HECKE ALGEBRAIC PROPERTIES OF DYNAMICAL $R$-MATRICES.
APPLICATION TO RELATED QUANTUM MATRIX ALGEBRAS

L.K. Hadjiivanov
Division of Theoretical Physics, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria,
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy,

A.P. Isaev
Dipartimento di Fisica, Università di Pisa, I-56100 Pisa, Italy
and
Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, 141 980 Moscow Region, Russian Federation,

O.V. Ogievetsky
Centre de Physique Théorique, Luminy, F-13288 Marseille, France
and
P.N. Lebedev Physical Institute, Theoretical Department, Leninsky prospect 53, 117924 Moscow, Russian Federation,

P.N. Pyatov
Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, 141 980 Moscow Region, Russian Federation

and

I.T. Todorov
Erwin Schrödinger Institute for Mathematical Physics (ESI), A-1090 Wien, Austria
and
Division of Theoretical Physics, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria.

MIRAMARE – TRIESTE
December 1997

1 On leave of absence from. E-mail address: lhadji@inrne.acad.bg
2 On leave of absence from. E-mail address: isaevap@thsun1.jinr.ru
3 On leave of absence from. E-mail address: oleg@cpt.univ-mrs.fr
4 E-mail address: pyatov@thsun1.jinr.ru
5 On leave of absence from. E-mail address: toodorov@inrne.acad.bg
The quantum dynamical Yang–Baxter (or Gervais–Neveu–Felder) equation defines an $R$-matrix $R(p)$, where $p$ stands for a set of mutually commuting variables. A family of $SL(n)$-type solutions of this equation provides a new realization of the Hecke algebra. We define quantum antisymmetrizers, introduce the notion of quantum determinant and compute the inverse quantum matrix for matrix algebras of the type $R(p)_{a_1a_2} = a_1a_2 R$. It is pointed out that such a quantum matrix algebra arises in the operator realization of the chiral zero modes of the WZNW model.
Introduction

Let \( \{ v^{(i)} , i = 1, \ldots , n \} \) be a "barycentric basis" in a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{sl}(n) \). Viewed as operators in the \( n \)-dimensional complex space \( V = \mathbb{C}^n \), \( v^{(i)} \) can be realized as real traceless diagonal \( n \times n \) matrices:

\[
\left( v^{(i)} \right)_{jj} = \delta_{ij} - \frac{1}{n} , \quad \sum_{i=1}^{n} v^{(i)} = 0 . \tag{0.1}
\]

Let further \( \{ p_i \}_{i=1}^{n} \) span the dual Lie algebra \( \mathfrak{h}^* \). Introduce the traceless diagonal matrix

\[
p = \sum_{i=1}^{n} p_i v^{(i)} \quad \text{with} \quad \sum_{i=1}^{n} p_i = 0 . \tag{0.2}
\]

We define a Hecke-type quantum dynamical R-matrix \( \hat{R}(p) \) as a map from \( \mathfrak{h}^* \) to \( \text{End} (V \otimes V) \) satisfying the twisted braid relation

\[
\hat{R}_{12}(p - v_1) \hat{R}_{12}(p) = \hat{R}_{23}(p - v_1) \hat{R}_{12}(p) \hat{R}_{23}(p - v_1) \tag{0.3}
\]

and the Hecke condition

\[
\hat{R}(p)^2 = 1 + (q - q) \hat{R}(p) , \quad \bar{q} := q^{-1} . \tag{0.4}
\]

(Although the notation is tailored to the special case in which the parameter \( q \) takes values on the unit circle, we shall not use this property in the main body of the paper.) The subscripts in (??) refer to the, by now standard, tensor product notation of Faddeev et al. (see, e.g., [?]); in particular, \( \hat{R}_{23}(p - v_1) \in \text{End} (V^o \otimes V) \) has matrix elements

\[
\left( \hat{R}_{23}(p - v_1) \right)_{j_1 j_2 j_3}^{i_1 i_2 i_3} = \delta_{j_1}^{i_1} \hat{R}(p - v^{(i_1)}) \delta_{j_2}^{i_2} \delta_{j_3}^{i_3} . \tag{0.5}
\]

The twisted braid relation (??) is equivalent to the quantum dynamical [?, ?] (or deformed [?]) Yang–Baxter equation (QDYBE) for the matrix \( \hat{R}(p) \) related to the braid operator \( \hat{R}(p) \) by \( \hat{R}(p) = PR(p) \) where \( P \) stands for the permutation operator \( Px_1y_2 = y_1x_2 , \quad x, y \in V , \quad P^2 = 1 \). Abusing notation we shall also refer to Eq.(??) by the above abbreviation. The term "dynamical R-matrix" for \( \hat{R}(p) \) is suggested by the fact that in the physical applications its arguments play the role of (commuting) dynamical variables and that \( \hat{R}(p) \) satisfies a finite difference rather than a purely algebraic equation.

The important concept of a quantum matrix algebra \( \mathcal{A} = \mathcal{A}(\hat{R}(p), \hat{\mathcal{R}}) \) (Sec.4) can be introduced as a (complex) associative algebra with 1 generated by rational functions of \( q^{p_i} , \quad i = 1, \ldots , n \), and the (noncommuting) entries of an \( n \times n \) matrix \( a = (a_{ij}^{\alpha}) \) satisfying the quadratic exchange relations

\[
\hat{R}(p) a_{ij} = a_{ij} \hat{R}(p) \tag{0.6}
\]

where \( \hat{R} = R^{(1)2} \) in the right-hand side is a constant (i.e., \( p \)-independent) solution of (??), (??), and all entries in a matrix row \( a^i = \{ a_{ij}^{\alpha} \}_{\alpha=1}^{n} \) are acting equivalently as shift operators for \( p \):

\[
p a^i = a^{i} (p + v^{(i)}) \quad \text{or} \quad p_{jk} a^i = a^{i} (p_{jk} + \delta_j^{i} - \delta_k^{i}) \quad \text{for} \quad p_{jk} = p_j - p_k . \tag{0.7}
\]
Remark 1
It has been pointed out \cite{?, ?} that, in the \(su(2)\) case, the matrix \(a\) generates the \(q\)-Clebsh–Gordan coefficients while \(R(p)\) plays the role of a "quantum 6j-symbol" \cite{?, ?, ?}.

Remark 2
Eq.\(\text{(??)}\) is related to the one with indices 1 and 2 interchanged,
\[
\hat{R}(p)\ a_2\ a_1 = a_2\ a_1\ \hat{R},
\]
by the substitution \(\hat{R} \rightarrow P\hat{R}P\), \(\hat{R}(p) \rightarrow P\hat{R}(p)P\). It is, on the other hand, formally obtained from
\[
\hat{R}\ \bar{a}_2\ \bar{a}_1 = \bar{a}_2\ \bar{a}_1\ \hat{R}(p),
\]
by the substitution \(\bar{a} = a^{-1}\); the same substitution relates \(\text{(??)}\) to
\[
\hat{R}\ \bar{a}_1\ \bar{a}_2 = \bar{a}_1\ \bar{a}_2\ \hat{R}(p).
\]
Since \(P\hat{R}(p)P\) satisfies the same conditions as \(\hat{R}(p)\), we can start with either of these relations.

The QDYBE, introduced by Gervais and Neveu \cite{?} for the exchange algebra associated with the Liouville equation and applied to the zero mode algebra \cite{?} of the Wess–Zumino–Novikov–Witten (WZNW) model \cite{?, ?, ?} is attracting ever more attention. Its classical counterpart, introduced in \cite{?} (see also \cite{?}) has been displayed in \cite{?} for the \(sl(2)\) case and in \cite{?} for an arbitrary simple Lie algebra. The quantum \(\hat{R}(p)\) is central to a continuing study \cite{?} of \(q\)-deformed cotangent bundles on group manifolds and quantum model spaces. A quasiHopf-algebraic point of view is taken in \cite{?} where \(\hat{R}(p)\) is obtained by a Drinfeld twist of the constant \(R\)-matrix. Felder \cite{?} explores the more general case of (classical and) QDYBE depending on a spectral parameter and finds elliptic solutions of this equation. These solutions are applied in \cite{?} to quantize Calogero–Moser and Ruijsenaars–Schneider models. A class of \(SL(n|m)\)-type solutions of the QDYBE (and related trigonometric solutions of the equation with spectral parameter) are described in \cite{?}. A more mathematically minded approach to the subject in terms of "\(q\)-algebroids" is being developed in \cite{?}.

The present work was motivated in part by earlier study \cite{?, ?, ?} of the canonical quantization of the WZNW model (following \cite{?, ?, ?, ?, ?}). It was noticed, in particular, that the exchange relations \cite{?} for the chiral zero modes \(a_\alpha^i\) that diagonalize the \(U_q(sl(2))\) monodromy matrix can be written in the form \(\text{(??)}\). As a result, the operator realization of the chiral group valued field was understood as a quantization of the (deformed) classical Poisson bracket relations of \cite{?} thus opening the way to its generalization for \(SU(n)\). Here we show that a special solution of the QDYBE \(\text{(??)}\) yields a new matrix representation of the Hecke algebra. We concentrate on a general study of the Hecke algebra properties of this solution and the ensuing properties of the quantum matrices satisfying \(\text{(??)}\) relegating applications to the WZNW model to a subsequent publication \cite{?} which is highlighted in Section 6. A central result is the computation of the quantum determinant of \(a\) and the (based on it) evaluation of the inverse quantum matrix.

The paper is organized as follows. We review and extend in Section 1 results of Gurevich \cite{?} on quantum (anti)symmetrizers and illustrate them in Section 2 on the known example of a constant \(\hat{R}\). We proceed in Section 3 to a study of a family of
SL(n)-type dynamical $R$-matrices and describe two types of symmetry transformations for this family: the twist transformation (a version of Drinfeld’s twist for dynamical $R$-matrices) and the canonical shifts. In Section 4 we show that these dynamical matrices provide a new realization of the Hecke algebra. This allows to define "dynamical" ($p$-dependent) analogs of quantum antisymmetrizers, including the Levi–Civita $E$ tensor. In Section 5 we study the quantum algebra $A$, define the quantum determinant $\det (a)$ and compute the inverse matrix $a^{-1}$. We demonstrate that $A$ provides a realization of a reflection equation algebra which is interpreted as a quantum monodromy algebra in the WZNW theory. An Appendix is devoted to deriving some useful identities for the parameters determining the solution of the QDYBE found in [?] and to computing the normalization of the dynamical Levi–Civita tensor.

1 Hecke algebras and $q$-antisymmetrizers

In this Section we collect some basic notions on Hecke algebras and describe the $q$-antisymmetrizers technique which is to be applied later on. We follow closely the approach of D. Gurevich [?].

In the present context by a Hecke algebra $H_k(q)$ we understand a $\mathbb{C}$-algebra with generators $1, g_1, g_2, \ldots, g_{k-1}$, a nonzero parameter $q \in \mathbb{C}$, and defining relations

\begin{align*}
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \quad \text{for } 1 \leq i \leq k - 2, \quad (1.1) \\
g_i^2 &= 1 + (q - \bar{q}) g_i \quad \text{for } 1 \leq i \leq k - 1, \quad (1.2) \\
g_i g_j &= g_j g_i \quad \text{if } |i - j| \geq 2, \quad (1.3)
\end{align*}

where $\bar{q} := q^{-1}$.

We shall consider the set of idempotents $A^{(j)} \in H_k(q)$, $j = 1, \ldots, k$, associated with single column Young diagrams containing $j$ nodes – the so-called $q$-antisymmetrizers. Their inductive definition is given by

\begin{equation}
A^{(1)} = 1, \quad A^{(j)} = \frac{1}{[j]} A^{(j-1)} \left( q^{j-1} - [j-1] g_{j-1} \right) A^{(j-1)}. \quad (1.4)
\end{equation}

Here $[j] = (q^j - \bar{q}^j)/(q - \bar{q})$ and we assume $[j] \neq 0$, for $j = 2, \ldots, k$. Note, that $A^{(k)}$ is a central idempotent in the algebra $H_k(q)$.

Equivalently, one may write

\begin{equation}
A^{(j)} = \frac{1}{[j]} A^{(2j-1)} \left( q^{j-1} - [j-1] g_1 \right) A^{(2j-1)}, \quad (1.5)
\end{equation}

where we have adopted the notation $A^{(i,j)}$, $1 \leq i \leq j$, for the central idempotent of the subalgebra $H_{i,j}(q) \subset H_j(q)$ generated by the subset $1, g_i, \ldots, g_{j-1}$. In particular, $A^{(1,j)} = A^{(j)}$, $A^{(j,j)} = 1$.

**Remark 1.1** All the subalgebras $H_{i,r+i}(q) \subset H_k(q)$, $i = 1, \ldots, k-r$, are isomorphic by definition. Moreover, they are related by inner $H_k(q)$-automorphisms. For example, the automorphism $\phi_i : H_{i,r+i}(q) \to H_{i+1,r+i+1}(q)$ is given by

\begin{equation}
\phi_i(t) = g_i g_{i+1} \cdots g_{r+i} t (g_i g_{i+1} \cdots g_{r+i})^{-1}, \quad \forall t \in H_{i,r+i}(q). \quad (1.6)
\end{equation}
The term $q$-antisymmetrizer for the elements $A^{(j)}$ is justified by the following properties:

\begin{align*}
(g_i + q)A^{(j)} &= A^{(j)}(g_i + q) = 0 \quad \text{for } 1 \leq i \leq j - 1, \quad (1.7) \\
A^{(j)}A^{(i,l)} &= A^{(i,l)}A^{(j)} = A^{(j)} \quad \text{for } 1 \leq i \leq l \leq j. \quad (1.8)
\end{align*}

**Remark 1.2** Replacing $q$ by $(-q^{-1})$ in (1.7) leads to another sequence of projectors, called symmetrizers. Abstractly, inside the Hecke algebra, it is a matter of convention - which projectors one calls symmetrizers, and which - antisymmetrizers. We use the common convention. However, on the level of representations, when one can calculate the ranks of projectors and see which sequence of projectors terminates, the distinction between symmetrizers and antisymmetrizers becomes meaningful.

Consider a representation $\rho_{W,k} : H_k(q) \to \text{Aut}(W)$ of the algebra $H_k(q)$ in a vector space $W$.

**Definition 1.1** We shall say that $\rho_{W,k}$ is a representation of height $n$ in one of the following two cases:

a) $n < k$ and the conditions

\begin{align*}
\rho_{W,k}(A^{(n+1)}) &= 0, \quad (1.9) \\
\text{rank } \rho_{W,k}(A^{(n)}) &= 1. \quad (1.10)
\end{align*}

are fulfilled, or

b) $n = k$ and $\text{rank } \rho_{W,n}(A^{(n)}) = 1$.

**Remark 1.3** The notion of height of a Hecke algebra representation was introduced in [?] for the special case of the representations generated by constant $R$-matrices. There it was named the rank of the $R$-matrix. We have changed the notation here in order to avoid a possible confusion with the standard notion of rank of a matrix. Note that the use of term ‘height’ is suggested by the fact that imposing condition (1.7) for the representation $\rho_{W,k}$ results in vanishing of any central (and primitive) idempotent related to a Young diagram (standard tableaux) containing more than $n$ boxes in one of its columns.

**Remark 1.4** In view of Remark 1.1, the whole sets of $q$-antisymmetrizers $\{\rho_{W,k}(A^{(i,n+i)})\}_{i=1,\ldots,k-n}$ and $\{\rho_{W,k}(A^{(j,n+j-1)})\}_{j=1,\ldots,k-n+1}$ satisfy conditions (1.7) and (1.8), respectively.

**Remark 1.5** Instead of using (1.7) one can impose the condition

\begin{equation}
A^{(n+1)} = 0 \quad (1.11)
\end{equation}

at the algebraic level. This is the way how generalized Temperley-Lieb-Martin algebras are defined (cf. [?]). Below we present several useful equivalent forms of this condition.

**Lemma 1.1** The condition (1.11) is equivalent to any of the following relations

\begin{align*}
A^{(n)}g_1 \cdots g_{n+1} &= (-1)^n q [n] A^{(n)} A^{(2,n+1)}, \quad (1.12a) \\
g_1g_2 \cdots g_n A^{(n)} &= (-1)^{n-1} q [n] A^{(2,n+1)} A^{(n)}, \quad (1.12b)
\end{align*}
\begin{align}
  g_n \cdots g_2 g_1 A^{(2,n+1)} &= (-1)^{n-1} q [n] A^{(n)} A^{(2,n+1)}, \\
  A^{(2,n+1)} g_1 g_2 \cdots g_n &= (-1)^{n-1} q [n] A^{(2,n+1)} A^{(n)}, \\
  A^{(n)} A^{(2,n+1)} A^{(n)} &= [n]^{-2} A^{(n)}, \\
  A^{(2,n+1)} A^{(n)} A^{(2,n+1)} &= [n]^{-2} A^{(2,n+1)}. 
\end{align}

**Proof.** Applying repeatedly (1.12) for the $q$-antisymmetrizers that appear as last factors in the resulting products and using (1.12), (1.13) we find

\[ A^{(n+1)} = \frac{1}{[n+1]} A^{(n)} \left( q^n - [n] g_n A^{(n)} \right) \]

Next, we apply $A^{(2,n+1)}$ to both sides of Eq.(1.13). Using again (1.12) and (1.13) we obtain

\[ A^{(n+1)} = \frac{1}{[n+1]} \left\{ q^n A^{(n)} + A^{(n)} g_n(q^{n-1} - [n-1] g_{n-1}) A^{(n-1)} \right\} \]

Taking into account the relation $(g_n g_{n-1} \cdots g_1) A^{(2,n+1)} = A^{(n)} (g_n g_{n-1} \cdots g_1)$ which is a consequence of (1.12) we end up with

\[ A^{(n+1)} = \frac{1}{[n+1]} \left\{ q^n A^{(n)} A^{(2,n+1)} + (-1)^n A^{(n)} g_n g_{n-1} \cdots g_1 \right\}. \]

This proves the equivalence of Eqs.(1.12a) and (1.12a). A similar argument using iteratively a substitution of the first $q$-antisymmetrizer in the right-hand side of (1.12) implies the equivalence of (1.12b) and (1.12d) in the same manner starting from Eq.(1.12).

To show equivalence of (1.12e) to (1.12e) and to (1.12f) one should employ Eqs.(1.12e) and (1.12f), respectively. We shall treat the case of (1.12e) here.

Consider the difference

\[ [n] A^{(n)} A^{(2,n+1)} A^{(n)} - A^{(n)} = A^{(n)} \left\{ [n] A^{(2,n+1)} - 1 \right\}. \]

where we have again used the definition (1.12) and the relations (1.12). Comparing the first and the last lines of the calculation we deduce the equivalence of conditions (1.12) and (1.12c).

Eqs.(1.12a–f) display properties of the rank 1 idempotents $\rho_{N,k} (A^{(n)})$ that are hidden in (1.12). In fact, they are the basic technical tools which one needs to effectively deal with the height $n$ Hecke algebra representations.
In the rest of the paper we make use of a special type of representations of the algebras $H_k(q)$ for which the representation space is given by $k$-th tensor power of an $(n\text{-dimensional, in the case of } SL(n))$ vector space $V$: $W = V^\otimes k$. These representations are generated by constant or dynamical $R$-matrices of Hecke type. The representations we are dealing with have the specific feature that their height, when defined (i.e., for $k \geq n$), coincides with the dimension of the space $V$. Below we first illustrate the general notions introduced above on the well known case of constant $SL(n)$-type $R$-matrices relegating the study of dynamical $R$-matrices to Sections 3 and 4.

2 Representations generated by a constant $R$-matrix of $SL(n)$ type

The $R$-matrix corresponding to the Drinfeld-Jimbo deformation of $SL(n)$ is an operator acting in a tensor square of an $n$-dimensional vector space $V$ and given by

$$\hat{R}^\alpha_{\beta_1 \beta_2} = q^{\delta_{\alpha_1 \alpha_2}} \delta_{\beta_1}^{\beta_2} + (q - \overline{q}) \theta_{\alpha_2 \alpha_1} \delta_{\beta_1}^{\alpha_2} \delta_{\beta_2}^{\alpha_1},$$

(no summation in the right-hand side) where the indices $\alpha, \beta$ take values from 1 to $n$, and $\theta_{\alpha \beta} = \{1$ if $\alpha > \beta$, 0 if $\alpha \leq \beta \}$. This $R$-matrix is a particular representative of a family of constant Hecke $R$-matrices, i.e., it satisfies the braid relation and the Hecke condition

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23},$$
$$\hat{R}^2 = 1 + (q - \overline{q}) \hat{R}. \tag{2.2}$$

Eqs. (2.1), (2.2) imply that the matrices $\hat{R}_{12}, \hat{R}_{23}$ generate a representation of $H_3(q)$ in $V^\otimes 3$. For an arbitrary $k$ the representation $\rho_{\hat{R},k} : H_k(q) \rightarrow \text{Aut}(V^\otimes k)$ generated by a constant Hecke $R$-matrix is defined by

$$\rho_{\hat{R},k}(g_i) = \hat{R}_{i+1}. \tag{2.4}$$

For representations $\rho_{\hat{R},k}$ generated by the $R$-matrix (2.1) we have

$$\text{height } \rho_{\hat{R},k} = n$$

if $k \geq n$. The rank 1 $q$-antisymmetrizers $\rho_{\hat{R},k}(A^{\{a_{n+i-1}\}})$ are most conveniently described in terms of $q$-analogues of (co- and contravariant) Levi–Civita tensors which are solutions of the equations

$$\varepsilon_{a_1 \ldots a_{i+1} \ldots a_n} \hat{R}^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_{i+1} \ldots \beta_n} = -\overline{q} \varepsilon_{a_1 \ldots a_{i+1} \ldots a_n},$$
$$\varepsilon_{a_1 \ldots a_{i+1} \ldots a_n} \hat{R}^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_{i+1} \ldots \beta_n} = -\overline{q} \varepsilon_{a_1 \ldots a_{i+1} \ldots a_n}, \quad i = 1, 2, \ldots, n-1.$$

It is straightforward to prove that these equations have unique (up to normalization) solutions. The rank 1 condition (2.2) follows as a corollary.

\textsuperscript{6}In general, this need not be the case. Examples with height $\rho \neq \text{dim } V$ were constructed in [?].
In the special case of representations $\rho_{R,k}$ generated by the $R$-matrix (2.1) the only nonvanishing components of the $\varepsilon$-tensors have pairwise different indices $\alpha_1, \alpha_2, \ldots, \alpha_n$, and can be chosen as

$$\varepsilon^{\alpha_1 \alpha_2 \cdots \alpha_n} = q^{n(n-1)/2} (-q)^{\ell(\sigma)} , \quad \varepsilon_{\alpha_1 \alpha_2 \cdots \alpha_n} = (-q)^{\ell(\sigma)} . \tag{2.5}$$

Here $\ell(\sigma)$ is the length of the permutation $\sigma = (1,2,n)$. The rank 1 $q$-antisymmetrizers are expressed in terms of the $\varepsilon$-tensors as

$$\rho_{R,k}(A(i,n+i-1)) = \frac{1}{[n]!} \varepsilon^{[i \cdots n+i-1]} \varepsilon_{[i \cdots n+i-1]} , \quad i = 1, \ldots, k - n + 1 . \tag{2.6}$$

Here (by analogy with the matrix notation) we substitute the vector space indices of $\varepsilon$-tensors by their labels: $\alpha_i \to i$. The "bra" and "ket" notation of $\varepsilon$-tensor indices is used in order to distinguish labels of matrix spaces from those of vector spaces. One should have in mind the following symbolic decomposition for the matrix space label: $i = |i⟩ \otimes 〈i|$. For example, the equation $A_i^j |b⟩ \equiv A^i_{(i)} |b⟩ = v^i |b⟩$ is to be understood as $\sum_{\sigma} A^\sigma_{(i)} u^\sigma = v^i$.

Finally, we shall adapt for $\rho_{R,k}$ those formulas (1.12a-f) which will be used in Section 5. Written in terms of the $\varepsilon$-tensors the relations (1.12b), (1.12c), (1.12e) and (1.12f) assume the form

$$\rho_{R,n+1}(g_1 \cdots g_n) \varepsilon^{[1 \cdots n]} = \hat{R}_{12} \cdots \hat{R}_{n+1} \varepsilon^{[1 \cdots n]} = q \varepsilon^{[2 \cdots n+1]} N^{[1]}_{(n+1)} , \tag{2.7}$$

$$\rho_{R,n+1}(g_n \cdots g_1) \varepsilon^{[2 \cdots n+1]} = \hat{R}_{n,n+1} \cdots \hat{R}_{12} \varepsilon^{[2 \cdots n+1]} = q \varepsilon^{[1 \cdots n]} K^{[n+1]}_{(1)} , \tag{2.8}$$

$$KN = NK = 1 . \tag{2.9}$$

Here the matrices $N$ and $K$ are defined as

$$N^{[1]}_{(n+1)} = \frac{(-1)^{n-1}}{[n-1]!} \varepsilon^{[1 \cdots n]} , \tag{2.10}$$

$$K^{[n+1]}_{(1)} = \frac{(-1)^{n-1}}{[n-1]!} \varepsilon^{[2 \cdots n+1]} , \tag{2.11}$$

and for the $\varepsilon$-tensors given by Eq.(2.5) we have just

$$N = K = 1 . \tag{2.12}$$

3 $SL(n)$-type dynamical $R$-matrices

We now turn to the dynamical $R$-matrix defined in the Introduction. In order to present the QDYBE in a form suitable for our purposes we shall introduce a set of commutative variables

$$X^i , i = 1, \ldots, n , \quad [X^i, X^j] = 0 , \quad \prod_{i=1}^{n} X^i = 1 . \tag{3.1a}$$
which play the role of elementary shift operators for \( p_i : \)
\[
p X^i = X^i ( p + v(i) ) .
\] (3.1b)

The elements \( X^i \) and \( q^p_i \) provide a realization of (the Weyl’s form of) the canonical commutation relations. Note that in concrete applications they can be naturally identified with (a subset of) dynamical variables of a model (see, e.g., [?]).

Let us arrange the auxiliary variables \( X^i \) into a unimodular diagonal matrix,
\[
X = diag \{ X^1, \ldots, X^n \} , \quad \det(X) = 1 .
\] (3.2)

The Hecke-type dynamical \( R \)-matrix is characterized by the following set of relations:
\[
R_{12}(p)(X_1 R_{23}(p)X_1^{-1})R_{12}(p) = (X_1 R_{23}(p)X_1^{-1})R_{12}(p)(X_1 R_{23}(p)X_1^{-1}) ,
\] (3.3)
\[
\hat{R}(p)^2 = 1 + (q - \overline{q}) \hat{R}(p) ,
\] (3.4)
\[
\hat{R}_{12}(p)X_1X_2 = X_1X_2 \hat{R}_{12}(p) .
\] (3.5)

Here the first and second relations are the dynamical Yang-Baxter equation and the Hecke condition, respectively. A condition of type (??), although not always imposed on dynamical \( R \)-matrices, is also necessary in our treatment. As we shall see below, it ensures that conditions (??) for the Hecke algebra representations generated by \( \hat{R}(p) \) are satisfied.

Following [?] we shall consider dynamical \( R \)-matrices of the form
\[
\hat{R}^{ij}_{j_1j_2}(p) = a_{i_1i_2}(p) \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} + b_{i_1i_2}(p) \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} , \quad i_{1,2}, j_{1,2} = 1, \ldots, n
\] (3.6)

(there is no summation over repeated upper and lower indices in the right-hand side); in order to have a unique decomposition in terms of the unit and the permutation matrices in the tensor square of spaces we impose the condition \( b_{ii}(p) = 0 \). This special class of \( p \)-dependent Hecke \( R \)-matrices will be called dynamical \( R \)-matrices of an \( SL(n) \)-type 7.

The unknown functions \( a_{ij}(p), b_{ij}(p) \) in the Ansatz (??) are to be fixed by the conditions (??-??). The Hecke condition (??) gives
\[
b_{ii} = 0 , \quad b_{ij} + b_{ji} = q - \overline{q} , \quad \text{for} \quad i \neq j ,
\] (3.7)
\[
a_{ij}a_{ji} - b_{ij}b_{ji} = 1 , \quad \text{for} \quad i \neq j ,
\] (3.8)
\[
a_{ii}^2 - (q - \overline{q}) a_{ii} = 1 .
\] (3.9)

The last equation has two solutions: \( a_{ii} = \pm q^{\pm 1} \) for each \( i \). Below we consider only the case \( a_{ii} = q, \ \forall i \) (the other cases correspond, in particular, to quantum supergroups and have been considered in [?]). Finally, the dynamical Yang-Baxter equation (??) and Eq.(??) impose the constraints
\[
a_{ij}(p_1, \ldots, p_n) = a_{ij}(p_{ij}) , \quad b_{ij}(p_1, \ldots, p_n) = b_{ij}(p_{ij}) ,
\]

7In [?] these \( R \)-matrices were called \( GL(n) \)-type \( R \)-matrices. We call them \( SL(n) \)-type \( R \)-matrices instead since we impose the additional condition \( \sum_{i=1}^n p_i = 0 \) (see (??)) on the variables \( p_i \).
\[ b_{ij} b_{jk} b_{ki} + b_{ik} b_{kj} b_{ji} = 0 , \quad (3.10) \]

\[ b_{ij}(p_{ij} + 1) = \frac{b_{ij}(p_{ij}) q}{q + b_{ij}(p_{ij})} , \quad (3.11) \]

where \( p_{ij} := p_i - p_j \). For \( a_{ii} = q \) the general solution of (3.10) - (3.11) can be written as [?]

\[ a_{ij}(p) = a_{ij}(p_{ij}) \xi_{ij}(p_{ij}) , \quad b_{ij}(p) = q - \xi_{ij}(p_{ij}) , \quad (3.12) \]

where \( \xi_{ij}(p) \) are expressed as the following ratios:

\[ \xi_{ij}(p) = \frac{f(p_{ij} - 1, \beta_{ij})}{f(p_{ij}, \beta_{ij})} , \quad f(p, \beta) = \beta^p + |p|\beta . \quad (3.13) \]

Here \( \beta_{ij}(p_{ij}) = \beta_{ij}(p_{ij} + 1) \). We shall consider \( \beta_{ij} \) as constant parameters since their functional dependence does not change any of the results below. The function \( f(p, \beta) \) satisfies the finite difference equation

\[ f(p + 1, \beta) + f(p - 1, \beta) = [2]f(p, \beta) \quad (3.14) \]

with the initial conditions

\[ f(0, \beta) = 1 , \quad f(1, \beta) = \bar{q} + \beta . \quad (3.15) \]

Equations (3.14) can be deduced from (3.15).

Not all of the remaining in (3.10), (3.11) parameters \( \alpha_{ij}(p_{ij}) \) and \( \beta_{ij} \) are independent. The relations between them are given by

\[ \alpha_{ii} = 1 , \quad \alpha_{ij}(p_{ij}) \alpha_{ji}(p_{ji}) = 1 , \quad (3.16) \]

\[ \beta_{ii} = 0 , \quad \beta_{ij} + \beta_{ji} = q - \bar{q} \quad \text{for} \quad i \neq j , \quad (3.17a) \]

\[ \beta_{ij} \beta_{jk} \beta_{ki} + \beta_{ik} \beta_{kj} \beta_{ji} = 0 . \quad (3.17b) \]

An easy way to solve Eqs.(3.10), (3.11) is to make the substitution

\[ \beta_{ij} = \frac{q - \bar{q}}{1 - \pi_{ij}} \iff \pi_{ij} = \frac{\beta_{ij} - q + \bar{q}}{\beta_{ij}} , \quad \text{for} \quad i \neq j . \quad (3.18a) \]

We stress that the parameters \( \pi_{ij} \) are not fixed here and can be chosen as arbitrary constants. In terms of new variables \( \pi_{ij} \) equations (3.10), (3.11) take the simple form

\[ \pi_{ij} \pi_{ji} = 1 , \quad \pi_{ij} \pi_{jk} \pi_{ki} = 1 , \quad (3.18b) \]

and are solved by \( \pi_{ij} = \prod_{k=1}^{j-1} \pi_{kk+1} = \pi_{ji}^{-1} \), for \( i < j \). Hence,

\[ \beta_{ij} = \frac{(q - \bar{q}) \prod_{k=1}^{j-1} \beta_k}{\prod_{k=1}^{j-1} \beta_k - \prod_{k=1}^{j-1} (\beta_k - q + \bar{q})} , \quad \text{for} \quad i < j , \quad (3.19) \]

where the remaining \((n - 1)\) parameters \( \beta_i \equiv \beta_{i+1} \) are independent.

We shall describe two types of transformations on the set of \( SL(n) \)-type dynamical \( R \)-matrices.
1. a simple version of *twisting* for dynamical $R$-matrices:

$$\hat{R}(p) \rightarrow \hat{F} R_{21} = \hat{F} \hat{R}(p) \hat{F}^{-1},$$

where $\hat{F}(p) = F_{12}(p)P_{12}$ and

$$F_{12} \equiv F_{j_1 j_2}^{i_1 i_2}(p) = \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \psi_{i_1 i_2}(p),$$

(3.20)

(analog of Drinfeld’s twist, see [?]). An explanation on how this twist works is given in the two Lemmas below.

**Lemma 3.1** Let $\hat{F}(p)$ be an operator acting in $V \otimes V$. If

$$\hat{F}_{12}^{-1} X^{-1}_{1} \hat{F}_{23} = \hat{A}_{123} X_{3}^{-1} \hat{A}_{123}$$

(3.21)

for some operator $\hat{A}(p)$ acting in $V \otimes V \otimes V$ and

$$\hat{R}(p)_{12} \hat{A}_{123} = \hat{A}_{123} \hat{R}(p)_{23}$$

(3.22)

then the matrix $\hat{F} R_{21} \equiv \hat{F} \hat{R}(p) \hat{F}^{-1}$ satisfies the QDYBE (??).

**Proof.** Substitute $\hat{F} R_{21}$ in the QDYBE taking into account that the QDYBE has two more equivalent forms:

$$\hat{R}(p)_{23} X_{3}^{-1} \hat{R}(p)_{12} X_{3} \hat{R}(p)_{23} = X_{3}^{-1} \hat{R}(p)_{12} X_{3} \hat{R}(p)_{23} X_{3}^{-1} \hat{R}(p)_{12} X_{3},$$

$$\hat{R}(p)_{21} X_{1}^{-1} \hat{R}(p)_{32} X_{1} \hat{R}(p)_{21} = X_{1}^{-1} \hat{R}(p)_{32} X_{1} \hat{R}(p)_{21} X_{1}^{-1} \hat{R}(p)_{32} X_{1}.$$

The first equation results from repeated application of (??) to (??). The second equation is obtained from the first one by simply permuting the subscripts 1 and 3.

**Lemma 3.2** Let $F_{12}$ be the diagonal matrix (??) where $\psi_{ij} \psi_{ji} = 1$ and $\psi_{ii} = 1$. Assume that $\psi_{ij}$ depends on $p_{ij}$ only. Then, Eqs. (?), (??) are satisfied by $\hat{A} = AP_{23}F_{12}$ where the matrix $A$ is diagonal, $A_{ik}^{ab} = a_{ijk} \delta_{i}^{a} \delta_{j}^{b} \delta_{k}^{c}$, the elements $a_{ijk}$ being given by

$$a_{ijk} = \begin{cases} \psi_{ik} \psi_{jk} & \text{if } i \neq j, \\ [\psi_{ik}(p_{ik} + 1)]^{2} & \text{if } i = j. \end{cases}$$

(3.23)

**Proof.** The operator $A_{123}$ is symmetric in the first two indices, $A_{123} = A_{213}$ which implies that it commutes with any $R$-matrix of the form (??). Therefore, (??) is satisfied.

Eq.(??) can be checked directly.

These Lemmas demonstrate that the operator (??) indeed generates a twist leading to the changes $\alpha_{ij} \rightarrow \alpha_{ij} \psi_{ij}^{2}$, $\beta_{ij} \rightarrow \beta_{ij}$ of the parameters in (??), (??).

2. **canonical transformations** of the dynamical parameters $p_{i} \rightarrow p_{i} + c_{i}$, where $c_{i}, i = 1, \ldots, n$ are arbitrary constants satisfying the condition $\sum_{i=1}^{n} c_{i} = 0.$
We conclude the Section by a brief discussion of the structure of the family of $SL(n)$-type dynamical $R$-matrices (??), (??), (??). There are two essentially different domains for the parameters $\beta_i$ of this family.

a) $\beta_i \neq 0$ and $\beta_i \neq q - \bar{q}$, for all $i$.
In this case the whole family (??), (??), (??) can be generated starting from any particular representative with the use of the two types of transformations described above.

Indeed, the parameters $\alpha_{ij}$ can be excluded with the help of a twist type transformation; then, performing a canonical transformation of the form

$$q^{2p_{ij}} \rightarrow q^{2p_{ij}} \pi_{ij} = q^{2p_{ij}} \prod_{k=1}^{j-1} \frac{\beta_k - q + \bar{q}}{\beta_k}$$

for instance, one excludes the parameters $\beta_{ij}$ from the Ansatz (??), (??), (??) and passes to a dynamical $R$-matrix with

$$\xi_{ij}(p) = \frac{|p_{ij} - 1|}{|p_{ij}|}$$

instead of (??). This $R$-matrix is the limiting case $\beta_i \rightarrow \infty$ of our family, and it is this type of dynamical $R$-matrix which is discussed in [?].

b) Either all $\beta_i = q - \bar{q}$, or all $\beta_i = 0$.
We shall consider the first case $\beta_i = q - \bar{q}$ for which

$$\beta_{ij} = \begin{cases} q - \bar{q} & \text{for } i < j \\ 0 & \text{for } i \geq j \end{cases}$$

and put $\alpha_{ij}(p_{ij}) = \text{const}_{ij}$. In this case the $R$-matrix (??), (??), (??) becomes independent on the dynamical variables $p_i$ and is reduced to the constant $R$-matrix describing the multiparametric [?] deformations of $GL(n)$ which are all twist-equivalent.

With the particular choice

$$\alpha_{ij} = \begin{cases} q & \text{for } i < j \\ 1 & \text{for } i = j \\ \bar{q} & \text{for } i > j \end{cases}$$

one reproduces the standard $SL(n)$-type $R$-matrix (??).

Remark 3.1 In the intermediate cases (where only a part of the parameters $\beta_i$ are equal to 0 or $q - \bar{q}$) the corresponding dynamical $R$-matrix $\hat{R}(p)$ contains the (dynamical and constant) $R$-matrices described in a) and b) as submatrices.

4 Representations generated by $SL(n)$-type dynamical $R$-matrices

Now we are in a position to introduce the Hecke algebra representations associated with Hecke-type dynamical $R$-matrices.
Proposition 4.1 Let \( \hat{R}(p) \) be a dynamical \( R \)-matrix of the Hecke type. The matrices
\[
\rho_{\hat{R}(p),k}(g_i) = (X_1X_2 \ldots X_{i-1})R_{i,i+1}(p)(X_1X_2 \ldots X_{i-1})^{-1}, \quad i = 1, \ldots, k-1,
\] (4.1a)
generate a Hecke algebra representation, \( \rho_{\hat{R}(p),k} : H_k(q) \to \text{Aut}(V^\otimes k) \).

**Proof.** Obviously, equation (4.1a) imply that the matrices \( \rho_{\hat{R}(p),k}(g_i) \) and \( \rho_{\hat{R}(p),k}(g_{i+1}) \) satisfy the braid relations (4.1a). Then, the conditions (4.1a) ensure that the matrices (4.1a) satisfy (4.1a) and, therefore, (4.1a) represent the generators of the braid algebra \( B_k \). Finally, the Hecke conditions (4.1a) for the generators (4.1a) follow from the Hecke property (4.1a) of the dynamical \( R \)-matrix (4.1a).

**Remark 4.1** In contrast with the case of constant Hecke \( R \)-matrix (4.1a) the representations generated by a dynamical Hecke \( R \)-matrix are nonlocal; in other words, the matrices \( \rho_{\hat{R}(p),k}(g_i) \) act nontrivially as diagonal matrices on \( V_j \) with \( j < i \) (and not merely on \( V_i \otimes V_{i+1} \)). Only the representation of the first generator with \( i = 1 \) has the usual ‘locality’ property.

**Remark 4.2** One can construct representations equivalent to \( \rho_{\hat{R}(p),k} \) in which some other generator is ‘localized’ instead \( g_1 \). For instance, the representation \( \overline{\rho}_{\hat{R}(p),k} \) which localizes \( \rho_{\hat{R}(p),k}(g_{k-1}) \) is given by
\[
\overline{\rho}_{\hat{R}(p),k}(t) = (X_1X_2 \ldots X_k)^{-1} \rho_{\hat{R}(p),k}(t)(X_1X_2 \ldots X_k), \quad \forall \ t \in H_k(q)
\] (4.1b)
so that
\[
\overline{\rho}_{\hat{R}(p),k}(g_i) = (X_{i+2} \ldots X_k)^{-1} \hat{R}_{i,i+1}(p)(X_{i+2} \ldots X_k).
\] (4.1c)
Note that in addition to the nonlocal property the representation matrices of \( \overline{\rho}_{\hat{R}(p),k} \) depend explicitly on \( k \).

From now on we shall restrict ourselves to discussing those representations \( \rho_{\hat{R}(p),k} \) which are generated by the \( SL(n) \)-type dynamical \( R \)-matrices (4.1a), (4.1a), (4.1a). For \( k \geq n \) all these representations are of height \( n \). The rank 1 \( q \)-antisymmetrizers are conveniently expressed in terms of dynamical \( E \)-tensors \( E^{[1..n]}(p) \) and \( E_{[1..n]}(p) \), which are the unique (up to normalization) solutions of the equations
\[
\rho_{\hat{R}(p),k}(g_i) E^{[1..n]}(p) = -q E^{[1..n]}(p), \quad 1 \leq i \leq n-1.
\] (4.2)
The only nonvanishing components of these \( E \)-tensors have pairwise different indices \( i_1, i_2, \ldots, i_n \) and look like
\[
E^{i_1i_2\ldots i_n}(p) = (-1)^{\ell(\sigma)} \prod_{(j,a) \in J(\sigma)} \alpha_{ji}(p_{ji}) \prod_{1 \leq a < b \leq n} \xi_{i_a i_b}(p_{i_a i_b}) ,
\] (4.3)
\[
E_{i_1i_2\ldots i_n}(p) = (-1)^{\ell(\sigma)} \prod_{(j,a) \in \widetilde{J}(\sigma)} \alpha_{ij}(p_{ij}) .
\] (4.4)
Here \( \ell(\sigma) \) is the length of the permutation \( \sigma = (1,2,\ldots,n) \), and
\[
J(\sigma) = \{(i_a, i_b) : a < b, i_a > i_b \}.
\]

The dynamical \( E \)-tensors (4.1a), (4.1a) are normalized so that they would coincide with the constant \( \varepsilon \)-tensors (4.1a) in the case (4.1a), (4.1a).

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Now the expressions for rank 1 $q$-antisymmetrizers in the representations $\rho_{R(p),k}$ are given by
\[
\rho_{R(p),k}(A^{(i,n+i-1)}) = \frac{1}{[n]!} (X_1 \ldots X_{i-1}) \mathcal{E}^{[i,n+i-1]}(p) \mathcal{E}^{(i,n+i-1)}(p) (X_1 \ldots X_{i-1})^{-1}.
\]
The numeric coefficient in this formula is calculated with the use of the relation
\[
\mathcal{E}^{(1..n)}(p) \mathcal{E}^{[1..n]}(p) = [n]!,
\]
which is proved in the Appendix.

We conclude the discussion on dynamical $R$-matrices by writing down formulas (1.12a), (1.12d), and (1.12e), (1.12f) for the representation $\rho_{R(p),n+1}$:
\[
\mathcal{E}^{(1..n+1)}(p) \mathcal{E}^{(1..n)}(p) = q K^{[n+1]}(p) X_1 \mathcal{E}^{(2..n+1)}(p) X_1^{-1},
\]
\[
X_1 \mathcal{E}^{(2..n+1)}(p) X_1^{-1} \mathcal{E}^{(1..n)}(p) = q N^{[1]}(p) \mathcal{E}^{(1..n)}(p),
\]
\[
K(p) N(p) = N(p) K(p) = I. \tag{4.9}
\]
Here the matrices $N(p), K(p)$ are defined as
\[
N^{[1]}(p) = \frac{(-1)^{n-1}}{[n-1]!} X_1 \mathcal{E}^{(2..n+1)}(p) X_1^{-1} \mathcal{E}^{(1..n)}(p), \tag{4.10}
\]
\[
K^{[n+1]}(p) = \frac{(-1)^{n-1}}{[n-1]!} \mathcal{E}^{(1..n)}(p) X_1 \mathcal{E}^{(2..n+1)}(p) X_1^{-1}. \tag{4.11}
\]
For the $SL(n)$-type dynamical $R$-matrices the matrices $N(p), K(p)$ are diagonal. Inserting formulas (??), (??) for the dynamical $\varepsilon$-tensors into (??), (??) and using (??), one ends up with the following expressions for their diagonal components:
\[
N_{ij}^{[1]}(p) = (K_{ij}^{[1]}(p))^{-1} = \prod_{j \neq i} (p_{ij} - \theta_{ji}) \xi_{ij}(p_{ij}), \tag{4.12}
\]
where $\theta_{ji} = \{1$ if $j > i$, $0$ if $j < i\}$.

5 Quantum matrix algebra $\mathcal{A}(\hat{R}(p), \hat{R})$: quantum determinant and inversion formula

We shall apply the above technique to the quantum matrix algebra $\mathcal{A}$ which is defined as follows (cf. Introduction).

**Definition 5.1** Let $\mathcal{F}$ be the field of the complex meromorphic functions of the (commuting) variables $p_j$, $j = 1, \ldots, n$. Let $\hat{R}(p)$ be a dynamical $n^2 \times n^2$ $R$-matrix and $\hat{R}$ is a $m^2 \times m^2$ constant $R$-matrix. Then $\mathcal{A} = \mathcal{A}(\hat{R}(p), \hat{R})$ is a complex algebra with 1 that is generated by $\mathcal{F}$ and the $mn$ elements $a_{i}^{\alpha}$ ($i = 1, \ldots, n$ and $\alpha = 1, \ldots, m$), satisfying the relations
\[
\hat{R}(p)_{12} a_{1} a_{2} = a_{1} a_{2} \hat{R}(p)_{12}, \tag{5.1}
\]
\[
af(p) = Xf(p)X^{-1}a, \quad \forall f(p) \in \mathcal{F}, \tag{5.2}
\]
where $X$ is a unimodular diagonal matrix (??) whose diagonal elements $X^i$ satisfy (3.1a), (3.1b).
Remark 5.1 The definition above is given for arbitrary \( m \) and \( n \). However in the sequel we shall discuss the case \( m = n \) only.

Remark 5.2 For the applications envisaged here, the field \( \mathbb{F} \) of meromorphic functions of \( p_j \) can be replaced by its subfield of rational functions of \( q^{p_j} \) (as stated in the Introduction). Then we should just require

\[
X^k q^{p_j} (X^k)^{-1} = q^{p_{ij} + \delta_{jk} - \delta_{ik}}
\]

instead of (3.1b). Note that for \( q \) a root of unity, \( q^{2h} = 1 \) (cf. Eq.(6.8) below) \( p_{ij} \) are only determined up to an additive integer multiple of \( 2h \).

Remark 5.3 More general matrix algebras are of interest in which the \( R \)-matrices on both sides of the quadratic relations (5.3) are allowed to depend on possibly different sets of commuting variables \( p \) and \( p' \)

\[
\hat{R}(p)_{12} Q_1 Q_2 = Q_1 Q_2 \hat{R}'(p')_{12},
\]

while the shift properties assume the form

\[
p_{kl} Q_j^i = Q_j^i (p_{kl} + \delta_k^i - \delta_l^i), \quad p'_{kl} Q_j^i = Q_j^i (p'_{kl} + \delta_{jk} - \delta_{jl}).
\]

Such \( \mathcal{A}(\hat{R}(p), \hat{R}'(p')) \) can be treated in much the same way or reduced to the study of two matrix algebras of the above type setting \( Q_j^i = a_a^i a_j^i \), where \( a \) and \( a' \) satisfy exchange relations of the type (5.3) and (5.4), respectively (see [?]). Note that dynamical quantum groups (introduced in [?]) are defined by relations similar to (5.3) and (5.4) but with the dynamical \( R \)-matrices (and momenta \( p, p' \)) related to each other by some equivalence transformation \( \hat{R}'(p') = X^{-1} \hat{R}(p) X \). Another desirable modification of the matrix algebra (5.3) corresponds to the case when \( \hat{R} \) is an \( SO_q \) or \( Sp_q \) constant \( R \)-matrix. In this case \( \hat{R} \) and \( \hat{R}(p) \) satisfy a third order (Birman–Wenzl) condition instead of the Hecke property (5.4) and the QDYBE (5.4) have to be modified correspondingly.

Remark 5.4 The algebra \( \mathcal{A} \) differs from the one considered in [?] where the counterpart of a matrix \( b = X^{-1} a \) is used for changing the basis of chiral vertex operators. It is assumed in [?] that the elements of \( b \) only depend on \( p \) and hence commute among themselves while in our case they belong to the reflection equation (monodromy matrix) algebra \( \mathcal{M}(\hat{R}) \) defined in Proposition 5.5 below. The difference is essential: as a result, Cremmer and Gervais do not recover the standard (constant) \( SL_q(n) \) \( R \)-matrix for \( n > 2 \) but introduce instead new solutions of the Yang-Baxter equation. One of the authors (I.T.) would like to thank J.-L. Gervais and E. Cremmer for an enlightening discussion on this point.

The term ”matrix algebra” ? for the algebra \( \mathcal{A} \equiv \mathcal{A}(\hat{R}(p), \hat{R}) \) is justified by the fact that we shall be able to (define and) compute the determinant of \( a \) – as a function of \( p \) – and to find the inverse of \( a \). In the case of \( 2 \times 2 \) matrices the determinant of \( a \) was constructed in [?] (see also [?]) for the special choice \( \beta_i \to \infty, \alpha_{ij} = 1 \) of the parameters. We shall present the definition of the determinant in a general setting.

? More conventional quantum matrix algebras have been introduced in [?, ?] and their matrix nature were further investigated in [?, ?].
Definition 5.2 Let $a = \|a_{\alpha i}\|$ be the matrix of generators of the algebra $A(\hat{R}(p), \hat{R})$. The determinant of the matrix $a$ is given by

$$\det(a) = \frac{1}{|n|!} \varepsilon_{\{1 \ldots n\}(p)a_1a_2 \ldots a_n} e^{[1 \ldots n]}.$$  \hfill (5.5)\]

The meaning of this definition is made clear by the following three Propositions. The first and the third of them are the quantum analogues of the basic determinant properties. The second one allows to perform an $SL(n)$-reduction in the algebra $A(\hat{R}(p), \hat{R})$.

**Proposition 5.1** The product $(a_1 a_2 \ldots a_n)$ intertwines between constant and dynamical $\varepsilon$-tensors:

$$\varepsilon_{\{1 \ldots n\}(p)a_1a_2 \ldots a_n} = \det(a) \varepsilon_{\{1 \ldots n\}},$$  \hfill (5.6)

$$a_1 a_2 \ldots a_n e^{[1 \ldots n]} = \varepsilon^{[1 \ldots n]}(p) \det(a).$$  \hfill (5.7)

**Proof.** First, observe that due to the relations (??), (??) the product of $k$ matrices $(a_1 a_2 \ldots a_k)$ intertwines between the representations $\rho_{\hat{R},k}$ and $\rho_{\hat{R}(p),k}$ of the algebra $H_k(q)$. Indeed,

$$\begin{align*}
(a_1 \ldots a_k) \rho_{\hat{R},k}(g_i) &= (a_1 \ldots a_k) \hat{R}_{i(i+1)} \\
&= a_1 \ldots a_{i-1} (\hat{R}_{i(i+1)}(p) a_i a_{i+1}) a_{i+2} \ldots a_k \\
&= (X_1 \ldots X_{i-1}) \hat{R}_{i(i+1)}(p) (X_1 \ldots X_{i-1})^{-1} (a_1 \ldots a_k) = \rho_{\hat{R}(p),k}(g_i)(a_1 \ldots a_k).
\end{align*}$$  \hfill (5.8)

In particular, one has

$$(a_1 \ldots a_n) \rho_{\hat{R},n}(A^{(n)}) = \rho_{\hat{R}(p),n}(A^{(n)})(a_1 \ldots a_n).$$

Multiplying both sides by $\rho_{\hat{R},n}(A^{(n)})$ from the right or by $\rho_{\hat{R},n}(A^{(n)})$ from the left and using projector property of the $q$-antisymmetrizer one comes to the equations

$$\begin{align*}
\rho_{\hat{R}(p),n}(A^{(n)})(a_1 \ldots a_n) &= \rho_{\hat{R}(p),n}(A^{(n)})(a_1 \ldots a_n) \rho_{\hat{R},n}(A^{(n)}), \\
(a_1 \ldots a_n) \rho_{\hat{R},n}(A^{(n)}) &= \rho_{\hat{R}(p),n}(A^{(n)})(a_1 \ldots a_n) \rho_{\hat{R},n}(A^{(n)}).
\end{align*}$$  \hfill (5.9), (5.10)

Finally, expressing (??), (??) for constant and dynamical $q$-antisymmetrizers in terms of the $\varepsilon$-tensors, one transforms (??), (??) to the form (??), (??). \hfill $\blacksquare$

**Proposition 5.2** The element $\det(a)$ of the algebra $A(\hat{R}(p), \hat{R})$ commutes with the generators $p_i$ and its commutation with the generators $a^i_\alpha$ is described by

$$\det(a) a = K(p) a \det(a),$$  \hfill (5.11)

where the diagonal matrix $K(p)$ is given in (??), (??).

**Proof.** Consider the permutation of $\det(a)$ with an arbitrary function $h(p)$:

$$\begin{align*}
\det(a) h(p) &= \varepsilon_{\{1 \ldots n\}(p)a_1a_2 \ldots a_n} h(p) e^{[1 \ldots n]} / |n|! \\
&= \varepsilon_{\{1 \ldots n\}(p)(X_n \ldots X_1) h(p)(X_n \ldots X_1)^{-1} a_1 \ldots a_n} e^{[1 \ldots n]} / |n|!
\end{align*}$$  \hfill (5.12)

Since the only nonvanishing components of the tensor $\varepsilon_{\{1 \ldots n\}(p)}$ are those with pairwise different indices and due to the diagonal structure of the matrix $X$ one has

$$\varepsilon_{\{1 \ldots n\}(p)} X_n \ldots X_1 = \varepsilon_{\{1 \ldots n\}(p)} \det(X) = \varepsilon_{\{1 \ldots n\}(p)}.$$
where in the last equality the unimodularity of $X$ (see (??)) is taken into account. Now we can complete the transformation of (??):

$$\det(a)h(p) = \ldots$$

$$= h(p)\mathcal{E}_{(1\ldots n)}(p)(X_1\ldots X_n)^{-1}a_1 \ldots a_n \varepsilon^{[1\ldots n]}/[n]! = h(p)\det(a).$$

This proves commutativity of $\det(a)$ and $p_i$.

Consider now permutation of $\det(a)$ with the matrix $a$. It is technically convenient to take $a$ living in the matrix space with label $(n+1)$:

$$\det(a)a_{n+1} = \mathcal{E}_{(1\ldots n)}(p)\left\{a_1 \ldots a_n a_{n+1}\right\}\varepsilon^{[1\ldots n]}/[n]!$$

$$= \left\{\mathcal{E}_{(1\ldots n)}(p)\rho_{R(p),n+1}(g_n \ldots g_1)\right\}a_1 \ldots a_{n+1} \left\{\rho_{R,n+1}^{-1}(g_n \ldots g_1)\varepsilon^{[1\ldots n]}\right\}/[n]!$$

$$= K^{[n+1]}(p)\left\{X_1\mathcal{E}_{(2\ldots n+1)}(p)X_1^{-1}a_1\right\}\left\{a_2 \ldots a_{n+1} \varepsilon^{[2\ldots n+1]}\right\}K^{-1[1]}_{(n+1)}/[n]!$$

$$= K^{[n+1]}(p)\left\{a_1 \mathcal{E}_{(2\ldots n+1)}(p)\mathcal{E}^{[2\ldots n+1]}(p)\right\}\det(a)K^{-1[1]}_{(n+1)}/[n]!$$

$$= \left(\mathcal{K}(a)K^{-1}\right)_{n+1}\det(a).$$

The following formulae are used in the course of the calculation: (??) and (??) in the first line, (??) and (??) in the second line, (??) and (??) in the third line, and (??) in passing to the last line. For clarity we put into braces those expressions which are to be transformed in the next step.

Finally, substituting $I$ for the matrix $K$ (see (??)) we obtain (??).

**Corollary 5.1** The element

$$\Delta = \det(a)\prod_{i<j} \varphi_{ij}(p_{ij})^{-1},$$

where $f(p_{ij}) = \varphi^{p_{ij}} + [p_{ij}]\beta_{ij}$ and the functions $\varphi_{ij}$ are defined by the relations

$$\alpha_{ij}(p_{ij}) = \varphi_{ij}(p_{ij}+1)\varphi_{ij}(p_{ij})^{-1}$$

belong to the center of the algebra $\mathcal{A}(R(p),\hat{R})$. The $SL(n)$-reduction in the algebra $\mathcal{A}(R(p),\hat{R})$ can be performed by imposing the condition $\Delta = 1$.

**Proof.** We shall search for the central element in $\mathcal{A}(R(p),\hat{R})$ in the form $\Delta = U(p)\det(a)$, where $U(p)$ is some function of $p_i$ which is to be fixed. As follows from Proposition 5.2 the element $\Delta$ commutes with $p_i$ and its commutativity with the generators $a^i_n$ imposes the following conditions on the function $U$

$$X^iU(p)(X^i)^{-1} = U(p)K_i^1(p), \quad i = 1,\ldots,n.$$

Now using (??), (??) and (??) it is straightforward to check that with the choice (??), (??) one satisfies conditions (??).

**Proposition 5.3** Let the algebra $\mathcal{A}(R(p),\hat{R})$ be completed with the inverse determinant of $a$: $(\det a)^{-1} = \det(a)(\det a)^{-1} = 1$. Then the left and right inverse of $a$ is given by

$$\left(a^{-1}\right)_{(n+1)}^{[1]} = \frac{(-1)^{n-1}}{[n-1]}(\det a)^{-1}\mathcal{E}_{[2\ldots n+1]}(p)a_2 \ldots a_n \varepsilon^{[1\ldots n]}.$$
Proof. We first check that the expression (5.11) is a left inverse of $a$:
\[
(a^{-1})^{(1)}_{n+1} a_{n+1} = \frac{(-1)^{n-1}}{[n-1]!} \left( \det a \right)^{-1} \left\{ \mathcal{E}(2,\ldots,n+1)(p) a_2 \ldots a_n \right. a_{n+1} \left. \right\} e^{[1\ldots n]}
\]
\[
= \frac{(-1)^{n-1}}{[n-1]!} \left( \det a \right)^{-1} \det(a) e^{[2\ldots n+1] e^{[1\ldots n]}} = N^{(1)}_{(n+1)} = 1^{(1)}_{(n+1)} .
\]
Here we have used successively Eqs. (5.13), (5.14) and (5.15).
Checking that (5.11) is also a right inverse is slightly more complicated:
\[
a_1 (a^{-1})^{(1)}_{n+1} = \frac{(-1)^{n-1}}{[n-1]!} \left\{ a_1 \left( \det a \right)^{-1} \mathcal{E}(2,\ldots,n+1)(p) a_2 \ldots a_n \right\} e^{[1\ldots n]}
\]
\[
= \frac{(-1)^{n-1}}{[n-1]!} \left( \det a \right)^{-1} K_1(p) \left\{ a_1 \mathcal{E}(2,\ldots,n+1)(p) \right\} a_2 \ldots a_n e^{[1\ldots n]}
\]
\[
= \frac{(-1)^{n-1}}{[n-1]!} \left( \det a \right)^{-1} K_1(p) \mathcal{E}(2,\ldots,n+1)(p) X_1^{-1} \left\{ a_1 \ldots a_n \right\}
\]
\[
= \left( \det a \right)^{-1} K_1(p) N^{(1)}_{(n+1)}(p) \det(a) = 1^{(1)}_{(n+1)} ,
\]
where we have applied successively Eqs. (5.13), (5.14), (5.15), (5.16) and (5.17).

The existence of inverse matrix $a^{-1}$ is needed in many applications of the algebra $\mathcal{A}(\hat{R}(p), \hat{R})$. As an example of such application we shall construct a realization of a reflection equation algebra $\mathcal{M}(\hat{R})$ in terms of the generators of $\mathcal{A}(\hat{R}(p), \hat{R})$. We have to use here the following general property of $SL(n)$-type dynamical $R$-matrices (which has been noticed in [9] for the $SL(2)$ case, see also [10], [11]):

**Proposition 5.4** The dynamical matrix $\hat{R}(p)$ (5.13), (5.14), (5.15) satisfies the equation
\[
D_1 \hat{R}(p) D_2^{-1} = \hat{R}(p)^{-1} \sigma_{12} ,
\]
where the diagonal matrices $D$ and $\sigma$
\[
D^i_j = q^{d_i} \delta_j^i , \quad (\sigma_{12})^{i_1}_{i_2} = \delta_{i_1}^{i_2} \delta_{j_2}^{j_1} \sigma_{i_1 i_2} .
\]
are fixed by (5.16) as
\[
q^{d_i - d_j} = q^{-2\pi i} \pi_{ij} \quad (i \neq j) ,
\]
\[
\sigma_{ij} = q^{2b_{ij}}
\]
so that $d_i$ are functions of $p$.

**Proof.** First of all we note that from the Hecke condition (5.1) (and (5.2)) one can deduce
\[
\hat{R}(p)^{-1} = (a_{i_1 i_2}(p) - (q - \bar{q}) \delta_{i_1 i_2}) \delta_{j_2}^{j_1} \delta_{j_1}^{i_2} - b_{i_2 i_1}(p) \delta_{i_2}^{i_1} \delta_{j_2}^{j_1} ,
\]
Substitution of (5.13), (5.14) and (5.15) into (5.16) gives the following equations for the parameters $\sigma_{ij}$ and $d_i$
\[
a_{ij} = (a_{ij} - (q - \bar{q}) \delta_{ij}) \sigma_{ji} ,
\]
\[
q^{d_i - d_j} b_{ij} = -b_{ji} \sigma_{ij} .
\]
Equation (5.16) leads to (5.17) while (5.20) is equivalent (in view of (5.16)) to (5.18).
Now we construct the matrix $M^{\alpha \beta}$ which is diagonalized with the help of the matrix $a^i_\alpha$ and the spectrum of which is defined by the matrix $D$ (??), (??)

$$M = a^{-1} D a \quad .$$

(5.24)

It is clear that $[D_1, D_2] = 0$ and therefore the spectrum of the matrix $M$ gives a commutative set of elements.

**Proposition 5.5** The elements of the matrix $M$ (??) satisfy a reflection equation of the form

$$M_2 \tilde{R}^{-1} M_2 \tilde{R}^{-1} = \tilde{R}^{-1} M_2 \tilde{R}^{-1} M_2 \quad ,$$

(5.25)

and thus provide a realization of a reflection equation subalgebra $\mathcal{M}(\tilde{R})$ in $\mathcal{A}(\tilde{R}(p), \tilde{R})$. The matrix elements of $M$ satisfy the following exchange relations with the generators of $\mathcal{A}(\tilde{R}(p), \tilde{R})$:

$$[D_2, M_1] = 0 \ , \ M_1 a_2 = q^{2/n} a_2 \tilde{R}^{-1} M_2 \tilde{R}^{-1} .$$

(5.26)

**Proof.** Using (??), one can bring the commutation relations of the matrix $D$ with the elements $a^i_\alpha$ to the form

$$a_1 D_2 = q^{-2/n} \sigma_{12} D_2 a_1 \quad ,$$

(5.27)

where the diagonal matrix $\sigma_{12}$ is given by (??), (??). Eqs. (??) and (??) imply $[D_2, M_1] = 0$. Then one proves (??) and the second relation in (??) by using (??), (??), (??) and (??).

6 Application to the $SU(n)$ WZNW model

As an application of quantum matrix algebras we briefly describe here a typical problem of the two dimensional conformal field theory in which such matrices arise (see [?] for more details).

Let $G$ be a connected compact Lie group and $g = g(t, x)$ be a map from the cylinder $\mathbb{R} \times S^1$ into $G$ which satisfies the Wess-Zumino-Novikov-Witten (WZNW) equations of motion. The general periodic solution $g(t, x) = g(t, x + 2\pi)$ of these equations factorizes into a product of group valued chiral fields

$$g^A_B(t, x) = u^A_B(x - t)u^B_A(x + t) \quad \text{(classically, } g, u, u \in G),$$

(6.1)

each of which satisfies a twisted periodicity condition; in particular,

$$u(x + 2\pi) = u(x) M \quad , \quad (M \in G)$$

(6.2)

where $M$ is the monodromy.

Furthermore, the quantum chiral fields obey quadratic exchange relations [?, ?, ?, ?, ?, ?, ?, ?, ?]

$$u(y)_2 u(x)_1 = u(x)_1 u(y)_2 \tilde{R}(x - y) \Leftrightarrow P u(y)_2 u(x)_1 = u(x)_2 u(y)_1 \tilde{R}(x - y) .$$

(6.3)
Here the matrix $R(x)$ is a solution of the quantum Yang–Baxter equation whose $x$-dependence is given by a step function, while $\hat{R}(x)$ is the associated braid operator:

\[ \hat{R}(x) = \hat{R}(x) + \hat{R}^{-1} \theta(-x) , \quad \hat{R}(x) = P R(x) = \hat{R}^{\varepsilon(x)} \]  

(6.4)

Since $\hat{R}$ enters Eq.(??) in pair with $P$ it should be normalized to have determinant $\det \hat{R} = \det P$. For $G = SU(n)$ this implies the relation

\[ \hat{R}_{ii+1} = q^{\frac{1}{n}} \rho(g_i) \left( \text{for } g_i^2 = \mathbb{1} + (q - q)g_i \right) \implies \det \hat{R} = \det P = (-1)\binom{n}{2} \]  

(6.5)

so that we have to renormalize $\hat{R}$ of (??) by multiplying it by $q^{\frac{1}{n}}$. (The resulting $\hat{R}$ has eigenvalues $q^{1-\frac{1}{n}}$ and $-q^{1+\frac{1}{n}}$ of multiplicities $\binom{n+1}{2}$ and $\binom{n}{2}$, respectively; thus the product of all $n^2$ eigenvalues of $\hat{R}$ is indeed $(-1)\binom{n}{2}$.)

We expand, following [?] and [?], $u(x)$ into a basis of zero modes that diagonalizes the monodromy matrix $M$:

\[ u_\alpha^A(x) = a_\alpha^A u(x,p) , \quad a M = D a , \quad D^i_j = q^d \delta^i_j . \]  

(6.6)

Here $d_i = -2p_i - 1/n + 1$, $p = \{ p_i \}$ are central elements of the reflection equation algebra $\mathcal{M}(\hat{R})$; in the quantum field theoretic representation at hand they form a commuting set of operators such that Eq.(??) takes place. The eigenvalues of the differences $p_{i+1} - p_i$ are natural numbers that can be identified with the extended weights, $\lambda_i + 1$ labeling the (finite dimensional) irreducible representations of $SU(n)$. The labels of the $\binom{n}{j}$ dimensional fundamental representation are given by $\lambda^{(j)}_i = \delta^i_j$, $1 \leq i, j \leq n - 1$. Under these assumptions Eq.(??) implies exchange relations of the type

\[ R(p) a_2 a_1 = a_2 a_1 \hat{R} \]  

(6.7)

where $R(p)$ obeys the QDYBE (??). Moreover, since $\hat{R}$ and $R(p)$ enter (??) homogeneously, they can be identified with the Hecke algebra solutions of the QDYBE displayed in Sections 2 and 3 (the factor $q^{\frac{1}{n}}$ of (??) cancelling in the two sides). Thus we can apply the results of Section 4 to the (chiral zero mode) quantum matrix algebra $\mathcal{A}$ of the $SU(n)$ WZNW model. It should be noted that in this case $q$ is a root of $-1$ associated with the level $k$ $\tilde{su}(n)$ Kac–Moody algebra:

\[ q = e^{i\pi/h} , \quad [2] = q + \bar{q} = 2 \cos \frac{\pi}{h} , \quad h = n + k (\geq n + 1) . \]  

(6.8)

The eigenvalues $q^{d_i}$ of the diagonal matrix $D$ can be expressed as differences of conformal dimensions. Indeed, according to [?], the chiral vertex operators $u_j(x,p)$ satisfy

\[ u_j (x + 2\pi, p) = u_j (x, p) e^{2\pi i (\Delta_h (p) - \Delta_h (p + x^{(j)}))} , \]  

(6.9)

where the matrices $x^{(j)}$ and $p$ are defined by (??) and (??). Here the conformal dimensions are expressed in terms of the $SU(n)$ Casimir operator,

\[ 2h \Delta_h (p) = C_2 (p) = \frac{1}{n} \sum_{1 \leq k \leq n} p_{ik}^2 - \frac{n(n^2 - 1)}{12} , \]  

(6.10)
so that
\[ d_j = C_2(p) - C_2(p + v(j)) = -2(p|v(j)| - |v(j)|^2) = \frac{1}{n} - 1 - 2p_j. \] (6.11)

Inserting this in (??), we deduce that \( \pi_{ij} = 1 \) so that we arrive at the special solution (??) for \( R(p) \), allowing to present (??) in the form
\[ \frac{\varphi(p_{ij} - 1)}{\varphi(p_{ij})} [p_{ij} - 1] a^i_{\alpha} a^j_{\beta} = [p_{ij}] a^i_{\alpha} a^j_{\beta} - q^{\alpha\beta p_{ij}} a^i_{\alpha} a^j_{\beta} \] (6.12)

(here \( \epsilon_{\alpha\beta} \) is equal to 1 for \( \alpha > \beta \), 0 for \( \alpha = \beta \) and -1 for \( \alpha < \beta \)).

An important consequence of (??) and (??) is the existence of an ideal \( \mathcal{I}_h \) of \( \mathcal{A} \) generated by \( n^2 \) elements \( (a^i_{\alpha})^h \) such that the factor algebra \( \mathcal{A}/\mathcal{I}_h \) is finite dimensional [?].

This allows to define a finite dimensional "Fock space representation" of \( \mathcal{A} \) with a unique vacuum vector \(|\text{vac}\rangle\) corresponding to trivial \( su(n) \) weight \( \lambda_i = 0 \) \( (p_{i+1} = 1, i = 1, \ldots, n-1) \) such that
\[ a^i_{\alpha}|\text{vac}\rangle = 0 \quad \text{for} \quad i > 1, \quad \mathcal{I}_h |\text{vac}\rangle = 0. \] (6.13)

**Acknowledgements**

This work started while three of us, L.K.H., O.V.O. and I.T.T. were visiting the Bogoliubov Laboratory of Theoretical Physics of the JINR, Dubna and was completed while A.P.I., L.K.H. and I.T.T. were visiting the Department of Physics of the University of Pisa, ABICTP (Trieste) and the Erwin Schrödinger Institute for Mathematical Physics, respectively. We thank all these Institutions for hospitality and support. This work was also supported in part by RFFI (grant 97-01-01041), INTAS (grant 93-127-ext), by the exchange program between INFN and JINR (Dubna) and by the Bulgarian National Foundation for Scientific Research under contract F-404.

**Appendix. Normalization of dynamical Levi-Cività tensors**

The definitions (??), (??) lead to the expression
\[ \mathcal{E}_{i_1 \ldots i_n}(p) \mathcal{E}^{i_1 \ldots i_n}(p) = \prod_{1 \leq a < b \leq n} \xi_{i_a i_b} \] (A.1)

(there are no summations over the indices \( i_k \)), and the normalization condition (??) for the dynamical \( \mathcal{E} \)-tensors follows from

**Proposition A** Let \( \xi_{ij} = d - b_{ij} \) where \( d \) is a constant (comparing with (??), one gets \( d = q \)) and the elements \( b_{ij} \) satisfy (??), (??). Then the following identity holds:
\[ I_k \equiv \sum_{S_k} \prod_{1 \leq a < b \leq k} \xi_{i_a i_b} = |k|_d!, \] (A.2)
where \( k \leq n \), \([k]_d = \frac{(d^{k}-(d-\lambda)^k)}{\lambda} \) (\( \lambda = q - \tilde{q} \)) and \( S_k \) denotes all permutations of indices \((i_1, \ldots, i_k) \) \((i_a \neq i_b \text{ for } a \neq b)\). Note that \([k]_d = [k] \) for \( d = q \) (as it is needed in (??)).

**Proof** We shall proceed by induction. For \( k = 2 \) we have

\[
I_2 = \xi_{i_1i_2} + \xi_{i_2i_1} = 2d - \lambda = [2]_d .
\]

Let (??) be correct for some \( k \) (\( 1 < k < n \)), then for \( k + 1 \) we derive

\[
I_{k+1} = \sum_{S_{k+1}} \left( \prod_{l=1}^{k} \xi_{\ell_{i_{l_1}} i_{l_1}} \prod_{1 \leq a < b \leq k} \xi_{i_{a} i_{b}} \right) =
\sum_{r=1}^{k+1} \left( \prod_{l=1}^{k+1} \xi_{\ell_{i_{l}} r} \sum_{S_{k}} \prod_{a \neq r \neq b} \xi_{i_{a} i_{b}} \right) = I_k \sum_{r=1}^{k+1} \prod_{l=1}^{k+1} \xi_{\ell_{i_{l}} r} .
\]

Therefore we should prove the identity

\[
[k + 1]_d = \sum_{r=1}^{k+1} \prod_{l=1}^{k+1} \xi_{\ell_{i_{l}} r} = \sum_{r=1}^{k+1} \prod_{l=1}^{k+1} (d - b_{i_{l} i_{r}}) . \tag{A.3}
\]

This identity follows from the relation

\[
\sum_{r=1}^{m} \prod_{l=1}^{m} b_{i_{l} i_{r}} = \lambda^{m-1} \quad (m \leq k + 1) , \tag{A.4}
\]

which can be obtained by induction. Indeed, from Eqs.(??), (??) we have for \( m = 2, 3 \)

\[
b_{i_1i_2} + b_{i_2i_1} = \lambda , \quad b_{i_1i_2} b_{i_2i_1} + b_{i_1i_2} b_{i_2i_3} + b_{i_1i_3} b_{i_2i_3} = \lambda^2 .
\]

Then we deduce

\[
\sum_{r=3}^{m} \prod_{l=3}^{m} b_{i_{l} i_{r}} = \left( \lambda^{m-2} - \sum_{r=3}^{m} (\lambda - b_{i_{r} i_{r}}) \prod_{l=3}^{m} b_{i_{l} i_{r}} \right) =
\] \[
b_{i_2i_3} \sum_{r=3}^{m} b_{i_{r} i_{1}} \prod_{l=3}^{m} b_{i_{l} i_{r}} = \sum_{r=3}^{m} (\lambda^2 - b_{i_1i_2} b_{i_{r} i_{2}} - b_{i_1i_2} b_{i_{r} i_{2}}) \prod_{l=3}^{m} b_{i_{l} i_{r}} =
\] \[
= \lambda^{m-1} - \sum_{r=2}^{m} \prod_{l=1}^{m} b_{i_{l} i_{r}} ,
\]

which proves (??). Expanding the right-hand side of (??) in power series of \( d \) and taking into account (??) we verify the relations (??) and, thus, complete the proof.

One can reformulate the statement of Proposition A in a more concise form (only in terms of elements \( \xi_{ij} \))
Proposition B  Let $\xi_{ij}$ satisfy

$$\xi_{ij} + \xi_{ji} = [2] = \xi_{i1} \xi_{i2} \xi_{k1} + \xi_{i2} \xi_{k2} \xi_{ji} \quad (i \neq j \neq k \neq i).$$

We rewrite these conditions as

$$\sum_{r=1}^{k} \prod_{l=1, i \neq r}^{k} \xi_{i,r} = [k] \quad \text{for } k = 2, 3. \quad (A.5)$$

Then, equation (??) is also valid for $4 \leq k \leq n$, and the following identity holds:

$$I_k \equiv \sum_{S_k} \prod_{1 \leq a < b \leq k} \xi_{i_a i_b} = [k]!,$$  \quad (A.6)

where $S_k$ denotes all permutations of the indices $(i_1, \ldots, i_k)$ and $i_a \neq i_b$ for $a \neq b$.

**Proof**  The proof is similar to that of Proposition A.  □

**Remark**  There are many other interesting relations among the elements $b_{ij}$ (??), (??) (as well as among $\xi_{ij}$). For example, one can easily deduce the identity

$$b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_{k-1} i_k} b_{i_k i_1} = (-1)^k b_{i_k i_k} b_{i_{k-1} i_{k-1}} \cdots b_{i_2 i_2} b_{i_1 i_1},$$

which generalizes (??) and follows from the relation

$$-\frac{b_{ji}(p)}{b_{ij}(p)} q^{2p_{ij}} = \pi_{ij}.$$  \quad (A.7)

Note that we consider $\pi_{ij}$ as constants which are independent of $p_i$.

**References**


