A simple criterion is derived in order that a number sequence $S_n$ is a permitted spectrum of a quantized system. The sequence of the prime numbers fulfills the criterion and the corresponding one-dimensional quantum potential is explicitly computed in a semi-classical approximation. The existence of such a potential implies that the primality testing can in principle be resolved by the sole use of physical laws.
In the realm of arithmetic, or rather number theory, one often encounters interesting number sequences $S_n$ (see, for instance [?, ?, ?]). For many of the sequences, it is their recreational aspect which renders them interesting, as in the case of the polygonal numbers $p_n$ [?]. For other sequences, instead, their interesting aspects consist in the influence which they still exert over devotees of number theory as sources of delightful surprises or intriguing properties, such as the perfect number sequence [?]

$$P_n = 1, 6, 28, 496, 8128, \ldots$$  \hspace{1cm} (1)

or Fibonacci numbers [?]

$$F_n = 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$  \hspace{1cm} (2)

Finally, it is probably fair to say that the most important sequence of integers in number theory – and still one of the most elusive – is represented by the class of primes

$$P_n = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$$  \hspace{1cm} (3)

No closed formula is known for the $n$-th prime number. However, by accepting an approximation of their value within a 10% margin, it is known that the $n$-th prime number may be expressed in terms of the simple formula

$$P_n \simeq n \log n .$$  \hspace{1cm} (4)

In the realm of physics, or rather quantum mechanics, it is also very natural to come upon discrete sequences of numbers. For instance, these may be the allowed energy values of a given system, once its quantization is carried out. This being so, it is then natural to wonder whether it would always be possible to design a physical device in such a way that the resulting quantized energy levels (in appropriate units) coincide with an assigned sequence of numbers taken from the arithmetic realm$^1$.

The answer to the above question is actually negative, i.e. not all number sequences $S_n$ can play the role of discrete energy eigenvalues of a quantized system. The easiest way to show this result is to derive a bound on the growing rate of energy eigenvalues of one-dimensional quantum hamiltonian by employing the semi-classical quantization formula$^2$

$$\int p(x)dx = (n + \alpha)\hbar ,$$  \hspace{1cm} (5)

$^1$By physical device we intend any apparatus (made by electric field traps, etc.) which can be parameterized in terms of a potential $V(x)$ entering the Schrödinger equation. For simplicity we consider only the one-dimensional case in this paper. In the following, the energy levels will often be denoted by their corresponding number sequences since the correct physical dimensions can always be restored by appropriate dimensional factors.

$^2$To simplify the following expressions we will assume $V(x) = V(-x)$. The constant $\alpha$ in eq.(??) will not play any significative role in the following and can be taken equal to 0.
where \( p(x) = \sqrt{2m [E - V(x)]} \). Let us initially consider the class of potentials

\[
V_r(x) = \lambda |x|^r .
\]

Substituting (6) into eq. (1), it is easy to obtain that the energy levels associated to these potentials scale as

\[
E_n = A_r n^{2r/2} ,
\]

where

\[
A_r = \left[ \frac{\lambda^{1/r} \hbar \Gamma \left( \frac{1}{r} + \frac{3}{2} \right)}{2 \sqrt{2m \pi \Gamma \left( \frac{1}{r} + 1 \right)}} \right]^{2r}.
\]

From (7) we see that the quantized energies relative to such potentials cannot grow faster than \( E_n \sim n^2 \), i.e. they always satisfy the bound

\[
E_n \leq C n^2 .
\]

The faster the variation of the potential, the faster is of the increase of the energy eigenvalues and the bound is therefore saturated for the quickest increasing function of the family (6), i.e. the potential well

\[
V(x) = \lim_{r \to \infty} V_r(x) = \begin{cases} 0, & \text{if } |x| < 1 \\ \infty, & \text{otherwise} \end{cases}
\]

It is easy to show that the validity of the bound (9) extends beyond the polynomial potentials which we used for its derivation\(^3\). In fact, it can be easily argued that the bound (9) is actually a general feature of all discrete spectra of one-dimensional quantum hamiltonian.

It should be clear at this stage that in order for a number sequence \( S_n \) to be a permitted spectrum of a quantized system, it must increase at a rate slower than \( n^2 \). This implies, for instance, that a one-dimensional quantum mechanical system that has Fibonacci numbers (3) as energy eigenvalues cannot exist. By the same token, other exponentially increasing number sequences are ruled out as well. Viceversa, quantum mechanical systems which allow, for instance, the polygonal numbers (4) as possible energy eigenvalues must exist. The same is also true for the sequence of the prime numbers, since their behaviour (5) is sufficiently mild so as not to violate the bound (9).

How can the potential corresponding to an allowed sequence of numbers be explicitly computed? As it is well known, the answer can be given in the context of semi-classical quantization, i.e. by inverting eq. (1)

\[
x(V) = \frac{\hbar}{\sqrt{2m}} \int_{E_0}^V \frac{dE}{dn \sqrt{V - E}} ,
\]

\(^3\)It is simple to see, for instance, that even for a potential which increases exponentially \( V(x) \sim \mu \exp(\beta |x|) \) for \( |x| \to \infty \), the eigenvalues cannot grow more than quadratically.
where \( E_0 \) sets the zero of the energy scale. The potential \( V(x) \) is thus finally obtained in terms of the inverse function of \( x(V) \).

Let us now apply the above semi-classical formula (??) in order to find the quantum mechanical potential which has the prime numbers \( P_n \) as its energy eigenvalues\(^4\) \( E_n \). Let us denote this potential as \( W(x) \). As mentioned before, a closed formula for the \( n \)-th prime number is not known but this is not an obstacle to obtain \( W(x) \). In fact, there is a great deal of information about \( \pi(N) \), i.e. the function which counts the prime numbers smaller or equal to a given number \( N \). Since the formula of the \( n \)-th prime number is nothing but the inverse of the function \( \pi(N) \), the density of states \( \frac{dE}{dn} \) in (??) can be computed by means of the inverse function derivative rule, i.e. \( \frac{dE}{dn} = 1/\frac{d\pi}{dE} \). To proceed further in the explicit computation of \( W(x) \), let us briefly recall some properties of the function \( \pi(E) \) (see, for instance [?]). Its first estimate was obtained by Gauss and Legendre [?]

\[
\pi(E) \sim \frac{E}{\ln E}.
\]

A more precise version of the above estimate is given by

\[
\pi(E) \simeq \text{Li}(E) \equiv \int_2^E \frac{dE'}{\ln E'} ,
\]

while a further refinement is provided by the series

\[
\pi(E) \simeq R(E) = \sum_{m=0}^{\infty} \frac{\mu(m)}{m} \text{Li}(E^{1/m}) ,
\]

with the Moebius numbers \( \mu(m) \) defined by

\[
\mu(m) = \begin{cases} 
1 & \text{if } m = 1 \\
0 & \text{if } m \text{ is divisible by a square of a prime} \\
(-1)^k & \text{otherwise}
\end{cases}
\]

where \( k \) is the number of prime divisors of \( m \). \( R(E) \) is the smooth function which approximates \( \pi(E) \) more efficiently (see, e.g. [?]). For its derivative we have

\[
\frac{d\pi}{dE} \sim \frac{1}{\ln E} \sum_{m=1}^{\infty} \frac{\mu(m)}{m} E^{1-m} .
\]

Hence, in the semiclassical approximation the potential whose energy eigenvalues are prime numbers is expressed in terms of the series

\[
x(W) = \frac{\hbar}{\sqrt{2m}} \sum_{m=1}^{\infty} \frac{\mu(m)}{m} \int_{E_0}^{W} \frac{E^{1-m}}{\ln E \sqrt{W - E}} \, dE .
\]

\(^4\)To restore the correct dimension, it may be convenient to introduce – in perfect analogy with the harmonic oscillator – a frequency \( \omega \) and express the energy eigenvalues as \( E_n = \hbar \omega P_n \).
The plot of this function is drawn in Fig. 1 (with $E_0 = 0$): the series (??) rapidly converges to a limiting function, which can be regarded as the potential $W(x)$, solution of the problem$^5$.

The existence of a potential which admits all the prime numbers as its only eigenvalues has some important implications. For instance, a long-standing issue in number theory such as the primality test of a given number $N$ can be completely resolved by the sole application of physical laws. To do this, it would be sufficient to design an apparatus $G$, which provides the potential shown in Fig. 2, where the central part is nothing else but the potential $W(x)$, truncated however at an energy cutoff $e_0$ that can be controlled by an external handle (see Fig. 3). If the barriers $B$ are shaped such that their penetration is strongly inhibited, the original energy levels of $W(x)$ are essentially left unperturbed: the device $G$ then allows the typical resonance phenomena of quantum mechanics to be observed (see, e.g. [?]). Hence, to determine whether a given number $N$ is prime or not, we merely have to send an incident plane wave of energy $E = N\hbar\omega$ into the apparatus $G$: if the number $N$ is a prime number (i.e. if the energy $E$ belongs to the spectrum of the potential $W(x)$), then a sharp resonance peak in the transmission amplitude $T(E)$ is observed, i.e. the plane wave will be in practice completely transmitted, otherwise it will be simply reflected. For the success of this experiment, we have only to pay attention to the tuning of the threshold $e_0$: this has to be chosen so that the resonance peaks of the tunnelling effects remain visible but, at the same time, without affecting the energy eigenvalues which are lower or close to the value $E = N\hbar\omega$ under the probe. This condition can always be satisfied by tuning $e_0$ to be sufficiently larger than the input energy $E$.

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$^5$The degree of accuracy which can be reached in the determination of $W(x)$ is purely a question of practical considerations.
The polygonal numbers $P_s^n$ count the number of balls that can be arranged in a regular polygonal pattern of $s$ side of increasing rank $n$. Their closed formula is given by $P_s^n = \frac{n}{2} [(n - 1)s - 2(n - 2)]$.

A perfect number is one which is equal to the sum of its aliquot divisors. A close formula for even perfect number is due to Euclid, $P_n = 2^{n-1}M_n$ where $n$ may be any positive integer exceeding unity which makes the second factor $M_n = (2^n - 1)$ a prime number. Integers of the form $M_n = (2^n - 1)$ are called Mersenne’s numbers and they provide the largest prime numbers known so far. No odd perfect numbers have been presently found.

Fibonacci numbers are defined by the recursive equation $F_{n+2} = F_{n+1} + F_n$, with initial condition $F_1 = 1, F_2 = 1$. Their closed expression is provided by Binet’s formula $F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$.

Although formula (??) is just a coarse approximation, it is known however that it is able to capture the asymptotic behaviour of $\pi(E)$ – a result which constitutes the content of the “Prime Number Theorem” [?]

$$\lim_{E \to \infty} \frac{\pi(E) \ln E}{E} = 1.$$ 


Figure Captions

**Figure 1.** Profiles of $W(x)$ (in units of $\hbar \omega$) versus $x$ (in units of $\sqrt{\hbar/2m\omega}$), obtained by including 100 terms of the series (??).

**Figure 2.** The potential $V(x)$ as realized by the apparatus $G$, i.e. the prime-number potential $W(x)$ with an energy cut-off $\epsilon_0$ and external barriers.

**Figure 3.** Resonance experiment to implement the primality testing: the device $G$ gives the transmission amplitude $T(E)$ of the incident plane wave as an output signal.
Figure 1
Figure 3