AMPLITUDE AND PHASE DIFFERENTIATION
OF SYNTHETIC SEISMOGRAMS:
A MUST FOR WAVEFORM INVERSION AT REGIONAL SCALE

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We study systematically, using the differential seismogram technique developed by Du et al. (1997b, hereafter cited as Paper 1), the effects of structural model perturbation on both amplitude and phase of synthetic seismograms computed with the modal summation. The detailed frequency and time domain analysis shows that the seismogram amplitude differentiation, which is neglected in more conventional structure waveform inversion schemes, plays a critical role in the computation of the differential seismogram, i.e. of the structural linear constraint, especially for higher modes. Since waveform inversion with numerical procedures for the differentiation of the amplitude of the seismogram is computationally very expensive, a fully analytical waveform inversion scheme is developed. As a test, the synthetic seismograms corresponding to the fundamental and first few higher modes of Rayleigh waves are computed for a series of geophysically realistic structural models and then inverted. The results obtained with amplitude and phase differentiations are compared to those obtained with only phase differentiation and the possible biases resulting from the use, as a constraint on the structural model, of the phase differentiation only are discussed.
1. Introduction

Various waveform inversion schemes have been developed, using modal summation as an efficient forward waveform modelling, to invert the properties of a laterally inhomogeneous Earth. In a laterally homogeneous medium modal summation is an exact forward method, and can be used to construct complete broadband seismograms (e.g. Woodhouse & Dziewonski, 1984; Panza, 1985), but the extension of this technique to laterally inhomogeneous media is based upon several not well justified assumptions (Snieder, 1996). Furthermore, in order to simplify the algorithms, most of the existing waveform inversion schemes make use only of the information contained in the seismogram phase.

Waveform inversion with the inclusion of the information contained in the amplitude of the seismogram is computationally very demanding and, therefore, it is frequently assumed that the amplitude change of the seismogram due to the perturbation of the structural model is negligible. However, this assumption is not well justified, since the perturbation of the structural model can affect the eigenfunctions. The first effort to simultaneously invert phase and amplitude information is made by Yamogida & Aki (1987) with a paraxial ray approximation and using the Gaussian beam method. They adopted a scalar wave equation, as a basis for the inversion, rather than the elastic dynamics equation. In an attempt to invert for the mantle transition zone, in the range from 400 to 1000 km of depth, Stutzmann & Montagner (1993) designed a two-step waveform inversion scheme for surface wave fundamental and higher modes. To fit waveforms, they demonstrate that it is necessary to take into account not only the phase but also the amplitude change of the seismogram caused by the structural model perturbation, especially for higher modes. Although they retrieve the phase change of the seismogram from the eigenfunctions of the reference model, Stutzmann & Montagner (1993; 1994) provide some physical insight about the effects of considering the seismogram amplitude in waveform inversion. Both methods, however, do not give
an efficient and accurate formalism that can be implemented in an inversion scheme, when amplitude is used as a constraint.

The change of the seismogram amplitude, due to structure perturbation, reflects the changes of the related eigenfunctions. In a laterally varying medium, not only the change but also the coupling of the eigenfunctions needs to be taken into account. For media which do not vary perpendicularly to the direction of the wave propagation, Kennett (1984) has introduced a theory to describe surface wave mode coupling due to 2-D heterogeneities. The details about the extension of the WKBJ approximation to include surface-wave modes coupling have been derived by Li & Romanowicz (1995) for a global scale waveform inversion, using the Born approximation. On a regional scale, Marquering & Snieder (1995) obtain similar results using the surface wave mode scattering theory (Snieder, 1986). The validity of using the eigenfunctions of the reference model to perform waveform inversion has been investigated by Kennett (1995). He roughly determines some frequency limits, and the maximum amount of structural perturbation that can be applied to the reference model through a visual inspection of the shape changes of the used reference eigenfunctions. Of course, this approach is not suitable for waveform inversion, but it gives a general idea as to the likely variations of eigenfunctions.

Instead of computing the eigenfunctions for the perturbed model to estimate the change of the amplitude of a seismogram caused by the structural model perturbations, it is possible to expand the eigenfunctions, for a given mode, in terms of the eigenfunctions of the neighbouring modes (Maupin & Kennett, 1987); this method, however, requires a large quantity of computer time.

In this paper, we present an efficient estimation of the amplitude change, caused by a structural model perturbation, through the accurate calculation of the eigenfunction partial derivatives with respect to model parameters (Du et al., 1997a). In order to quantitatively analyse the effect of the structural model perturbation on the amplitude and on the phase of the seismogram, we decompose the differential seismogram (DS)
into its amplitude (ADS) and phase (PDS) differentials. The frequency domain analysis combined with the analysis of the waveforms of DS, ADS and PDS show clearly and accurately assess the relative sensitivity of DS, ADS and PDS on the structural model parameters, as well as their behaviour as a structural constraint.

At last, we extend the Partitioned Waveform Inversion (PWI) scheme (Nolet at al., 1986; Nolet, 1990) to include the effect of the structural model perturbation on the amplitudes of the seismogram, and we compare the waveform inversion results obtained with amplitude and phase differentiations (DS) to those obtained with phase differentiation (PDS) only. Our new approach does not require significant additional computation time with respect to the original PWI inversion scheme, because the analytical calculation of the differential seismogram (presented in Paper 1) allows us a time saving of about 95%, in comparison with usually adopted numerical approaches.

2. Differential seismograms

For a given Rayleigh-mode, the formalism of a differential seismogram for an assigned double-couple source is given in Paper 1. In the frequency domain the expressions are:

\[ \frac{\partial}{\partial p_j} U_r(\omega) = U_r(\omega) \left[ \frac{1}{\chi(\Theta, h)} \frac{\partial}{\partial p_j} \chi(\Theta, h) - \left( \frac{1}{c} \frac{\partial c}{\partial p_j} + \frac{1}{u} \frac{\partial u}{\partial p_j} + \frac{1}{I_1} \frac{\partial I_1}{\partial p_j} \right) + \left( \frac{1}{2k} \cdot i \cdot r \right) \frac{\partial k}{\partial p_j} + \frac{1}{\varepsilon_0} \frac{\partial}{\partial p_j} \varepsilon_0 \right] \]  

\[ \frac{\partial}{\partial p_j} U_z(\omega) = U_z(\omega) \left[ \frac{1}{\chi(\Theta, h)} \frac{\partial}{\partial p_j} \chi(\Theta, h) - \left( \frac{1}{c} \frac{\partial c}{\partial p_j} + \frac{1}{u} \frac{\partial u}{\partial p_j} + \frac{1}{I_1} \frac{\partial I_1}{\partial p_j} \right) + \left( \frac{1}{2k} \cdot i \cdot r \right) \frac{\partial k}{\partial p_j} \right] \]  

where \( \chi(\Theta, h) \) is the source radiation pattern, \( c \) and \( u \) are respectively the phase and group velocity, \( k = \frac{\omega}{c} \) is the wave number, \( r \) is the epicentral distance, \( I_1 \) is the energy integral and \( \varepsilon_0 \) is the ellipticity. The model parameter \( p_j \) (\( j \) indicates the layer sequential
number) can be either chosen as S-wave velocity, \( \beta_j \), P-wave velocity, \( \alpha_j \), or density \( \rho_j \). The physics behind each term in equations (1) and (2) is discussed in Paper 1.

The seismogram expressions, used to derive equations (1) and (2) in Paper 1, can be written as:

\[
U(\omega) = A(\omega)e^{-ikr} \quad (3)
\]

Accordingly, by grouping the terms in equations (1) or (2) into two groups corresponding to the derivatives of the amplitude and of the phase, the partial derivatives corresponding to expression (3) can be obtained. If (3) represents a vertical component seismogram, we obtain from (2):

\[
\frac{\partial}{\partial p_j} A(\omega) = \frac{1}{A} \frac{\partial A}{\partial p_j} U(\omega)
= \left[ \frac{1}{\chi(\Theta, h)} \frac{\partial}{\partial p_j} \chi(\Theta, h) + \frac{1}{E} \frac{\partial E}{\partial p_j} + \frac{1}{2k} \frac{\partial k}{\partial p_j} \right] U(\omega) \quad (4)
\]

where \( \frac{1}{E} \frac{\partial E}{\partial p_j} = -\left( \frac{1}{c} \frac{\partial c}{\partial p_j} + \frac{1}{\mu} \frac{\partial \mu}{\partial p_j} + \frac{1}{\lambda} \frac{\partial \lambda}{\partial p_j} \right) \), and \( E = \frac{1}{2c\alpha\mu} \);

\[
\frac{\partial}{\partial p_j} e^{-ikr} = -ik \frac{\partial}{\partial p_j} U(\omega)
= -ik \frac{1}{c} \frac{\partial c}{\partial p_j} U(\omega) \quad (5)
\]

The radial component can be obtained easily by including in equation (4) the additional term due to the ellipticity, \( \epsilon_0 \), derivative (see eq. (1)), or by directly grouping equation (1) following the same procedure as that just described.

The first term on the right-hand side (r.h.s) of equation (4) indicates that the term depending on the source parameters is affected by the structural model perturbation (Kennett, 1995), since \( \frac{\partial}{\partial p_j} \chi(\Theta, h) \) is formed by the inner product of the derivative of the Green's function with the source time function. The second term is the partial
derivative of the amplitude response factor (Harkrider, 1970), and it depends only on the structural model, while the last term is due to the assumption of a point source in a three-dimensional space (Aki & Richards, 1980). Both the radiation pattern, \( \chi(\Theta, h) \), and the energy integral, \( I_1 \), are explicit functions of the stress-displacement vectors, i.e. of the eigenfunctions; \( \chi(\Theta, h) \) varies with the changes of the stress-displacement vector at the source depth, while \( I_1 \) varies with the changes of the displacement vector over the entire structure.

Compared with (4), the derivative of the phase of the seismogram equation (5), is relatively simple. It only involves one term, which is the product of the epicentral distance, \( r \), with either the partial derivative of the wavenumber, or the wavenumber times the partial derivative of the phase velocity. One prominent property of this derivative is that it can become large when a very large epicentral distance is used. Most of the global and larger regional scale waveform inversions take advantage of this property and neglect the amplitude constraint (e.g. Woodhouse & Dziewonski, 1984; Zielhuis & Nolet, 1994), but there are also a few applications of this property to waveform inversion for relatively small regional or local scale (e.g. Gomberg & Masters, 1988; Das & Nolet, 1995).

3. Numerical results

In this section, we examine the behaviour of each term in equations (4) and (5), and assess their sensitivity both in frequency and in time domain.

The upper 200 km of the S-wave velocity models used to generate the synthetic seismograms are shown in Fig. 1. While \textit{iasp91} global model (Kennett & Engdahl, 1991) is used for S-wave velocity at depths greater than 200 km, the density model in \textit{CAL 8} (Bullen & Bolt, 1985) is used for all depths. The \( Q \) value is adopted from the model used in Paper 1. Our structural models have continental properties and are characterized by two major discontinuities. The first occurs at the Moho depth: Model A has a sharp
Moho, marked by a S-wave velocity increase of 0.6 km/s, while Model B has a relatively smooth crust-mantle transition. The second discontinuity marks the boundary between the lid and the low-velocity layer in the asthenosphere. The point source is located in the crust at a depth of 33 km. A double-couple mechanism is adopted, and the same source parameters of $\delta = 37^\circ$ and $\lambda = 283^\circ$, as used in Paper 1, is adopted here for the computation of synthetic seismograms. For conciseness, we show only the results for the shaded layers in Fig. 1.

**Frequency domain analysis**

In Fig. 2, we show $\frac{1}{E} \frac{\partial E}{\partial p_j}$, $\frac{1}{c} \frac{\partial c}{\partial p_j}$, $\frac{1}{u} \frac{\partial u}{\partial p_j}$ and $\frac{1}{I_1} \frac{\partial I_1}{\partial p_j}$ as a function of frequency for Model A. For the fundamental mode (Figure 2a), the largest contribution to $\frac{1}{E} \frac{\partial E}{\partial p_j}$ comes from $\frac{1}{I_1} \frac{\partial I_1}{\partial p_j}$, and the smallest one from $\frac{1}{c} \frac{\partial c}{\partial p_j}$. The main lobe of $\frac{1}{E} \frac{\partial E}{\partial p_j}$ depends on frequency and on layer depth. As we could expect, in the deeper layers, it moves towards lower frequencies with a decreasing amplitude. For the higher modes, as shown in Fig. 2b and 2c, the contribution to $\frac{1}{E} \frac{\partial E}{\partial p_j}$ comes mainly from $\frac{1}{I_1} \frac{\partial I_1}{\partial p_j}$.

Next, we examine the behaviour of the amplitude and of the phase of the spectrum of a complete derivative of a seismogram, representative of ground velocity. In Fig. 3 the absolute value of the spectrum of the amplitude derivative, $\frac{1}{A} \frac{\partial A}{\partial p_j}$, defined by equation (4), is plotted together with the spectrum of the phase derivative, $r \cdot \frac{\partial k}{\partial p_j}$, for $r = 1500$ km. In the same figure $\frac{1}{E} \frac{\partial E}{\partial p_j}$ and $\frac{\partial k}{\partial p_j}$ are plotted for comparison. $\frac{\partial k}{\partial p_j}$ is small regardless of layer depth and mode number. As shown in Fig. 3a, for the fundamental mode, the phase derivative, $r \cdot \frac{\partial k}{\partial p_j}$, is much larger than the corresponding amplitude derivative, $\frac{1}{A} \frac{\partial A}{\partial p_j}$, in a broad frequency range. Depending on the layer depth, the pattern of $r \cdot \frac{\partial k}{\partial p_j}$ changes, its main lobe moves toward low frequencies with a decreasing amplitude. For the higher modes, as shown in Fig. 3b and 3c, the spectral amplitudes of $\frac{1}{E} \frac{\partial E}{\partial p_j}$ and $r \cdot \frac{\partial k}{\partial p_j}$ are comparable. The contribution from the source depending term, $\frac{1}{\chi(\Theta, h)} \frac{\partial}{\partial p_j} \chi(\Theta, h)$, to $\frac{1}{A} \frac{\partial A}{\partial p_j}$ is not significant. The main contribution to $\frac{1}{A} \frac{\partial A}{\partial p_j}$ comes from $\frac{1}{E} \frac{\partial E}{\partial p_j}$. The results obtained for structural Model B are nearly the same, and all the main features just described are confirmed.
Time domain analysis

To perform time domain analysis, we compute DS by Fourier transform of equation (2), and ADS and PDS, by Fourier transform of equations (4) and (5) respectively, for \( r = 1500 \ km \).

In Figs. 4 and 5 are shown DS, ADS and PDS computed for Model A, by using two different upper frequency limits, 0.03 \( hz \) and 0.05 \( hz \), respectively. For the fundamental mode (the upper most trace in each panel), PDS is almost identical to DS when the upper frequency limit is 0.05 \( hz \), as shown in Figure 5. When the upper frequency limit is 0.03 \( hz \) (Fig. 4), PDS is comparable to ADS for layers (8 and 12) around the Moho discontinuity. This frequency dependent property of PDS can easily be understood from Fig. 3a, where it is shown that when frequencies are higher than 0.04 \( hz \), the phase term is dominant. For the higher modes (middle and lower traces in each panel), since their spectra vary very irregularly with frequency and layer depth (Fig. 3b and 3c), the time domain waveforms have a relatively complex pattern. In most cases either ADS dominates DS, or is comparable to PDS, but in some case, ADS is smaller than both DS and PDS. In the example shown in Fig. 6 (upper frequency limit 0.1 \( hz \) and \( r = 500 \ km \)), a similar pattern can be observed with a general dominance of ADS. The obvious conclusion is that both ADS and PDS are required in order to correctly compute DS.

The analysis made for structural Model B confirms all these results.

Discussion

From the frequency and time domain analysis a quite complex relationship between DS and its two decompositions ADS and PDS is recognized. The main result is that ADS role in the computation of the complete derivative of the seismogram (DS), especially for higher modes, cannot be neglected. Although the use of a large epicentral distance, \( r \), can significantly amplify the effect of \( \frac{dk}{dp_i} \), the PDS dominance is not warranted for higher modes, and even for the fundamental one the dominance of PDS cannot be achieved consistently, depending upon the used frequency band and the properties of the structural model. Only at very large epicentral distances, we see from
equation (5) that PDS dominantes DS.

4. Waveform inversion

The kernel of the PWI (Nolet et al., 1986; Nolet, 1990) depends only on PDS. We extend the PWI so that its kernel depends from DS, i.e. both from ADS and PDS, using the algorithm to compute the seismogram derivative, presented in Paper 1. This leads to a fully analytic waveform inversion scheme, and establishes a first-order relationship between the perturbation of the wavefield, and that of the sampled medium. To show the similarities and differences between our new fully analytical scheme and the scheme of the PWI, we construct our notations in agreement with the PWI (Nolet, 1990).

In the PWI, the structure deviations from the reference model can only result in a change of the seismogram phase term defined by an averaged wavenumber perturbation $\delta k_n(\omega)$. Starting with this approximation, from equation (3), we obtain:

$$ U(\omega) = \sum_{n=1}^{N} A_n^0(\omega) \exp\{-i[k_n^0(\omega) + \delta k_n(\omega)]r\} \tag{6} $$

where the superscript '0' indicates quantities that are calculated for the reference model, and $N$ is the number of modes considered in the computation of the synthetic seismogram. If we assume that the perturbations of the wavefield are caused by variations in the S-wave velocity, the relationship between the averaged wavenumber perturbation and the average model perturbation $\delta \beta(z)$ can be written as:

$$ \delta k_n(\omega) = \int_0^\infty (\frac{\partial k_n(\omega)}{\partial \beta(z)})\delta \beta(z)dz \tag{7} $$

Following Nolet (1990), the averaged velocity perturbation $\delta \beta(z)$ can be parametrized with $M$ functions, $h_i(z)$, of depth:

$$ \delta \beta(z) = \sum_{i=1}^{M} \gamma_i h_i(z) \tag{8} $$
Substitution of (8) into (7) and then (6) yields an expression for the wavefield as a function of the model vector $\gamma$. The optimal model (or model vector $\gamma$) that produces the synthetic waveforms which fit, within a preassigned band, the data can be found by minimizing the penalty function $F(\gamma)$ defined by

$$F(\gamma) = \int [s(t) - u(t, \gamma)]^2 dt$$

where $s(t)$ is the recorded seismic waveform and $u(t, \gamma)$ is the synthetic waveform constructed as the Fourier transform of equation (6). In practice, both the waveforms $s(t)$ and $u(t, \gamma)$ are windowed, filtered and, furthermore, a small damping term is added to the r.h.s. of equation (9).

Equation (6) can be rewritten in order to take into account the change of the amplitude term as follows:

$$U(\omega) = \sum_{n=1}^{N} \left[ A_n^0(\omega) + \delta A_n(\omega) \right] \exp\left\{ -i[k_n^0(\omega) + \delta k_n(\omega)]r \right\}$$

The relationship of $\delta A_n(\omega)$ with the velocity perturbation $\delta \beta(z)$ can be written, in analogy with (7), as:

$$\delta A_n(\omega) = \int_{0}^{\infty} \left( \frac{\partial A_n(\omega)}{\partial \beta(z)} \right) \delta \beta(z) dz$$

where $\frac{\partial A_n(\omega)}{\partial \beta(z)}$ can be determined from (4).

Although the system of equations (7), (10) and (11) appears to be more complicated than the system of equations (6) and (7), the search for the minimum of the misfit function (9) can be carried out in the same manner as in the PWI, using a conjugate gradient or other minimization methods.

The physical meaning of (10) is that the velocity perturbation can modify the phase $k_n^0$ and at the same time the amplitude $A_n^0$ for each single-mode synthetic seismogram.
calculated using the reference model. The phase change of the seismogram is related to wavenumber (eigenvalue) variation, whereas the amplitude change is mainly connected to eigenfunctions variations. These two changes are first-order effects of structural perturbation, as demonstrated in section 4.3. In the inversion, the complete spectrum of the starting model, instead of just the phase velocity as in the PWI, is updated through iterations, and the gradient of the penalty function $F(\gamma)$, the structural linear constraint, is calculated with a fully analytical formalism. In our experiment to simultaneously invert changes of seismogram amplitude and phase, and to test the purely numerical aspects of the problem, we consider synthetic seismograms constructed for six geophysically realistic structures.

The 'real' and the starting models are shown in Fig. 7 (solid lines and dashed lines, respectively). The synthetic waveforms of the fundamental and first two higher Rayleigh modes calculated for the 'real' model are used as 'data'. The source mechanism and azimuth are the same as in section 4.3. The configuration of these 'real' structural models is relatively more complex than the starting ones, and the maximum S-wave velocity difference between starting and 'real' models is as large as about ±6%. The structural models are characterized by several well pronounced discontinuities: the Moho (at 30 km) and two mantle boundaries (located at depths of 400 km and 640 km).

Although with the adoption of the analytical technique we can easily compute the differential seismogram for a fine scale multi-layered structure, the inversion of the large dimension model (many parameters) is impracticable. Following Nolet (1990), we impose the following restriction on the structural model: using equations (8), we model the S-wave velocity with a total of 13 depth functions, i.e. 13 basis functions $h_i(z)$. The crust is defined by 3 step functions whereas the mantle is parametrized with 10 linear triangular functions. Since our major interest is the upper mantle, we use 5 to 6 pivots (i.e. support points for linear interpolation to parametrize the depth dependence of the model) above the 400 km discontinuity and only 2 to 3 pivots below it. Below the 640 km discontinuity we use 2 pivots. The positions of the used pivots are flexible,
and the S-wave velocity jump at each discontinuity is obtained by using a pair of pivots at the same depth. The multi-layered (N layers) structure required for the forward computation is obtained from the expansion, \( h_m(z) \), of \( h_i(z) \):

In the crust:

\[
h_m(z) = h_i(z) \quad \text{if} \quad z_{i-1} < z_m \leq z_i
\] (12)

where \( i = 1, 2, 3 \)

In the mantle:

\[
h_m(z) = \begin{cases} 
\frac{z_m}{z_i} h_i(z) & \text{if} \quad z_{i-1} < z_m \leq z_i \\
\frac{z_{i+1} - z_m}{z_i} h_i(z) & \text{if} \quad z_i < z_m < z_{i+1}
\end{cases}
\] (13)

where \( i = 4, 5, \cdots 13, m = 1, 2, 3 \cdots N \), and \( z_i (z_0 = 0) \) and \( z_m \) are respectively the pivot and the layer depths.

In order to avoid that the inversion ends up at a local minimum, we start inversion from a low frequency. At each iteration we gradually increase the frequency band towards higher frequencies. A low-pass filter at 0.03 Hz is applied to the fundamental, while for the higher modes the low-pass frequency limit is 0.06 Hz. The conjugate gradient technique is used for the minimization of the penalty function.

From Fig. 7, we can see a good matching between 'real' and inverted models (dotted lines). Using an epicentral distance of 1000 km, we retrieve the model in the depth range from 30 km (just below the Moho) to 400 km (Models 1 and 2), while a deeper resolution, to a depth of about 600 km, can be reached considering an epicentral distance of 1800 km.

In Fig. 8 the inversion results (dotted lines) obtained using PDS as the structural constraint are shown. For the shorter epicentral distance, significant misfit is observable both for Model 1 and 2. The misfit in Model 2 can be explained by the fact that the amplitude changes are important for higher modes, as we have shown in section 3. The
significant misfit in the upper part of Model 1 is due to the presence of two major discontinuities, which are affecting the fundamental mode as well (see the upper traces in Fig. 4). The middle two (Models 3 and 4) both show large misfit, especially at large depth. A low-velocity channel is present in the upper mantle of 'real' Model 4, while the starting model is a very simple one, therefore the use of only the phase is not sufficient to retrieve a major structural feature represented by a velocity inversion with depth.

Comparing Models 5 and 6 (the lower two panels) in Figs. 7 and 8, we find that the differences between the two inversions obtained with the use of the structural constraints of DS and PDS are small. This is due to the fact that for Model 5 the difference between the starting and 'real' model is a constant shift of 0.20 km/s in the depth range from 30 km to 640 km. The normalized eigenfunctions for both models are shown in Fig. 9. In this special case of constant shift, the eigenfunction of the two models, the starting and the 'real', have nearly the same shape. Model 6, in the depth range from 30 km to 400 km, is more complex than Model 5, but the starting model is also parallel to the 'real' in the depth range from 30 km to 640 km. As expected, the increased complexity of the structural configuration doesn't spoil the main property: the eigenfunction of the starting model and of the 'real' one have nearly the same shape, and therefore the agreement between the 'real' and inverted model is satisfactory, as shown in Fig. 8. However, the discrepancies visible in the upper 30 km and below 640 km demonstrate that we need to use the amplitude as an additional constraint even in the case when the eigenfunction changes are small.

The differences between the 'real' and the inverted models can be quantified by the percent difference (i.e. (\('real' - inverted')/'real' \times 100), shown in Fig. 10. The differences between the 'real' and the inverted model, when using DS as the constraint on the structural model, are small and within ±1%. In contrast, the inversion with PDS as the constraint on the structural model reveals relatively larger differences: in more detail, for Model 1 it gives a difference of about ±2% in the upper 220 km, whereas larger than and of the same order differences as Model 1 are visible in the deeper part.
of Model 2, and Models 3 and 4. Although the differences between Models 5 and 6 in
the middle part are small: comparable with and a little larger than the inversion results
obtained with DS as the constraint on the structural model, the artifacts introduced at
the top and at the bottom of the models are around ±2%. From Fig. 8, we see that the
percent differences given by Models 2, 3 and 4 are larger than 50% of the pre-designed
differences between the ‘real’ and the starting model.

In Fig. 11 we show examples of the final fit to Model 1 that confirm the findings of
Stutzmann & Montagner (1993): the use of the amplitude information brings consistent
good fittings both in frequency and in time domains.

5. Conclusions

Through a detailed analysis of the differential seismogram (given in Paper 1), we
present a systematic study of the effects of structure perturbation on the amplitude
and on the phase of seismograms constructed with modal summation. The amplitude
differentiation (ADS) is critical for the correct computation of the differential seismogram
(DS), i.e. of the linear structural constraint, especially for the higher modes. By
including ADS in the waveform inversion, we obtain an efficient fully analytical scheme,
that is, the extension of the PWI (Nolet et al., 1986; Nolet, 1990). The new scheme
can be used, with an improved resolution with respect to the original formalism, for
waveform inversion at regional scale.

For geophysically realistic structures, at \( r = 1500 \) \( km \), we have found that for the
fundamental mode, in general, PDS is more important than ADS, and vice versa for the
higher modes. As PDS depends on the epicentral distance, ADS becomes increasingly
relevant when this parameter decreases.

Compared with the numerical approach, our analytical calculation of ADS requires
an insignificant amount of computation time. By taking this advantage, we extend the
phase depending kernel (PDS) of the PWI and obtain a new waveform inversion scheme,
which has a DS (both ADS and PDS) dependent kernel.

To describle the numerical aspects of the problem, we consider synthetic seismograms constructed for six geophysically realistic structures. Within a regional epicentral distance, less than 2000 km, we perform waveform inversion adopting two different structural constraints: PDS and DS. The inversion results show that satisfactory models can be retrieved only from the inversion that uses as structural constraint DS. Although with the use of PDS, as a constraint on the structural model, some relatively simple models, i.e. a systematic shift from the reference models, can be retrieved, the inadequate constraint gives rise to a large bias both at the top and at the bottom of the inverted models.

The method, in its present stage of development, can be applied to smoothly varying media, since for wave propagation in strongly inhomogeneous media, mainly the waveform amplitude, is affected by several well known but not easy to model factors, such as focusing, defocusing and scattering. The extension to media containing strong lateral heterogeneities, using the forward modelling based on the formalism developed by Vaccari et al. (1989), will be the subject of forthcoming studies.

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References


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Figure 1. Upper 200 km of the velocity models A and B. Model A contains two pronounced discontinuities in S-wave velocity at 20 km and 39 km of depth in the crust; Model B contains a smooth increase of S-wave velocity with depth in the crust. The whole structure used to determine the mode-spectrum extends to a depth of 1060 km.
first higher mode (layer 8)

first higher mode (layer 12)

first higher mode (layer 17)

first higher mode (layer 26)

(b)
Figure 2. Partial derivative spectra of the single-mode computed for Model A. $\frac{1}{F} \frac{\partial F}{\partial p_j}$ (solid line), $\frac{1}{I_i} \frac{\partial I_i}{\partial \nu_i}$ (long dashed line), $\frac{1}{c} \frac{\partial c}{\partial \nu_i}$ (dotted line) and $\frac{1}{u} \frac{\partial u}{\partial \nu_i}$ (dashed line) are computed for the layers 8, 12, 17 and 26 shaded in Fig. 1. The different quantities are defined in the text. (a) fundamental mode, (b) first higher mode, and (c) fourth higher mode.
fundamental mode (layer 8)

fundamental mode (layer 17)

fundamental mode (layer 12)

fundamental mode (layer 26)
Figure 3. As Fig. 2, but for $\frac{1}{A} \frac{\partial A}{\partial P_j}$ (solid line), $\frac{1}{E} \frac{\partial E}{\partial P_j}$ (dotted line), $1500 \cdot \frac{\partial k}{\partial P_j}$ (long dashed line), and $\frac{\partial k}{\partial P_j}$ (dashed line).
Figure 4. Vertical component of the single-mode seismogram partial derivative with respect to S-wave velocity, low-pass filtered at 0.03 Hz for the layers 8, 12, 17, and 26 shaded in Fig. 1 (Model A). The top trace in each group of three seismogram derivatives corresponds to the fundamental mode, the middle trace to the first higher mode, whereas the bottom one to the fourth higher mode. The solid line is for DS, the dotted line for PDS and the dashed line for ADS. The maximum amplitude is reported on the r.h.s. of each trace, and peak values are in units of $10^{-9}$. 
Figure 5. As Fig. 4, but low-pass filtered at 0.05 hz.
Figure 6. As Fig. 4, but for $r = 500$ km, and low-pass filtered at 0.1 Hz. Peak values are in units of $10^{-7}$. 
Figure 7. S-wave velocity Models 1 to 6, as indicated by the number on the lower left-hand corner in each plot. The 'real' and the starting models are shown as solid and dashed lines, respectively, whereas the waveform inversion results obtained with structural constraint DS are denoted by dotted lines.
Figure 8. As Fig. 7, for waveform inversion with the constraint PDS.
Figure 9. Normalized eigenfunctions for structural Model 5. In each panel, from left to right we show radial, vertical displacements, normal and tangential stresses in the order. The solid line is for the 'real' and the dashed line for the starting model.
Figure 10. Percent differences between 'real' and inverted models, by using the structural constraint DS (dotted line) and PDS (dashed line). The dot-dashed line is 0%. The structural Model is indicated by the number on the lower left-hand corner in each plot.
Figure 11. Final waveform inversion for Model 1: the upper two panels show the waveforms and the spectra for the fundamental mode whereas the bottom two refer to the higher modes. In each group of two waveforms on the left, the upper trace is the result of the inversion with the structural constraint PDS whereas the trace below is obtained with the constraint DS. The solid line is the 'data' and the dotted line is the inverted seismogram. On the right the corresponding spectra are shown: the solid line represents the 'data', and the dotted and dashed lines represent the results obtained with the constraint DS and PDS, respectively.