

Extreme black hole solutions with non-constant scalar fields have also been obtained in four [7] and five dimensions [7]. Whereas the relevant four dimensional theory is described by special geometry, in five dimensions the coupling of $N = 2$ supergravity to abelian vector multiplets is based on the structure of very special geometry [7, 7]. As already mentioned above, moduli spaces of string compactifications have critical points where additional massless states appear. Sometimes this is related to topology changing phase transitions of the internal compact space. The moduli spaces relevant in the context of supersymmetric black hole solutions with finite horizons are the Kähler moduli spaces of four- and five dimensional theories with $N \geq 2$ supersymmetry. In four dimensions, there can be phase transitions between geometrical phases with a sigma model description and non-geometrical phases corresponding to more abstract conformal field theories like Landau-Ginzburg models. In five dimensions, on the other hand, it has been shown that all the phases of the M-theory compactification are geometrical and the phase transitions are sharp [7].

Generically, topological phase transitions among $N = 2$ string vacua occur at points in the moduli spaces where the non-perturbative BPS states become massless. The well-known example in four dimensions are the massless electrically or magnetically charged black holes at the conifold points in the Calabi-Yau moduli spaces, where certain homology cycles of the Calabi-Yau spaces shrink to zero size [7]. The other examples are enhancement of gauge symmetry [7]
and appearance of tensionless strings [1]. The interplay between Calabi-Yau phase transition and the behaviour of black hole like space time configurations has been studied in [2] in the context of four dimensional type II string compactifications. In this paper we will perform a similar analysis in five dimensional M-theory, taking advantage of the fact that here various simplifications occur. Namely, in the five dimensional theory resulting from the compactification of M-theory on Calabi-Yau threefold, there is no analogue of stringy $\alpha'$ corrections. In addition, there are no axionic fields and the moduli (the sizes of the two-cycles in the Calabi-Yau space) take values in a real Kähler cone. Unlike the four dimensional case, the transitions between various phases in five dimensions have to go through true singularities in the Calabi-Yau space and one obtains sharp phase transitions. There is a variety of possible phenomena that can occur as one approaches the boundary of the Kähler cone. The so-called ”flop transition” is one such example, where one of the moduli approaches the boundary of its original range of values and subsequently is analytically continued to a new region. Geometrically this corresponds to a transition into a new Kähler cone which is assigned to a birationally equivalent Calabi-Yau with different intersection numbers, but the same Hodge numbers [3].

A study of critical points and phase transitions in five dimensions in the case of double extreme black holes has been presented in [4]. In this article we study these phase transitions in the context of extreme black hole solutions in five dimensions, i.e. we construct and analyze solutions that break half of $N = 2$ supersymmetry, with scalar fields which are not constant throughout the entire space-time. Thus, the scalars represent dynamical degrees of freedom of the model.

The paper is organized as follows. In sections ?? and ?? we give a brief review of $D = 5$, $N = 2$ supergravity and very special geometry and its connection to M-theory compactified on a Calabi-Yau threefold, respectively. In section ??, we analyze the static, extreme black hole solutions in five dimensions. In section ??, we present the so-called $F_1$ model and the prepotentials of its two Kähler cones, which are connected by a flop transition. In sections ?? and ?? we present the black hole solutions in both regions, namely region II and III [5], by solving the stabilization equations and we obtain the ADM mass and the entropy of the black hole by evaluating the central charge at infinity and at the horizon, respectively. In section ??, we consider generic supersymmetric black hole solutions in five dimensions, which interpolate between flat space at $r = \infty$ and a Bertotti-Robinson geometry at the horizon at $r = 0$. We analyze the behaviour of solutions when the moduli at infinity and at the horizon take values in different regions (Kähler cones). In section ??, we find the conditions for obtaining massless black hole solutions at the flop transition. Finally we summarize our results in section ??.

## 2 $D=5$, $N=2$ Supergravity and Very Special Geometry

The action of five dimensional $N = 2$ supergravity coupled to $N = 2$ vector multiplets has been constructed in [6]. In the following we will consider compactifications of $N = 1$, $D = 11$ supergravity, i.e. the low energy limit of M-theory, down to five dimensions on Calabi-Yau 3-folds ($CY_3$) with Hodge numbers $(h_{(1,1)}, h_{(2,1)})$ and topological intersection numbers $C_{IJK}(I, J, K = 1, \ldots, h_{(1,1)})$. The five dimensional theory contains the gravity multiplet $(e^\mu, *M^\mu, A^I (I = 1, 2), h_{(1,1)} - 1)$ vector multiplets $(A^A_{\mu}, \lambda^A_I, \phi^A)$ ($A = 1, \ldots, h_{(1,1)} - 1$) and $h_{(2,1)} + 1$ hypermultiplets $(\zeta^m, A^F_m)$ ($m = 1, \ldots, 2(h_{(2,1)} + 1)$). The $N_V$-dimensional space $M$ ($N_V = h_{(1,1)} - 1$) of scalar components of $N = 2$ abelian vector multiplets coupled to supergravity can be regarded as a hypersurface of a $h_{(1,1)}$-dimensional manifold whose coordinates $X(\phi)$ are in correspondence with the vector bosons (including the graviphoton). The defining equation of the hypersurface is

$$V(X) = 1$$ (1)
and the prepotential $\mathcal{V}$ is a homogeneous cubic polynomial in the coordinates $X(\phi)$:

$$\mathcal{V}(X) = \frac{1}{6} C_{IJK} X^I X^J X^K, \quad I, J, K = 1, \ldots, h_{1,1}$$

(2)

In five dimensions the $N = 2$ vector multiplet has a single scalar and $\mathcal{M}$ is therefore real. The special case corresponding to perturbative heterotic compactifications on $K3 \times S^1$ has factorizable prepotential

$$\mathcal{V}(X) = X^I Q(X^{I+1}), \quad I = 1, \ldots, N_V$$

(3)

where $Q$ denotes a quadratic form. It follows that the scalar fields parameterize the coset space

$$\mathcal{M} = SO(1,1) \times \frac{SO(N_V - 1, 1)}{SO(N_V - 1)}.$$  

(4)

The bosonic action of five dimensional $N = 2$ supergravity coupled to $N_V$ vector multiplets is given by (with metric $(-,+,+,+,+)$ and omitting Lorentz indices)

$$e^{-1} \mathcal{L} = -\frac{1}{2} R - \frac{1}{2} g_{ij} \partial \phi^i \partial \phi^j - \frac{1}{4} g_{1J} F^I F^J + \frac{e^{-1}}{48} C_{KLM} e^{F K} F^L A^M.$$  

(5)

The corresponding vector and scalar metrics are completely encoded in the function $\mathcal{V}(X)$

$$G_{I J} = -\frac{1}{2} \frac{\partial}{\partial X^I} \frac{\partial}{\partial X^J} \ln \mathcal{V}(X)|_{\nu=1},$$

$$g_{ij} = G_{I J} \frac{\partial}{\partial \phi^i} X^I(\phi) \frac{\partial}{\partial \phi^j} X^J(\phi)|_{\nu=1}.$$  

(6)

(7)

Moreover, it is convenient to introduce “dual” special coordinates

$$X_I \equiv \frac{1}{6} C_{IJK} X^J X^K \Rightarrow X^I X_I = 1.$$  

(8)

### 3 Connection of $D=5$, $N=2$ Supergravity to M-theory and String Theory

As already mentioned above we consider $D = 5$, $N = 2$ supergravity as a compactification of eleven dimensional supergravity on a Calabi-Yau threefold ($CY_3$) [?]. The corresponding non-perturbative definition of eleven dimensional supergravity is provided by M-theory. Alternatively one can interpret this setup as type IIA string theory compactified to four dimensions on the same $CY_3$ in the limit of large Kähler structure and large type IIA coupling [?, ?]. In this limit, $CY_3$ completely decompactifies with respect to the IIA metric. However, in terms of M-theory variables, the volume of $CY_3$ is kept fixed whereas the M-theory circle decompactifies.

Going from $D = 4$ to $D = 5$, $N = 2$ supergravity results in various simplifications in the geometry of the Kähler moduli space and hence in the dynamics of vector multiplets. Since the key features are important in what follows, we briefly review them, following [?]:

(i) The M-theory limit of the IIA Kähler moduli space can be thought of as a zero slope limit in which stringy effects, i.e. $\alpha'$ corrections, are switched off. As a consequence the five dimensional prepotential is purely cubic. The coefficient of the cubic term is given by the triple intersection numbers $C_{IJK}$ of homology four-cycles.

(ii) The Kähler moduli in four dimensions are complex due to the presence of axion-like scalars, whereas they are real in five dimensions. Thus, the Kähler moduli space of the five dimensional theory is the standard real Kähler cone of classical geometry, in contrast to the complexified
“quantum” Kähler moduli space of string compactifications.\(^6\) Finally, the modulus corresponding to the total volume of the Calabi-Yau space sits in a hypermultiplet of the five dimensional theory.

Concerning the global structure of the Kähler moduli space, the M-theory limit has the effect of eliminating the non-geometric phases encountered in four dimensions, because these are due to \(\alpha'\) effects. On the other hand, the transition between different geometric phases is still possible through so-called “flop transitions” \(^7\). At a flop transition, one or several complex curves in the manifold degenerate to zero volume. After transforming the triple intersection numbers in a specific way depending on the particular degeneration one can continue into the Kähler cone of another Calabi-Yau space. Gluing together the Kähler cones of all Calabi-Yau spaces that are related by flop transitions, one obtains the so-called “extended Kähler cone”.\(^8\)

In physical terms one has additional massless hypermultiplets at the transition point which descend from M-theory branes wrapped on the vanishing cycles. Integrating these extra states out amounts to changing the triple intersection numbers in, precisely, the way predicted by the Calabi-Yau geometry. It will be important later that the low energy effective action is regular at the transition. In comparison with four dimensional type II compactifications we note that there flop transitions are “washed out” by \(\alpha'\) corrections and by the fact that the Kähler moduli are complexified. This has the consequence that one can smoothly interpolate between flopped Calabi-Yau spaces in four dimensional string theory, whereas in five dimensional M-theory the transition is sharp with singular geometry, but regular physics.

Finally, the Kähler moduli spaces of five dimensional M-theory have boundaries where a further continuation of the moduli is impossible. These boundaries are either related to the presence of additional massless states or to tensionless strings \(^9\). In the first case the effective action is regular, whereas it is singular in the second one.

Thus, the maximally extended Kähler moduli space in the M-theory limit is given by a hypersurface in the (partially) extended Kähler cone, which contains all Calabi-Yau spaces that can be related to one another by flop transitions. Since no stringy \(\alpha'\) corrections are present and since the prepotential is simply a cubic polynomial, we can compute exact solutions to the low-energy effective action and analyze them in detail, whereas in four dimensions one needs to expand around special points in moduli space, as was done in \(^7\).

4 Extreme Black Hole Solutions

In the following we will analyse spherical symmetric electrically charged BPS black hole solutions breaking half of \(N = 2\) supersymmetry. We will use the approach and general solution presented in \(^?\). The black hole solution is given by the following set of equations

\[
\begin{align*}
\text{ds}^2 &= -e^{-4V(r)} \, dt^2 + e^{2V(r)}(dr^2 + r^2 \, d\Omega_3^2), \\
F_{tm} &= -\partial_m(e^{-2V} X^I), \quad m = 1, 2, 3, 4, \\
3e^{2V} X_I &= H_I \equiv h_I + \frac{q_I}{r^2}.
\end{align*}
\]  

\(^6\)See for example \(^?\) for a review.

\(^7\)We do not consider phase transitions that involve tuning the hypermultiplets, because the solutions we study only depend on the vector multiplets. Transitions involving hypermultiplets have been studied extensively in F-theory \(^?, \)\)

\(^8\)This is sometimes called the partially extended Kähler cone, thus reserving the term extended Kähler cone to the case of stringy moduli spaces which include non-geometric phases.
where, $F^I_{mn}$ is the electric field strength. Near the horizon of the black hole the metric function $e^{2V}$, which is in general a function of harmonic functions, satisfies
\[
\lim_{r \to 0} e^{2V} = \frac{1}{3} \frac{Z_h}{r^2},
\]
where $Z = X^I q_I$ is the (electric) central charge, appearing in the supersymmetry algebra, and $Z_h$ is its value at the horizon. It follows that the Bekenstein-Hawking entropy \cite{Bekenstein} of extreme black holes in five dimensions is given by \cite{Hawking}
\[
S_{BH} = \frac{A}{4G_N} = \frac{\pi^2}{2G_N} \left| \frac{Z_h}{3} \right|^{3/2}.
\]
Moreover, the ADM mass of the black hole solution is determined by the central charge evaluated at spatial infinity:
\[
M_{ADM} = \frac{\pi}{4G_N} Z_{\infty}.
\]
It is useful to introduce rescaled special coordinates
\[
Y^I = e^V X^I, \quad Y_I = e^{2V} X_I
\]
such that $V(Y) = Y_I Y^I = e^{3V}$. It follows that the background metric of the black hole solution is given by
\[
ds^2 = -V^{-4/3}(Y) \, dt^2 + V^{2/3}(Y) (dr^2 + r^2 \, d\Omega_3^2).
\]
Moreover, the stabilisation equations take the simple form
\[
C_{IJK} Y^J Y^K = 2 H_I.
\]
Thus, near the horizon of the black hole one obtains
\[
\lim_{r \to 0} V(Y) = \left( \frac{1}{3} \frac{Z_h}{r^2} \right)^{3/2}
\]
\section{The Model}
In the following we will restrict ourselves to one family of Calabi-Yau spaces. The reason is that five dimensional black holes have a complicated model dependence, because \cite{Cvetic} is a system of coupled quadratic equations, which can only be solved on a case by case basis. Thus solutions are determined in terms of harmonic functions, but the way the metric and scalars depend on the harmonic functions is much more complicated as it is, for instance, for four dimensional axion-free black holes.
The particular model we chose was introduced in the context of F-theory \cite{Acharya}. We refer to it as the F\textsubscript{1} model, since one of its phases is an elliptic fibration over the first Hirzebruch surface $F_1$. We will use the notation introduced in \cite{Acharya}. The (partially) extended Kähler cone can be covered by one set of moduli (special coordinates) $X^{1,2,3} = (T, U, W)$\footnote{Eventually one has to restrict to a hypersurface in this cone as explained above.}. It consists of two Kähler cones, which are connected by a flop transition at $W = U$. Following \cite{Acharya} we call the cones region III and region II, respectively. Inside region III the moduli satisfy
\[
W > U > 0 \quad \text{and} \quad T > W + \frac{1}{2} U
\]
and the corresponding Calabi-Yau space is the above mentioned elliptic fibration over $F_1$. The prepotential is given by

$$V_{III}(X) = \frac{5}{24} U^3 + \frac{1}{2} U T^2 - \frac{1}{2} U W^2 + \frac{1}{2} U^2 W. \tag{18}$$

This Calabi-Yau space is a $K_3$ fibration and therefore it has a heterotic dual, which is obtained by compactifying the heterotic $E_8 \times E_8$ string on $K_3 \times S_1$ with instanton numbers $(13,11)$. In addition to the flop boundary region III has two more boundaries: at $W = T + \frac{1}{2} U$ the gauge symmetry is enhanced to $SU(2)$, whereas one finds tensionless strings at $W = U$. Both boundaries are boundaries of the extended cone, i.e. the moduli space ends at these boundaries. Passing through the flop transition at $U = W$ one enters region II, which is parametrized by

$$U > W > 0 \text{ and } T > \frac{3}{2} U. \tag{19}$$

Here the prepotential takes the form [2]

$$V_{II}(X) = \frac{3}{8} U^3 + \frac{1}{2} U T^2 - \frac{1}{6} W^3 \tag{20}$$

The corresponding Calabi-Yau space is not a $K_3$ fibration and, therefore, there is no dual (weakly coupled) heterotic description. Region II has two additional boundaries: The boundary $W = 0$ corresponds to an elliptic fibration over $P^2$. At the other boundary $T = \frac{3}{2} U$ strings become tensionless. The effective action is regular everywhere in the extended Kähler cone including those boundaries where no tensionless strings appear.

5.1 The Black Hole Solution in Region II

In region II the dual (special) coordinates are given by

$$X_1 = \frac{1}{3} T U, \quad X_2 = \frac{1}{6} T^2 + \frac{3}{8} U^2, \quad X_3 = -\frac{1}{6} W^2. \tag{21}$$

Let us denote the rescaled coordinates as, $Y^I = T, U, W$. The prepotential in terms of these quantities are given by,

$$V(Y) = Y_I Y^I = \frac{1}{6} C_{IJK} Y^I Y^J Y^K = \frac{3}{8} U^3 + \frac{1}{2} U T^2 - \frac{1}{6} W^3 \tag{22}$$

The stabilization equations in terms of these rescaled coordinates are given by,

$$\frac{1}{2} (T)^2 + \frac{9}{8} (U)^2 = H_T$$

$$-\frac{1}{2} (W)^2 = H_W \tag{23}$$

with solution

$$T = \sqrt{H_U (1 \pm \sqrt{1-\Delta})}$$

$$U = \frac{H_T}{\sqrt{H_U (1 \pm \sqrt{1-\Delta})}}$$

$$W = \sqrt{-2 H_W} \tag{24}$$
and \( \Delta = \frac{\sqrt{3}}{4} \). Using the equivalent form

\[
U = \frac{2}{3} \sqrt{H_T (1 \mp \sqrt{1 - \Delta})}.
\]  

(25)

one can check that the corresponding unrescaled scalar fields, when evaluated on the horizon, take the fixpoint values found in [?].

Any additional negative sign appearing inside the roots of the solutions yields an additional constraint in terms of charges/harmonic functions of the solution. In general, the solution has to satisfy at least two bounds

\[
H_W \leq 0, \quad H_T^2 \geq \frac{9}{4} H_T^2
\]  

(26)

Then it manifests that \( T, U, W > 0 \). The condition for the solution to be inside the Kähler cone is given by, \( \frac{2}{3} T > U > W > 0 \). For the condition \( U > W \), we get

\[
H_W^2 + \frac{4}{9} H_T H_W + \frac{1}{9} H_T^2 > 0
\]  

(27)

At the boundary, \( U = W \), so the condition for reaching the boundary is,

\[
H_W = -\frac{2}{9} H_T \pm \frac{1}{3} \sqrt{\frac{4}{9} H_T^2 - H_T^3}
\]  

(28)

If \( 1 - \Delta \geq 0 \), then we can have at least one solution with \( H_W < 0 \). This shows that the solution can cross the Kähler cone while still respecting the other constraints.

Next consider the condition \( T > \frac{3}{2} U \). In terms of harmonic functions, we get

\[
H_T + \sqrt{H_T^2 - \frac{9}{4} H_T^2} > \frac{3}{2} H_T
\]  

(29)

This is obviously satisfied if the term under the square root is positive. Note that this time at the boundary, the moduli of \( T, U \) would become imaginary when further continuing the solution. The behaviour at this boundary is different from that at the flop boundary and one does not expect to be able to extend the solution. Analogous remarks apply to the boundary \( W = 0 \).

The corresponding ADM mass of the black hole solution can be determined by the central charge evaluated at spatial infinity, i.e. using \( Z_\infty = q_1 Y_\infty^I = q_1 X_\infty^I \) one obtains for our particular black hole solution

\[
Z_\infty = q_T \sqrt{h_T} \left[ \sqrt{(1 \pm \sqrt{1 - \delta(h)})} + \frac{1}{\sqrt{(1 \pm \sqrt{1 - \delta(h)})}} \right] + q_W \sqrt{-2h_W}
\]  

(30)

with

\[
\Delta(r \to \infty) \equiv \delta(h) = \frac{9h_T^2}{4h_T^2}.
\]  

(31)

The entropy of the black hole solution is given by the central charge of the \( N = 2 \) supersymmetry algebra evaluated at the horizon \( r = 0 \):

\[
Z_h = \lim_{r \to 0} 3^{1/3} \left( r q_T Y^I \right)^{2/3}
\]  

(32)
Our solutions at the horizon are,

\[
\begin{align*}
\lim_{r \to 0} rT &= \sqrt{q_U \pm \sqrt{q_U^2 - \frac{9}{4} q_T^2}} \\
\lim_{r \to 0} r\bar{T} &= \frac{q_T}{\sqrt{q_U \pm \sqrt{q_U^2 - \frac{9}{4} q_T^2}}} \\
\lim_{r \to 0} r\bar{W} &= \sqrt{-2q_W}
\end{align*}
\]

Thus, in region II one obtains

\[
Z_h = 3^{1/3} \left( q_T \sqrt{q_U} \left[ \sqrt{(1 \pm \sqrt{1 - \delta(q)})} + \frac{1}{\sqrt{(1 \pm \sqrt{1 - \delta(q)})}} \right] - \frac{1}{2} (-2q_W)^{3/2} \right)^{2/3}
\]

with

\[
\Delta_h = \delta(q) = \frac{9q_T^2}{4q_U^2},
\]

which can be checked to coincide with the result of [?] by extremization of the central charge.

\subsection{5.2 The Black Hole Solution in Region III}

In region III a field identification of the form

\[
W = S' - \frac{1}{2} (T' - U'), \quad T = S' + \frac{1}{2} T', \quad U = U'
\]

yields the prepotential

\[
\mathcal{V}_{III}(X) = S'T'U' + \frac{1}{3} U'^3.
\]

The corresponding black hole solution of this prepotential within a different model\textsuperscript{10} has been already studied in [?]. Since we can express all the fields in both chambers using special coordinates \((T, U, W)\), we keep this parametrization of the prepotential which is given by,

\[
\mathcal{V} = \frac{5}{24} U^3 + \frac{1}{2} UT^2 - \frac{1}{2} UW^2 + \frac{1}{2} U^2 W
\]

It follows that the dual coordinates in region III are given by

\[
\begin{align*}
X_1 &= \frac{1}{3} TU \\
X_2 &= \frac{1}{6} T^2 - \frac{1}{6} W^2 + \frac{1}{3} UW + \frac{5}{24} U^2 \\
X_3 &= -\frac{1}{3} UW + \frac{1}{6} U^2
\end{align*}
\]

Again in terms of rescaled coordinates, the stabilization equations read

\[
\begin{align*}
TU &= H_T \\
(T)^2 - (W)^2 + 2UW + \frac{5}{4} (U)^2 &= 2 H_{U} \\
(U)^2 - 2UW &= 2 H_W
\end{align*}
\]

\textsuperscript{10}The fact that the prepotentials can be brought to the same form does not imply that the Kähler cones are the same. See [?] for a detailed discussion.
One obtains the following set of solutions

\[ \mathcal{T} = \frac{H_T}{\sqrt{a \pm \sqrt{a^2 + b}}} \]
\[ \mathcal{U} = \frac{U}{\sqrt{a \pm \sqrt{a^2 + b}}} \]
\[ \mathcal{W} = \frac{W}{\sqrt{a \pm \sqrt{a^2 + b}}} + \frac{1}{2} \sqrt{a \pm \sqrt{a^2 + b}} \]  \hspace{1cm} (41)

with

\[ a = \frac{1}{2} (H_U + \frac{1}{2} H_W), \quad b = \frac{1}{2} (H_W^2 - H_T^2) \]  \hspace{1cm} (42)

Again additional negative signs appearing in the roots of the solutions yield additional constraints. The reality of the moduli gives the constraint,

\[ a^2 + b \geq 0. \]  \hspace{1cm} (43)

In terms of harmonic functions, this condition reads as,

\[ 9H_W^2 + 4H_U^2 - 8H_T^2 + 4H_W H_U \geq 0 \]  \hspace{1cm} (44)

From the solutions, \( \mathcal{U} \geq 0 \) is automatically satisfied. Analysis of \( \mathcal{W} \geq \mathcal{U} \) implies that \( H_W < 0 \). Assuming \( H_U > 0 \) and \( -9H_W - 2H_U > 0 \), the constraint is obtained as

\[ 9H_W^2 + 4H_W H_U + H_T^2 \geq 0 \]  \hspace{1cm} (45)

Finally, let us consider \( \mathcal{T} > \frac{\mathcal{U}}{2} + \mathcal{W} \). It is useful to analyze the simpler and stronger constraint \( \mathcal{T} \geq \frac{3}{2} \mathcal{U} \), which will be relevant while constructing solutions interpolating between chambers II and III. Assuming \( 8H_T - 3H_W - 6H_U > 0 \), this implies,

\[ 17H_W^2 - 9H_W^2 - 6H_T H_W - 12H_T H_U \geq 0 \]  \hspace{1cm} (46)

The ADM mass of the solution is determined by

\[ Z_{\infty} = \frac{1}{\mathcal{U}_{\infty}} (q_T h_T - q_W h_W) + \mathcal{U}_{\infty} (q_U + \frac{1}{2} q_U). \]  \hspace{1cm} (47)

Again one can compute the black hole entropy from the central charge evaluated at the horizon at \( r = 0 \). The result is

\[ Z_h = 3^{1/3} \left( \frac{8q_T^2 + 4q_U q_W + 4q_U^2 - 7q_W^2 \pm (q_W + 2q_U) \alpha(q)}{4 \sqrt{q_W + 2q_U} \pm \alpha(q)} \right)^{2/3} \]  \hspace{1cm} (48)

with

\[ \alpha^2(q) = 9q_W^2 + 4q_W q_U + 4q_U^2 - 8q_T^2 \]  \hspace{1cm} (49)

### 5.3 Interpolating solutions

In this section we present an explicit example of a massive and regular supersymmetric black hole solution, where the scalar fields take values in region II at infinity but values in region III at the horizon. Thus the solution interpolates between the two \( N = 2 \) vacua, namely, the
supersymmetric 'flat space × CYII' at \( r = \infty \) and 'horizon × CYIII' at \( r = 0 \). The geometry of the horizon and the restoration of full \( N = 2 \) supersymmetry on it was found in [?]. Our point here is that one can take \( CYII \) and \( CYIII \) to be topologically distinct Calabi-Yau spaces associated with the regions II and III of the extended Kähler cone. Consequently the solution crosses the flop line \( U = W \) in moduli space for some specific value \( r = r^* \) of the radius. We will show explicitly that one can glue appropriate solutions of region II and region III such that the scalar fields are continuous but not smooth (i.e. not \( C^\infty \)) at \( r = r^* \).

The equations of motions of scalars coupled to gravity can be cast in the form of a generalized geodesic equation mapping space-time to moduli space. In the case of static supersymmetric four- and five dimensional black holes this equation shows a fix point behaviour [?, ?]. This means that the values of the scalar fields at the horizon are uniquely fixed by the charges. In contradistinction, the values of the scalars at infinity can be changed continuously. The equations of motion determine the flow from the values specified at infinity to their fix point values at the horizon.

We will now make a specific choice for the parameters \( h_i \) and \( q_i \), which control the asymptotic behaviour at \( r = \infty \) and \( r = 0 \), respectively. Our strategy will be to impose that the solution is in the interior of region II at infinity, but in the interior of region III at the horizon. Beside this we will make the explicit expressions we get at \( r = r^* \) as simple as possible, so that they allow for an exact analytical treatment. Note that with a generic choice of parameters one is likely to get higher order algebraic equations as matching conditions at \( r = r^* \), which need not have a general solution at all. We would, however, like to have an explicit example where we can compute analytically the behaviour of the scalar fields and of the space-time metric at the transition point \( r = r^* \). Our example is otherwise generic, because the scalar fields take values on a special line in moduli space only at a single radius \( r = r^* \). Therefore we can expect that the qualitative behaviour exhibited in the example is generic.

In order to be in region II at infinity we need to impose \( U_\infty > W_\infty > 0 \) and \( T_\infty > \frac{3}{2} U_\infty \). In addition we have to impose \( \mathcal{V}(Y)_{\infty} = 1 \) in order to have an asymptotically flat configuration. Using the solution (??) in region II, we convert these conditions into restrictions on the asymptotic values \( h_W, h_U, h_T \) of the harmonic functions.\(^{11}\) We will assume that \( h_W < 0, h_U, h_T > 0 \). Then being in region II implies

\[ h_U > \frac{3}{2} h_T, \quad h_T^2 > -4 h_W h_U. \] (50)

Next we have to impose that the scalars at \( r = 0 \) satisfy (??), i.e. that they are in region III at the horizon. In terms of our rescaled variables this becomes

\[ (rW)_{r=0} > (rU)_{r=0} > 0 \quad \text{and} \quad (rT)_{r=0} > (rW)_{r=0} + \frac{1}{2} (rU)_{r=0}. \] (51)

The constraints resulting from (??) have already been discussed in section ???. Here we will simply specify a set of charges which is easily seen to satisfy (??).

It turns out that the following choice of charges and asymptotic parameters is convenient:

\[ h_W = -\frac{1}{18} h, \quad h_U = \frac{5}{2} h, \quad h_T = h, \]

\[ q_W = -\frac{2}{5} q, \quad q_U = \frac{5}{2} q, \quad q_T = q, \] (52)

with \( h > 0 \) and \( q > 0 \). This satisfies both (??) and (??). Now \( h \) is not a free parameter but fixed by demanding asymptotic flatness \( \mathcal{V}(Y)_{\infty} = 1 \). We will not need the explicit value of \( h,\)

\(^{11}\) We take the '+' branch in (??).
but evaluation of $V(Y)_\infty = 1$ shows that it is positive, and therefore the set of parameters (53) is consistent. The advantage of this choice is that it drastically simplifies the solution in region II because $H_\U = \frac{1}{2} H_\T = \frac{5}{2} H$ and therefore

$$\mathcal{W} = \sqrt{-2H_\mathcal{W}}, \quad \mathcal{T} = \frac{\sqrt{9}}{2} H, \quad \mathcal{U} = \frac{\sqrt{2}}{9} H.$$  \hspace{1cm} (53)$$

for $r > r^\ast$.

Since the charges are chosen such that the solution is in region III at the horizon we expect to find $\mathcal{W}(r^\ast) = \mathcal{U}(r^\ast)$ for some value $r = r^\ast$ of the radius. For the above choice of data one indeed finds $r^\ast = \sqrt{\frac{2}{h}}$ with

$$\mathcal{W}(r^\ast) = \frac{\sqrt{3}}{3} \sqrt{h} = \mathcal{U}(r^\ast), \quad \mathcal{T}(r^\ast) = \frac{3}{2} \sqrt{3} \sqrt{h}.$$  \hspace{1cm} (54)$$

For $r < r^\ast$ the scalar fields further evolve according to the region III solution to the fix point at $r = 0$. Thus we need to specify a solution (53) in region III, such that it satisfies the boundary conditions (54) and reaches the fix point. Such a solution depends on a second set of harmonic functions, $H'_i = h'_i + \frac{h_i}{r}$. Note that the charges appearing in the functions $H'_i$ have to be the same as in the $H_i$, since we have already chosen our fix point. The asymptotic constants $h'_i$ have to be determined from (54). One finds that no matching is possible if one takes the '+' sign in (54) whereas with the choice of the '−' in (54) the matching condition (54) is satisfied if and only if $h'_i = h_i$. Thus we conclude that the harmonic functions $H'_i$ and $H_i$ have to be the same in both regions. This yields a unique continuation of the region II solution through the flop transition into region III, which reaches the fix point specified by the charges.

We can now analyze how the scalar fields, the prepotential $V(Y)$ and the space-time metric behave at the flop transition.\textsuperscript{12} The result is that scalar fields are $C^1$-functions with respect to the radius $r$, i.e. they are continuously differentiable. The second derivatives of the scalar fields are not continuous at $r = r^\ast$ but jump by a finite amount. In the prepotential $V(Y)$, which is a cubic polynomial in the scalars and which determines the space-time metric the discontinuities in the second derivatives precisely cancel. However, the third derivative is discontinuous at the flop, i.e. $V(Y)$ is a $C^2$-function. As a consequence the space-time Riemann tensor is continuous at the flop transition, because it depends on no higher derivative of the metric than the second. One also expects that the Riemann tensor is not continuously differentiable at the flop, because of the discontinuity of the third derivative of $V(Y)$. But since the Riemann tensor depends on $V(Y)$ and its derivatives in a complicated way, one could imagine a cancellation of the discontinuities, as we have seen happening for $V(Y)$ itself. However, one can check by explicit computation that the Riemann tensor is continuous, but not differentiable in $r$ at the transition point. Namely, the $r$-derivatives of those components of the Riemann tensor containing the index $r$ jump by a finite amount at $r = r^\ast$.

This shows that the space-time geometry is regular, but not smooth at the transition. The phase transition in the internal Calabi-Yau space manifests itself in a ‘roughening’ of the space-time geometry.

\textbf{5.4 Massless black holes at the flop transition}

In this subsection we analyse the conditions to obtain massless black holes at the flop transition. The relevant quantity to consider massless black holes is, of course, the ADM mass or, equivalently, the central charge evaluated at spatial infinity. For our particular model the latter reads

\textsuperscript{12}The results reported in the following have been obtained using Maple.
in general

\[ Z_\infty = q_T T_\infty + q_u U_\infty + q_W W_\infty \]  

(55)

At the flop transition the solution has to satisfy \( U_\infty = W_\infty \geq 0 \). Thus, at the flop transition the first constraint on the moduli at spatial infinity is given by

\[ \frac{2}{3} T_\infty > U_\infty = W_\infty \geq 0 \]  

(56)

and restricts the parameters \( h_I \), so that the moduli lie in the physical moduli space. It follows that one of the three parameters \( h_I \) can be eliminated at the flop transition. Moreover, one obtains from the stabilisation equations

\[ V_\infty = 1 = \frac{5}{24} W_\infty^3 + \frac{1}{2} W_\infty T_\infty^2. \]  

(59)

This second constraint yields

\[ h_T^2 = 2(-2h_W)^{1/2} - \frac{5}{12} (-2h_W)^2. \]  

(60)

Thus, the solution at the flop transition depends, in general, on three charges and one parameter, say \( h_W \), only. From the first constraint it now follows that the parameter \( h_W \) is constrained, i.e.

\[ 4(-2h_W)^{3/2} \leq 3. \]  

(61)

This means that \( |h_W| \) has to be sufficiently small, such that the set of parameters denotes a valid black hole solution. If the solution does not satisfy this constraint, then it is either not in the physical moduli space or cannot be asymptotically flat.

In order to obtain massless black holes we have to analyze the condition \( Z_\infty = 0 \). Writing this out in terms of independent parameters yields

\[ q_T \sqrt{2\sqrt{-2h_W} - \frac{5}{12} (-2h_W)^2} + (q_u + q_W)(-2h_W) = 0 \]  

(62)

As explained in [?] consistency of the model requires that there is an extra non-perturbative state present at the line \( U = W \). In order to identify this state with a massless black hole we need to have \( Z_\infty = 0 \) for the whole line \( U = W \), that is for all values of \( h_W \). This implies \( q_T = 0 \) and \( q_u + q_W = 0 \) and fits with the geometric picture of a M-theory two-brane wrapped on a vanishing two-cycle. As explained in [?] \( U - W \) is the modulus associated with this vanishing cycle. Wrapped two-branes with \( q_T = 0 \) and \( q_u + q_W = 0 \) wind around the cycle whose size is measured by \( U - W \), but do not wind around any other two-cycle. Therefore precisely these winding states become massless at the flop transition.\(^{13}\) We conclude that the spectrum of massless black holes agrees with the spectrum of massless non-perturbative states predicted by

\[^{13}\text{In order to have a single massless multiplet one has to assume that multiple winding states are multiple particle states, rather than multiply-charged single particle states. This is hard to verify directly, because the multiple winding states are bound states at threshold [?].}\]
M-theory. Note however that the charge configuration one has to take is such that the solution at the horizon is not inside the extended Kähler cone. This means that the solution meets the boundary at finite $r > 0$. Thus, the term massless black hole might be a misnomer. Clearly the behaviour of such solutions deserves further study.

One might also wonder whether there are additional massless black holes corresponding to zeros of $Z_{\infty}$ at special points within the line $U = W$. Such multicritical points would correspond to more complicated singularities of the Calabi-Yau space, occurring in higher codimension in moduli space. Assuming $q_T \neq 0$ one can solve for $h_W$ with the result:

$$h_W^3 = -\frac{1}{2} \left( \left( \frac{q_W + q_T}{q_T} \right)^2 + \frac{5}{12} \right)^{-2}$$

In addition these charges must satisfy all the constraints given above. It would be very interesting to find the explicit quantum numbers for an additional massless black hole at the flop transition. However, because of the number of constraints and the complicated form of the black hole solution, it seems that there is no straightforward analytical way to find such a solution. In addition, a possible relation to higher degenerations of the Calabi-Yau space is not clear to us. These questions are beyond the scope of this article and we leave them for future investigations.

6 Summary

In this paper we have presented generic extreme black hole solutions of M-theory compactified on a specific family of Calabi-Yau threefolds. In particular their ADM masses and BH entropies were computed. Our results contain those obtained from studying double extreme solutions through extremization of the central charge in [?] as a subset and they further confirm the attractor picture [?] of BPS black holes. The picture that we get of phase transitions and massless black holes nicely fits with the one found in four dimensions [?].

In particular, we showed by an explicit example that black hole solutions can interpolate between topologically distinct (though birationally equivalent) internal Calabi-Yau spaces. Despite the fact that the geometry of the internal space is truly singular at a flop transition, this is reflected in the space-time geometry only by a “roughening”, namely a discontinuity in the radial derivative of certain components of the Riemann tensor. We also showed that one can obtain massless solutions at the flop transition line, which can be identified with the extra massless states predicted by M-theory [?]. However these solutions have charge configurations which force the solution to run into the boundary of the extended Kähler cone before a horizon is reached. Clearly these massless solutions, as well as those related to the other boundaries of the Kähler cone, need further study.

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