CATEGORIES OF PROJECTIVE GEOMETRIES
WITH MORPHISMS AND HOMOMORPHISMS

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ABSTRACT

Projective geometries studied as Pasch geometries possess morphisms and homomorphisms. A homomorphic image of a projective geometry is shown to be projective. A projective geometry is shown to be Desarguesian iff it is a homomorphic image of higher dimensional one, which in a sense is dual to the classical imbedding theorem. Semi-linear maps induce morphisms which are homomorphisms iff the associated homomorphisms of skewfields are isomorphisms. Projective geometries form categories with morphisms as well as homomorphisms and Desarguesian ones form a subcategory with Desarguesian homomorphisms.

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1 Introduction:

The geometries of double cosets and orbits of groups are studied abstractly as Pasch geometries, hypergroups, multigroups etc. The basic theory of Pasch geometries has been developed in [1]. These geometries with properly defined morphisms and homomorphisms form categories in which the classical projective geometries form a full subcategory. Quotients, homomorphisms and isomorphism theorems are available in the category of geometries with homomorphisms. This paper will focus on some of the consequences of this theory on projective geometries and related orbit spaces.

It is well known that a Desarguesian projective space is given by a vector space $V$ over a skew field $F$. This space can be studied as the space of orbits $V/F^*$ of the group $V$ under the action of $F^* = F - \{0\}$. In a more general situation a geometry is shown to be represented by $V/K$ where $K$ is a subgroup of $F^*$ (see[2]). Projective geometries are also obtained from geometric spaces over geometric skewfields (see[3]).

It is also known that every semi-linear isomorphism between vector spaces induces isomorphism between corresponding projective spaces and the Fundamental theorem asserts that for proper dimension the converse is also true. However, the not necessarily bijective semi-linear maps induce morphisms between corresponding projective spaces which are homomorphisms if the associated homomorphism is an isomorphism of the skew fields. The Fundamental theorem has been generalized to the homomorphisms. It is shown that Desarguesian spaces with such homomorphisms form a category giving a functor from vector spaces with semi-linear homomorphisms to Desarguesian geometries with Desarguesian homomorphisms. A classical theorem states that a projective plane is Desarguesian iff it can be imbedded in a higher dimensional space (see[4]). It is shown dually that a projective plane is Desarguesian iff it is homomorphic image of a higher dimensional projective space.

It should be pointed out that in the category of projective geometries with morphisms, the image of a morphism may not form a subgeometry of the codomain and quotients, isomorphism theorems are not available. These morphisms (sometimes called homomorphisms in the literature [5]) are interesting for further investigation. Recently Faure-Frolicher [6] have defined morphisms on projective geometries as partial maps for similar study. These morphisms seem to be equivalent to the morphisms of Pasch projective geometries. It seems to us that the language of Pasch geometry is a convenience for the study of such geometries and their morphisms.
2 Preliminaries:

The basic concept of the Pasch geometry can be found in the references, particularly in [1]. For convenience we briefly state some relevant definitions and results here.

2.1 Definition: By a Pasch geometry it is meant a triple $(A, e, \Delta_A)$ where $A$ is a set, $e \in A$, and $\Delta_A = \Delta \subseteq A \times A \times A$ subject to the following axioms:
(I) $\forall a \in A, \exists$ a unique $b \in A$ with $(a, b, e) \in \Delta$. Let $b = a^\#$.
(II) $e^\# = e$ and $(a^\#)^\# = a^\# a \in A$.
(III) $(a, b, c) \in \Delta \Rightarrow (b, c, a) \in \Delta$.
(IV) $(a_1, a_2, a_3), (a_1, a_4, a_5) \in \Delta \Rightarrow \exists a_6 \in A$ with $(a_6, a_1^\#, a_2), (a_6, a_5, a_3^\#) \in \Delta$.

As a consequence of the above one gets:
(V) $(a, b, c) \in \Delta \Rightarrow (c^\#, b^\#, a^\#) \in \Delta$.
(VI) $a, b \in A \Rightarrow \exists c \in A$ with $(a, b, c) \in \Delta$.

Throughout this paper, geometry will mean a Pasch geometry.

A geometry is called **Abelian** if $(a, b, c) \in \Delta \Rightarrow (b, a, c) \in \Delta$. A geometry is called **sharp** if $(a, b, c), (a, b, d) \in \Delta \Rightarrow c = d$. In such, if one lets $a \cdot b = c^\# \iff (a, b, c) \in \Delta$ then it makes $A$ into a group with $e$ as identity. Conversely every group $G$ is naturally a sharp geometry with $(x, y, z) \in \Delta_G$ iff $xyz = e$, the identity of $G$.

2.2 Subgeometry: Let $A$ be a geometry (with $e$ and $\Delta$ given implied) and $S \subseteq A$.

Then $S$ is called a **subgeometry** if $e \in S$ and $(s_1, s_2, x) \in \Delta, s_1, s_2 \in S \Rightarrow x \in S$. Let $\Delta_S = \Delta_A \cap (S \times S \times S)$. Then $(S, e, \Delta_S)$ is a geometry. We use $\Delta$ for both $A$ and $S$ or any other geometry to let the context distinguish.

A subgeometry $S$ is called **normal** if $\forall a, b \in A, (s, a, b) \in \Delta$ for some $s \in S \Rightarrow \exists s_1 \in S$ with $(s_1, b, a) \in \Delta$.

2.3 Quotient Geometry: Let $S$ be a subgeometry of $A$. For $a, b \in A$ define $a \sim b$ if $\exists s_1, s_2 \in S$ and $x \in A$ such that $(a, s_1, x^\#), (x, b^\#, s_2) \in \Delta$. This defines an equivalence relation on $A$. For $a \in A$, let $[a] = \{x : a \sim x\}$. Let $A//S = \{[a] : a \in A\}$ be the set of all equivalence classes. Define $\Delta_{A//S} = \{(X, Y, Z) : X, Y, Z \in A//S$ and $\exists x \in X, y \in Y, z \in Z$ with $(x, y, z) \in \Delta_A\}$. Then one gets ([1]):

2.4 Proposition: $(A//S, S, \Delta_{A//S})$ is a geometry.
Remark If $S$ is normal in $A$, then the equivalence relation is equivalent to: $a \sim b$ if $\exists s \in S$ with $(a, b^#, s) \in \Delta$.

2.5 Example: Consider a group $G$ with corresponding sharp geometry. Then $S$ is a subgeometry of $G$ iff it is a subgroup of $G$. $S$ is a normal subgroup iff it is a normal subgeometry. Also, the set $G//S$ corresponds to the set of double cosets $\{SaS, a \in G\}$. If $S$ is normal then clearly $SaS = aS$ the usual coset. Thus the geometry of $G//S$ is the geometry of double cosets. It is sharp if $S$ is normal and hence gives the quotient group $G/S$.

2.6 Morphisms: Let $A$ and $B$ be geometries. A map $f : A \rightarrow B$ is called a geometry morphism if $f(e_A) = e_B$ and $(x, y, z) \in \Delta_A \Rightarrow (f(x), f(y), f(z)) \in \Delta_B$. For $f$ such a morphism let $K_f = \{a \in A : f(a) = e_B\}$, called the kernel of $f$ and $Imf = \{b \in B : b = f(x) \text{ for some } x \in A\}$, called the image of $f$. Then $K_f$ is a subgeometry of $A$ which may not be normal. Also $Imf$ may not be a subgeometry of the codomain.

For any subgeometry $S$ of $A$, the natural map $A \rightarrow A//S$ is a morphism.

A morphism $f : A \rightarrow B$ is called an isomorphism if $f$ is bijective and $f^{-1}$ is also a morphism. For such, one has $(x, y, z) \in \Delta_A \Leftrightarrow (f(x), f(y), f(z)) \in \Delta_B$. A bijective morphism may not be an isomorphism. As an example, let $A$ be the sharp geometry of a cyclic group $\{a, e\}$ and $B = Q/Q^* = \{[1], [0]\}$ be the orbit geometry of rationals acted by non-zero rationals. Let $f(a) = [1], f(e) = [0]$. Then $f$ is a bijective morphism but not an isomorphism.

2.7 Homomorphism: A map $f : A \rightarrow B$ is called a homomorphism if it is a morphism and $(f(x), f(y), b) \in \Delta_B \Rightarrow \exists z \in A \text{ with } b = f(z) \text{ and } (x, y, z) \in \Delta_A$.

It checks that a bijective homomorphism is an isomorphism.

With the kernel and image defined as above we have the following (see[1]):

2.8 Proposition: If $f : A \rightarrow B$ is a homomorphism, then $K_f$ is a normal subgeometry of $A$, $Imf$ is a subgeometry of $B$ and $A//K_f \cong Imf$.

In particular, for any normal subgeometry $S$ of $A$, the natural map $A \rightarrow A//S$ is a surjective homomorphism. Other homomorphism theorems are also valid [1].

2.9 Action of a Group: Let $A$ be a geometry and $G$ be a group written multiplicatively with 1. Then $G$ is said to act on $A$ if there is a homomorphism from $G$ to the group of geometry automorphisms of $A$. Thus for $\alpha \in G$ and $a \in A$ we get $\alpha a \in A$ with $ae_A = e_A$, $1a = a$, $(a, b, c) \in \Delta$ implies $(\alpha a, \alpha b, \alpha c) \in \Delta$; $\alpha(\beta a) = (\alpha \beta) a$ etc. If $G$ acts on $A$, then $A$ is called a $G$-geometry [1].

A subgeometry $S$ is called a $G$-subgeometry if $\forall \alpha \in G$ we get $\alpha s \in S \forall s \in S$. 

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If $A$ and $B$ are $G$-geometries, a morphism $f : A \rightarrow B$ is called a $G$-morphism if $f(\alpha a) = \alpha f(a) \forall \alpha \in G$ and $\forall a \in A$. The homomorphism theorems are valid when homomorphisms are replaced by $G$-homomorphisms. If $S$ is a $G$-subgeometry, then the action of $G$ on $A//S$ is obviously defined as $\alpha X = \{\alpha x : x \in X\}$.

2.10 Geometry of Orbits: Let $A$ be a $G$-geometry and $H$ be a subgroup of $G$ so that $A$ is also an $H$-geometry. For $a \in A$, let $[a] = \{ha : h \in H\}$ be the orbit of the element $a$ under $H$-action. Let $A/H = \{[a] : a \in A\}$ be the set of orbits. Let $([a],[b],[c]) \in \Delta_{A/H}$ iff $\exists x \in [a], y \in [b], z \in [c]$ with $(x,y,z) \in \Delta_A$. This makes $A/H$ a geometry with $[e] = \{e\}$, called the geometry of orbits of $A$ by $H$. If also $H$ is normal in $G$, then $A/H$ is a $G/H$-geometry in a natural way. The natural map $A \mapsto A/H$ is a morphism which, in general, is not a homomorphism.

We have the correspondence theorem between the $H$-subgeometries of $A$ and the subgeometries of the orbit space $A/H$ ([1]):

2.11 Proposition: Let $A$ be an $H$-geometry and $A/H$ be the geometry of orbits. For an $H$-subgeometry $S$ of $A$, let $S/H = \{[s] : s \in S\}$. Then $S/H$ is a subgeometry of $A/H$. The correspondence $S \leftrightarrow S/H$ is a bijection between $H$-subgeometries of $A$ and subgeometries of $A/H$. Moreover, $(A//S)/H \cong (A/H)/(S/H)$.

3 Projective Geometries and Related Results

3.1 Definition: Let $(A,e,\Delta)$ be a geometry. It is called projective if $a^# = a \forall a \in A$ and $(a,a,b) \in \Delta$ implies $b = a$ or $b = e$. It checks that a projective geometry must be Abelian.

Let $A$ be a projective geometry and $A^* = A - \{e\}$. Then taking the elements of $A^*$ as points and distinct points $a,b,c$ to be collinear if $(a,b,c) \in \Delta$, a usual projective space is obtained including degenerate ones (points and lines). Conversely, if $P$ is the set of points of a projective space, we let $A = P \cup \{e\}, e \notin P$, one defines $\Delta_A$ in a natural way and gets a projective geometry (see [2]). In particular, if $V$ is a vector space over a skew field $F$ then $V$ is naturally an $F^*$-geometry and the geometry of orbits $V/F^*$ as defined above is projective and corresponds to the classical projective space $\mathcal{P}(V)$ associated to $V$.

Clearly, subgeometry of a projective geometry is projective and corresponds to a subspace of the projective space and conversely.
For the quotient geometry, we have the following:
3.2 Proposition: Let $S$ be a subgeometry of a projective geometry $A$. Then $A//S$ is projective.

Proof: Since $A$ is projective, $X^# = X \forall X \in A//S$ is clear. So let $(X, X, Y) \in \Delta_{A//S}$. We show $Y = X$ or $S$. By definition,

$\exists x_1 \in X, x_2 \in X, y \in Y$ with $(x_1, x_2, y) \in \Delta_A$. Since $x_1, x_2 \in X, x_1 \sim x_2$, so $\exists s \in S$ with $(x_1, x_2, s) \in \Delta$. So $(x_1, x_2, y), (x_1, x_2, s) \in \Delta$ implies $\exists t \in A$ with $(t, x_2, x_2), (t, s, y) \in \Delta$. But $A$ is projective so $(t, x_2, x_2) \in \Delta$ implies either $t = e$ or $t = x_2$. If $t = e$, then $(e, s, y) \in \Delta$ implies $y = s^# = s \in S$, so $Y = S$. If $t = x_2 \in X$, then obviously $Y = X$.

Thus subgeometries and homomorphic images of projective geometries are projective.

Let $A$ be a projective geometry. Then its dimension can be defined as the length of maximal chain of proper subgeometries $(e) \subset S_1 \subset S_2 \subset S_3 \ldots \subset A$. If the dimension of $A$ is $n$, then the so-called projective dimension is $n - 1([2])$. For quotient geometries, we have

3.3 Proposition: If $A$ is a projective geometry of finite dimension and $S$ is a subgeometry, then $\dim(A//S) = \dim(A) - \dim(S)$.

The proof is obtained with standard arguments by extending a chain for $S$ to that of $A$.

The following is the generalized form of the so-called Desargue’s configuration:

3.4 Definition: A (not necessarily abelian) geometry $A$ is called a D-geometry if

$(x_0, x_1, x_2), (x_0, y_1, y_2), (x_0, z_1, z_2) \in \Delta \Rightarrow \exists s, t, u \in A$ such that $(s, y_1^#, x_1), (s, y_2, x_2^#), (t, z_2^#, y_1), (t, z_2, y_2^#), (u, x_1^#, z_1), (u, x_2, z_2^#), (s, t, u) \in \Delta$.

We point out that in [2] $(s, t, u)$ was misprinted as $(u, t, s)$.

It checks that every sharp geometry (i.e. a group) is a D-geometry. Also, if a geometry $A$ is a D-geometry and is acted by a group $H$, then the orbit geometry $A/H$ is also a D-geometry. In particular, the orbit space $V/F^*$ of a vector space over $F$ is a D-geometry and hence a Desarguesian projective space which is well known.

If a projective geometry $A$ has $\dim(A) \geq 4$ (projective $\dim(A) \geq 3$), then it is well known that $A \cong V/F^*$, where $V$ is a vector space over $F$ with $\dim(A) = \dim(V)$. Also, if $\dim(A) = 3$ (a projective plane) then such is true only when $A$ is a D-geometry (Desarguesian projective plane). A classical theorem states that a plane is Desarguesian iff it can be imbedded in a space of higher dimension. As a dual to this we have:
3.5 Theorem: Let $A$ be a projective geometry with $\dim(A) = 3$. Then $A$ is Desarguesian iff it is a homomorphic image of a projective geometry of dimension $\geq 4$.

Proof: Let $B$ be a projective geometry of dimension $\geq 4$ and let $f : B \mapsto A$ be a surjective homomorphism. Then as is well known $B$ is Desarguesian and there exists a vector space $V$ over a skew field $F$ such that $g : V/F^* \mapsto B$ is an isomorphism of geometries. Then $gf : V/F^* \mapsto A$ is a surjective homomorphism. The kernel of the homomorphism is a subgeometry of $V/F^*$ and so by proposition (2.11) $\ker(gf) = W/F^*$ where $W$ is a $F$-subgeometry of $V$ i.e. a subspace of $V$. So $(V/F^*)/(W/F^*) \cong A$. But by the same proposition $(V/F^*)/(W/F^*) \cong (V//W)/F^*$. So $A \cong (V//W)/F^*$. Now the quotient space $V//W$ is a vector space over $F$ and the isomorphism of geometries establishes that $A$ is Desarguesian.

The converse is easily obtained.

From the proof we get a useful corollary:

3.6 Cor: A subgeometry of $V/F^*$ is of the form $U/F^*$ where $U$ is a subspace of $V$.

In view of the the above theorem, we make the

3.7 Definition: We call a projective geometry Desarguesian if it is a homomorphic image of a Desarguesian projective geometry of dimension $\geq 3$.

3.8 Remark: The definition implies that if $A$ is a projective geometry with $\dim(A) = 2$, then $A \cong V/F^*$ for some vector space $V$ over $F$. There is no uniqueness of the field except in the finite case where two fields of the same cardinality are isomorphic. Also, for $\dim(A) = 1$, $A$ must be of the form $F/F^*$ which contains two elements. There are only two such non-isomorphic geometries; one is sharp when $F$ is $Z_2$ and the other non-sharp for $F$ different from $Z_2$.

4 Semi-linear Maps and Induced Morphisms

Let $V$ and $W$ be vector spaces over the skew fields $F$ and $K$ respectively. A pair of maps $(g, \hat{g}) : (V, F) \mapsto (W, K)$ is called a semi-linear map if $g : V \mapsto W$ is a group homomorphism: $g(v_1 + v_2) = g(v_1) + g(v_2)$ and $\hat{g} : F \mapsto K$ is a homomorphism of skewfields such that $g(\alpha v) = \hat{g}(\alpha)g(v) \forall \alpha \in F$ and $\forall v \in V$. Given such a semi-linear map, define $\hat{g} : V/F^* \mapsto W/K^*$ by $\hat{g}(< v >) = < g(v) > \forall < v > \in V/F^*$, where for $v \in V$ the corresponding element of $V/F^*$ is written as $< v >$. It easily checks that it is a well
defined map between the projective geometries $V/F^*$ and $W/K^*$.

4.1 Theorem: Let $(g, \hat{g}) : (V, F) \mapsto (W, K)$ be a semi-linear map. Then

(i) The induced map $\bar{g} : V/F^* \mapsto W/K^*$ is a morphism of projective geometries.

(ii) If the homomorphism $\hat{g}$ is surjective (so an isomorphism of the skew fields), then
the induced map $\bar{g}$ is a homomorphism.

(iii) The converse of (ii) is true if $\dim(\text{Im}(\hat{g})) \geq 2$.

Proof:

(i) Clearly, $\bar{g}(\langle 0 \rangle) = \langle 0 \rangle$. So let $\langle v_1, v_2, v_3 \rangle \in \Delta_V$. Then
$\langle v_1, av_2, bv_3 \rangle \in \Delta_V$ for some $a, b \in F^*$. So $v_1 + av_2 + bv_3 = 0$. Since $g$ is semi-linear, so
$g(v_1) + \hat{g}(a)g(v_2) + \hat{g}(b)g(v_3) = 0$ giving $g(v_1) + \hat{g}(a)g(v_2) + \hat{g}(b)g(v_3) \in \Delta_W$ which implies
$\langle g(v_1), g(v_2), g(v_3) \rangle \in \Delta_V/F^*$ i.e. $\langle g(v_1), g(v_2), g(v_3) \rangle \in \Delta$. So
$\bar{g}$ is a morphism of geometries.

(ii) To show that $\bar{g}$ is a homomorphism, let $\bar{g}(\langle v_1 \rangle), \bar{g}(\langle v_2 \rangle), \langle w \rangle \in \Delta_W/K^*$ so
that $\langle \hat{g}(v_1), \hat{g}(v_2), \langle w \rangle \rangle \in \Delta$. Then $\exists x, y \in K^*$ with $\hat{g}(a) = x$
so $g(v_1) + \hat{g}(a)g(v_2) + yw = 0$ giving $g(v_1) + \hat{g}(a)g(v_2) = -yw$ i.e. $g(v_1 + av_2) = -yw$ so
$\langle g(v_1), g(v_2), g(v_1 + av_2) \rangle = \langle -yw \rangle \in \Delta$, or $\bar{g}(\langle v_1 \rangle), \bar{g}(\langle v_2 \rangle), \langle v \rangle \in \Delta/V/K^*$ is clear, thus proving that $\bar{g}$ is a homomorphism of geometries.

(iii) Suppose the induced map $\bar{g}$ is a morphism and $\dim(\text{Im}(\hat{g})) \geq 2$. To show $\bar{g}$ is
surjective, let $x \in K, x \neq 0$. Let $v_1, v_2 \in V$ be such that $g(v_1), g(v_2)$ are independent in $W$.
Consider $g(v_1) + xg(v_2)$ in $W$. Clearly $\langle g(v_1), g(v_2), g(v_1) + xg(v_2) \rangle \in \Delta$.
So $\bar{g}(\langle v_1 \rangle), \bar{g}(\langle v_2 \rangle), \langle g(v_1) + xg(v_2) \rangle \in \Delta$. Since $\bar{g}$ is a homomorphism of geometries, $\exists \langle v \rangle \in V/F^*$ with $\langle g(v_1), g(v_2) \rangle = \langle g(v_1) + xg(v_2) \rangle = \langle g(v_1) \rangle$ for some $a, b \in F^*$. So
$\hat{g}(a) = y$ and $\hat{g}(b) = yx$. Now $\hat{g}(a^{-1}b) = \hat{g}(a)^{-1}\hat{g}(b) = y^{-1}yx = x$ and $\hat{g}$ is surjective.

Example: Let $\hat{g} : R \mapsto C$ be the imbedding of real field into complex field. It

gives a semi-linear map $(g, \hat{g}) : R^3 \mapsto C^3$. The induced map $\bar{g} : R^3/R^* \mapsto C^3/C^*$ is obviously a morphism of projective geometries which is not a homomorphism: if
e_1 = (1,0,0), e_2 = (0,1,0), then (\bar{g}(e_1), \bar{g}(e_2), e_1 + ie_2) \in \Delta but there exists no \nu \in \mathbb{R}^3 with \bar{g}(\nu) =< e_1 + ie_2 >.

In view of the above theorem, we make the

4.2 Definition: We call a semi-linear map \((g, \hat{g}): (V, F) \mapsto (W, K)\) a semi-linear homomorphism if the associated map \(\hat{g}\) is an isomorphism of skewfields.

Thus a semi-linear map induces a morphism of corresponding projective geometries and the morphism is a homomorphism iff the semi-linear map is a semi-linear homomorphism.

But for such induced homomorphisms, we make the

4.3 Definition: For vector spaces \(V\) and \(W\) over \(F\) and \(K\), we call a homomorphism \(f: V/F* \mapsto W/K*\) Desarguesian if there exists a semi-linear homomorphism \((g, \hat{g}): (V, F) \mapsto (W, K)\) such that \(\hat{g} = f\).

The Fundamental theorem of Projective Geometry states that an isomorphism of projective geometries of proper dimension is Desarguesian. The following generalizes it to homomorphisms.

4.4 Theorem: Let \(f: V/F* \mapsto W/K*\) be a homomorphism of projective geometries. If \(\text{dim}(\text{Im}f) \geq 3\), then \(f\) is Desarguesian.

Proof We may assume \(f\) surjective. By (3.) kernel \(K_f = U_1/F*\) where \(U_1\) is a subspace of \(V\). Let \(V = U_1 \oplus U_2\). Then \(v \in V \Rightarrow v = u_1 + u_2\) where \(u_1 \in U_1, u_2 \in U_2\). So \((u_1, u_2, -v) \in \Delta_V\) giving \(< u_1, u_2, v > \in \Delta\). Since \(f\) is homomorphism, \((f(< u_1 >), f(< u_2 >), f(< v >)) \in \Delta W/K*\). It gives \((< 0 >, f(< u_1 >), f(< v >)) \in \Delta\) which implies \(f(< u_2 >) = f(< v >)\). So the restriction \(f: U_2/F* \mapsto W/K*\) is surjective and easily seen to be an isomorphism of projective geometries. Since \(\text{dim}(W/K*) \geq 3\), by Fundamental theorem there is a semi-linear isomorphism \((g, \hat{g}): (U_2, F) \mapsto (W, K)\) such that \(\hat{g} = f\) on \(U_2/F*\). Define \(g(u_1) = 0 \forall u_1 \in U_1\) and extend \(g\) uniquely to \(V\). It clearly gives \(g = f\).

4.5 Proposition: If \(f: V/F* \mapsto W/K*\) is a homomorphism and \(\text{dim}(\text{Im}f) = 1\), then \(f\) is Desarguesian iff \(F \cong K\).

Proof If \(f\) is Desarguesian, \(F \cong K\), by definition. Conversely, let \(\hat{g}: F \mapsto K\) be an isomorphism. Let \(\text{Im}f = \{< w, 0 >\}\) and \(K_f = H/F*\). One chooses \(v \in V\) with \(f(< v >) = < w >\). Then \(V = H \oplus Fv\) and define \(g\) by \(g(h) = 0 \forall h \in H\), and \(g(xv) = \hat{g}(x)g(v)\). Then \(g\) is extended to get \(g = f\).
Remark: If $\text{dim}(\text{Im} f) = 2$, there may exist homomorphisms which are not Desarguesian even if $F \cong K$.

4.6 Categories of Projective Geometries:
From the preceding discussion, we arrive at the following categories of Projective geometries.

4.6.1 The Category $\mathcal{P}_m$: The objects of the category are projective geometries. For objects $A$ and $B$ the morphisms $\mathcal{P}_m(A, B)$ is the set of all morphisms of geometries (cf. 2.6). Let $\mathcal{V}_{sm}$ be the category with objects $(V, F)$ where $V$ is a vector space over $F$ and morphism the semi-linear maps (cf. 4.1). Then we have a functor $\Lambda_m : \mathcal{V}_{sm} \rightarrow \mathcal{P}_m$ given by $\Lambda_m(V, F) = V/F^*$ and $\Lambda_m(g, \tilde{g}) = \tilde{g}$.

4.6.2 The Category $\mathcal{P}_h$: The objects are projective geometries. The morphisms $\mathcal{P}_h(A, B)$ is the set of all homomorphisms of geometries (cf. 2.7). If $\mathcal{V}_{sh}$ is the category of vector spaces with semi-linear homomorphisms (cf. 4.2), then we have a functor from $\mathcal{V}_{sh}$ to $\mathcal{P}_h$, since by 4.1(ii) a semi-linear homomorphism induces a homomorphism of projective geometries.

4.6.3 The category $\mathcal{P}_{dh}$. Here the objects are Desarguesian projective geometries and the morphisms are Desarguesian homomorphisms (cf. 4.3). Then the functor from $\mathcal{V}_{sh}$ to $\mathcal{P}_{dh}$ is a full functor.

4.7 Remarks: A projective geometry is also obtained from a geometric space $V$ over a geometric skewfield $A$ as an orbit space $V/A^*$ [3]. Geometric semi-homomorphisms from $(V, A)$ to $(W, B)$ induce homomorphisms of these projective geometries. In the Desarguesian case, different groups of semi-linear isomorphisms of vector spaces give rise to semi-isomorphisms in a way similar to projective linear groups. Study of such will appear elsewhere.

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