INCOMPRESSIBLE FLUID BINARY SYSTEMS
WITH INTERNAL FLOWS - MODELS OF CLOSE BINARY NEUTRON
STAR SYSTEMS WITH SPIN

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MIRAMARE - TRIESTE
October 1997
ABSTRACT

In this paper we have examined numerically exact configurations of close binary systems composed of incompressible fluids with internal flows. Component stars of binary systems are assumed to be circularly orbiting each other but rotating nonsynchronously with the orbital motion, i.e. stars in binary systems have steady motions seen from a rotational frame of reference. We have computed several equilibrium sequences by taking fully into account the tidal effect of Newtonian gravity without approximation. We consider two binary systems consisting of either 1) a point mass and a fluid star or 2) a fluid star and a fluid star. Each of them corresponds to generalization of the Roche–Riemann or the Darwin–Riemann problem, respectively. Our code can be applied to various types of incompressible binary systems with various mass ratios and spin as long as the vorticity is constant.

We compare these equilibrium sequences of binaries with approximate solutions which were studied extensively by Lai, Rasio and Shapiro (LRS) as models for neutron star–neutron star (NS–NS) binary systems or black hole–neutron star (BH–NS) binary systems. Our results coincide qualitatively with those of LRS but are different from theirs for configurations with small separations. For these binary systems, our sequences show that dynamical or secular instability sets in as the separation decreases. The quantitative errors of the ellipsoidal approximation amount to $2 \sim 10\%$ for configurations near the instability point. Compared to the results of LRS, the separation of the stars at the point where the instability sets in is larger and correspondingly the orbital frequency at the critical state is smaller for our models. Since these sequences can be considered as evolutionary models of binary NS systems where component stars are approaching due to the effect of gravitational wave emission, we can expect that the final fate of such binary systems will be dynamical coalescence due to dynamical instability.
1 INTRODUCTION

Close binary systems composed of massive compact objects such as neutron stars or black holes are relevant systems in astrophysics, because they are possible sources of many varieties of astrophysical phenomena. A close binary system of compact objects will evolve quasi-statically by decreasing its orbital separation due to gravitational radiation emission until a final stage such as a violent merging or mass overflow will be reached. During inspiraling and coalescence phases, binary systems may be observed as sources of not only cosmological \( \gamma \)-ray bursts (see e.g. Paczynski 1986) but also gravitational waves which will be detectable by kilometer size laser interferometers such as LIGO and VIRGO (see e.g. Abramovici et al. 1992; Thorne 1994).

The evolution of binary systems prior to merging was discussed by Kochanek (1992) and Bildsten & Cutler (1992). In their papers, it was pointed out that, for most viscosities known thus far, the timescale of angular momentum transfer within a neutron star is longer than that of evolution due to gravitational radiation reaction. Therefore the effect of viscosity may be negligible during the evolution of binary systems toward coalescence. In other words, viscosity may not be powerful enough to synchronize spin motions nor change spin of component stars.

In the framework of Newtonian gravity without viscosity, circulation of a fluid remains constant even when the gravitational radiation reaction potential is included (Miller 1974). Therefore, stars in binary systems will not rotate synchronously but have internal flows or spin in the frame of reference rotating with the orbital angular velocity.

Because of a long timescale of energy dissipation due to gravitational radiation, such evolutionary sequences of binary systems can be followed by using equilibrium configurations even until stars come near to the innermost stable orbit (see for example, Shapiro & Teukolsky 1983). However, it is very difficult to construct stationary states of compressible stars with internal motions even today for non-axisymmetric configurations such as binary stars (Uryū & Eriguchi 1996, and references therein) except for the case of irrotational gaseous star binary systems (Uryū & Eriguchi 1997). Several years ago, Lai Rasio & Shapiro (hereafter LRS) developed a semi-analytical method to treat equilibrium of incompressible and compressible stars under the assumption that stellar shapes and equi-density contours are approximated by ellipsoids (Lai, Rasio & Shapiro 1993a) (LRS1 hereafter). In a series of papers, they applied their method to binary neutron star systems with spin and extensively
examined the innermost stable circular orbits where dynamical instability of binary systems sets in (Lai, Rasio & Shapiro 1994a and 1994b: hereafter LRS2 and LRS3, respectively).

Equilibrium configurations of incompressible binary fluid star systems with internal motions were computed numerically exactly by Eriguchi and Hachisu (1985). Furthermore, Eriguchi, Futamase and Hachisu (1990) computed sequences of binary disks with internal motions. In these papers they computed sequences of equal mass binary disks or stars which can be used to study evolutionary sequences due to gravitational wave emission by keeping the circulation of each component constant and showed that there exist turning points on equilibrium sequences. In other words, they indicated occurrences of secular and dynamical instabilities prior to the work of Lai, Rasio & Shapiro (LRS1).

In this paper we have extended the computational method developed by Eriguchi and Hachisu (1985) so as to compute Roche type binary configurations and/or binary systems with arbitrary mass ratios and internal motions, and have reexamined equilibrium models of incompressible binaries whose parameters are in the physically meaningful range. First, we have computed sequences of synchronously rotating binary systems consisting of a point mass and a fluid, i.e. Roche type binary systems, and have compared our results with those obtained by using the ellipsoidal approximation by LRS. Second, we have concentrated on configurations of irrotational Roche–Riemann and Darwin–Riemann problems for which vorticity seen from the inertial frame vanishes. We have compared our results again with ellipsoidal models computed in LRS1 and LRS2. Our results coincide qualitatively with those of LRS. For these binary systems, our sequences show that dynamical or secular instability sets in. The quantitative errors of the ellipsoidal approximation are usually $2 \sim 10\%$ for models near the instability point. These sequences of equilibrium binary stars can be used as quasi-evolutionary models of binary NS systems in which the separation of two component stars is decreasing due to the effect of gravitational wave emission. Consequently such binary systems will result in coalescence due to dynamical instability.

In the next section we present the basic equations and explain briefly the numerical method for incompressible fluid binary systems with spin, in section 3 we show computational results of binary sequences for several types, and in section 4 we discuss evolution of binary systems of compact objects by applying our results.
2 FORMULATION OF THE PROBLEM

2.1 Assumptions and basic equations

We assume plane symmetry of stellar structures about the x-y plane and the x-z plane in the rotational frame of reference. Here the Cartesian coordinates \((x, y, z)\) are used. The \(z\)-axis is chosen along the axis of rotation and centers of mass of two stars are located on the \(x\)-axis. In this paper we consider neutron stars which are approximated by incompressible fluids. Although this is a crude approximation, we have not found appropriate schemes to treat compressible self-gravitating gases with generic steady internal flows as mentioned in Introduction (see also Uryu & Eriguchi 1996). In particular, it is difficult to formulate a well-posed problem and derive proper basic equations for steady states of compressible fluids, when the flow is planer along the equatorial plane, in other words, the vorticity vector \(\mathbf{\zeta}\) of the fluid is parallel to the rotational axis, \(\mathbf{\zeta} = \zeta \mathbf{e}_z\) where \(\mathbf{e}_i\) \((i = x, y, z)\) denotes the unit basis vector along each coordinate axis.

On the other hand, for incompressible fluids we can formulate the problem properly by introducing the stream function for planer flows as follows:

\[
\begin{align*}
    u_x &= \frac{\partial \Psi}{\partial y}, & u_y &= -\frac{\partial \Psi}{\partial x},
\end{align*}
\]  

(1)

where \(\Psi\) and \((u_x, u_y)\) are the stream function and \(x\)- and \(y\)-components of the velocity vector in the rotational frame, respectively. Note that \(\Psi\) and \((u_x, u_y)\) depend only on \((x, y)\) and that the equation of continuity is automatically satisfied. Thus we can write down basic equations and boundary conditions consistently.

If the stream function is a function of the \(z\)-component of vorticity vector, \(\zeta\), which we will refer to as the vorticity, the Euler equation of fluid motion can be integrated to the Bernoulli equation. Since we consider the case with \(\zeta = \text{const}\) everywhere in the fluid, the Bernoulli equation becomes as follows:

\[
\frac{1}{2}(u_x^2 + u_y^2) + (\zeta + 2\Omega)\Psi - \frac{1}{2}\Omega^2(X^2 + Y^2) + \frac{p}{\rho} + \phi = C,
\]  

(2)

where \(\Omega\), \(p\), \(\rho\), \(\phi\) and \(C\) are the orbital angular velocity, the pressure, the density, the Newtonian gravitational potential and the Bernoulli constant, respectively. The quantity \(X^2 + Y^2\) means the square of the distance of fluid element from the rotational axis.

By substituting the velocity components Eq. (1) into the following definition of vorticity
the equation for the stream function can be derived as,

$$\Delta^{(2)}\psi = -\zeta,$$  \hspace{1cm} (4)

where

$$\Delta^{(2)} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \hspace{1cm} (5)$$

As for the gravitational potential, we use the integral form of the Poisson equation as follows:

$$\phi(r) = -G\rho \int_V \frac{1}{|r - r'|} \, d^3r'. \hspace{1cm} (6)$$

Here $G$ is the gravitational constant and $V$ is the total volume over which the fluid extends.

Boundary conditions for the equations described above are as follows:

$$p(x, y, z) = 0, \text{ on the surface of the star,} \hspace{1cm} (7)$$

and

$$\Psi(x, y) = 0, \text{ on the equatorial surface of the star.} \hspace{1cm} (8)$$

It should be noted that boundary conditions for the gravitational potential are already included in the integral representation Eq. (6).

### 2.2 Solving method

In actual computations, we introduce two spherical coordinate systems $(r_\pm, \theta_\pm, \varphi_\pm)$ whose origins are fixed on the $x$-axis at distances

$$d_\pm \equiv \frac{|R_{out}^\pm + R_{in}^\pm|}{2}, \hspace{1cm} (9)$$

for each fluid component, where $R_{out}^\pm$ and $R_{in}^\pm$ are distances from the rotational axis to the outer and inner edges of the star on the $x$-axis, respectively. Here subscripts $+$ and $-$ correspond to the primary and the secondary stars, respectively. For Roche type binary systems, the primary star corresponds to a fluid star and the secondary to a point mass. Therefore note that usually the mass of primary $M_+$ is smaller than that of secondary $M_-$, i.e. $M_+ \leq M_-$. The angle $\theta_\pm$ is the zenithal angle measured from the positive direction parallel to the $z$-axis and the angle $\varphi_\pm$ is the azimuthal angle measured from the positive direction of the $x$-axis. Relations between the Cartesian coordinates $(X_\pm, Y_\pm, Z_\pm)$ and the
polar coordinate systems \((r, \theta, \varphi)\) are expressed as follows:

\[
\begin{align*}
X_\pm &= \pm d_\pm + r_\pm \sin \theta_\pm \cos \varphi_\pm, \\
Y_\pm &= \pm d_\pm + r_\pm \sin \theta_\pm \sin \varphi_\pm, \\
Z_\pm &= r_\pm \cos \theta_\pm.
\end{align*}
\]

Hereafter we will omit subscripts \(\pm\) except for necessary cases in order to make expressions simple. Since deformation of the stellar surface from sphere is not so large for binary stars, we may consider the stellar surface as a function of \(\theta\) and \(\varphi\) and write it as \(R(\theta, \varphi)\) in this coordinate.

In the coordinate systems adopted here, we can analytically integrate the gravitational potential (6) along the \(r\)-direction from the center to the surface of the star \(R(\theta, \varphi)\) because the density \(\rho\) is constant. For incompressible stars, we only need to solve for the shape of the surface \(R(\theta, \varphi)\) because physical quantities such as the pressure or the gravitational potential are completely determined from the shape of the surface (see e.g. Eriguchi, Hachisu & Sugimoto 1982; Eriguchi & Hachisu 1983; Eriguchi & Hachisu 1985).

The stream function \(\Psi\) is also expressed in terms of the surface variable. Since \(\zeta\) is assumed to be constant, we can explicitly write down the stream function \(\Psi\) by using power series expansion on the equatorial plane as follows:

\[
\Psi(r \sin \theta, \varphi) = -\zeta \left\{ \frac{1}{4} (r \sin \theta)^2 + \sum_{m=0}^{\infty} a_m (r \sin \theta)^m \cos m \varphi \right\}.
\]

Imposing the boundary condition (8), i.e.

\[
\Psi(R(\pi/2, \varphi), \varphi) = 0, \tag{12}
\]

on the equatorial surface, we can obtain coefficients \(a_m\) for a given vorticity. Components of the fluid velocity in the rotational frame \((u_x, u_y)\) are computed by differentiating this expression.

Substituting the above formulae into the Bernoulli equation on the fluid surface and imposing the boundary condition (7) for the pressure there, we finally obtain the basic equations for numerical computations. They are basically the equations (2) and (12) (see Appendix A). The derived equations contain two unknown variables \(R(\theta, \varphi)\) and \(a_m\), and several unknown physical constants. These equations are calculated in the coordinate system \((\theta, \varphi)\) over the region \(\theta \in [0, \pi/2]\) and \(\varphi \in [0, \pi]\). This region is discretized equidistantly into grid points as follows: \((\theta_i, \varphi_i)\) \((0 \leq i_{\theta} \leq N_{\theta}, 0 \leq i_{\varphi} \leq N_{\varphi})\) and we choose \((N_{\theta}, N_{\varphi}) = \ldots\)
As for the coefficients $a_m$, we include terms up to $m = N_v/2$. The trapezoidal formula is used for the integration of the gravitational potential.

One model of a binary system can be numerically computed by specifying some physical quantities which characterize the binary system. We can choose the values of physically meaningful parameters so as to be able to compute an appropriate sequence by changing some of them. For the present purpose, it is convenient to specify the mass ratio and the separation of two component stars as two parameters. For other parameters, we choose relations for the vorticity of the fluid star (see Appendix B). Since we consider equilibrium sequences whose vorticity is conserved in the inertial frame as stars evolve adiabatically by the force of the gravitational radiation reaction potential, the following quantity must be constant in time as well as in space (Miller 1974):

$$\zeta + 2\Omega = \zeta_0 = \text{constant}, \quad (13)$$

where $\zeta_0$ is the vorticity in the inertial frame of reference. In particular, the stars with $\zeta_0 = 0$ are usually referred to as irrotational configurations.

The basic equations are discretized on the mesh points. Since these difference equations are nonlinear equations for $R(\theta, \varphi)$, $a_m$ and unknown constants, $C$ and $\Omega$, the Newton-Raphson method is applied to solve them. Since the Newton-Raphson scheme requires a rather large matrix, we were able to employ only the mesh number mentioned above. For typical models, our computational time was $\sim 3.5$ minutes for 1 iteration for a fluid star-fluid star binary system and $\sim 5.7$ seconds for a point mass-fluid star binary system by using a Fujitsu VX vector computer. A further discussion on the solving method is described in the appendices in detail.

### 3 RESULTS

It is convenient to use nondimensional quantities to compare our results with those of others. Thus physical quantities are normalized by using normalization factors as follows.

The nondimensional orbital angular velocity $\tilde{\Omega}$ is defined as:

$$\tilde{\Omega} = \frac{\Omega}{\sqrt{G\rho}}. \quad (14)$$

Concerning the separation, the nondimensional separation between centers of mass of two component stars $\tilde{d}_G$ is defined as

$$\tilde{d}_G = \frac{(d_{G+} + d_{G-})}{(3M_{\text{tot}}/4\pi \rho)^{1/3}}, \quad (15)$$
where the total mass is expressed as $M_{\text{tot}} = M_+ + M_-$, and we define $d_{G+}$ as follows:

$$d_{G+} = \rho \left| \int V X_+ dV \right| \frac{1}{M_+}.$$ (16)

The separation between two geometrical centers is normalized as

$$\ddot{d} = \frac{(d_+ + d_-)}{(3M_{\text{tot}}/4\pi \rho)^{1/3}}.$$ (17)

The total angular momentum $J$ and the total energy $E$ are normalized as

$$\ddot{J} = \frac{J}{\sqrt{G} M_{\text{tot}}^{5/3} \rho^{-1/6}},$$ (18)

and

$$\ddot{E} = \frac{E}{G M_{\text{tot}}^{5/3} \rho^{1/3}}.$$ (19)

In the Tables we also show the luminosity of gravitational radiation, which is related to the quadrupole momentum tensors of the binary system. In the rotating frame of reference, the quadrupole momentum tensors are defined as

$$I_{ij}^{(\text{rot})} = \rho \int_V (x_i x_j - \delta_{ij} \frac{|x|^2}{3}) dV,$$ (20)

where $i, j = 1, 2, 3$ correspond to the $x, y, z$-component of the Cartesian coordinates, respectively. By taking the symmetry of the system into account, the quadrupole formula for the energy generation rate is written as follows:

$$\frac{dE}{dt} = -\frac{32G}{5 c^5} \Omega^6 (I_{11}^{(\text{rot})} - I_{22}^{(\text{rot})})^2,$$ (21)

where $c$ is the velocity of light. Derivation of this formula is shown in the book of Misner, Thorne & Wheeler (1970) (see also Eriguchi, Futamase & Hachisu 1990). This energy generation rate is normalized as

$$\ddot{\overline{E}} = \frac{d\overline{E}}{dt} \left/ \left( \frac{v_0}{c} \right)^5 \right.,$$ (22)

where nondimensional time is defined as

$$\ddot{t} = \sqrt{G \rho t},$$ (23)

and

$$v_0 = \sqrt{G M_{\text{tot}}^{2/3} \rho^{1/3}}.$$ (24)

In the following subsections, our results for three different types of binary configurations are discussed. They are,

(i) Roche configurations with $M_+/M_- = 1$ and $0.1$. 


(ii) Irrotational Roche–Riemann configurations with several mass ratios.

(iii) Irrotational Darwin–Riemann configurations with several mass ratios.

LRS computed several types of binary configurations by using the ellipsoidal approximation. Concerning binary solutions, nothing but synchronously rotating fluid–fluid binary configurations with equal mass, i.e. Darwin models, are compared in their papers with fully deformed solutions. We compare our results with those of LRS for models listed above and clarify quantitative differences. These comparisons provide the evidence that our computational method works accurately.

3.1 Roche type binary systems

We show our results in Tables 1, 2 and 3 for Roche sequences with the mass ratio $M_+/M_- = 1, 0.5$ and $0.1$, respectively. These sequences can be regarded as simplified models of BH–NS binary systems in the limit of strong viscosity (Kochanek 1992) and in the frame of Newtonian gravity.

In Figures 1 and 2, we show our results and those of LRS. In Figures 1(a)–(c) and 2(a)–(c), physical quantities, $\tilde{J}$, $\tilde{E}$ and $\tilde{\Omega}$, are plotted against the separation of mass centers of two component stars $\tilde{d}_G$ and against the non-dimensional angular momentum. The models on a sequence at the smallest separation in Figures 1(a), (b) and 2(a), (b) almost correspond to those at the Roche limit. Here we define the Roche limit as the smallest separation of two centers of mass of two bodies. Note that models shown in the last entry of the Tables are not the exact critical models.

As seen from these figures, two results obtained by two different methods agree rather well even near the turning points of sequences. Differences of $\tilde{J}$ or $\tilde{E}$ between two results are less than 0.5% for most part of the sequence. It is also shown that our values of the separation between the centers of mass of the fluid star and the point mass agree with those of LRS to within $2 \sim 6\%$ around the models with the minimum of $\tilde{J}$ or $\tilde{E}$ where secular instability sets in and also around the models near the Roche limit stage. Our results show that the separations where the instability sets in increase by the amount of roughly $\sim 4\%$ and $\sim 2\%$ for $M_+/M_- = 1$ and $0.1$ sequences, respectively, and correspondingly the orbital frequency at the critical state decreases by the amount of 6% and 3%, respectively, compared to the results of LRS.

Here it should be noted that the secular instability occurs via angular momentum transfer
Table 1. Equilibrium sequence of Roche type binary systems with equal mass

<table>
<thead>
<tr>
<th>(d)</th>
<th>(dG)</th>
<th>(\Delta)</th>
<th>(J)</th>
<th>(\delta)</th>
<th>(\dot{dE}/dt)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>4.079</td>
<td>2.486(-1)</td>
<td>4.104(-1)</td>
<td>-3.525(-1)</td>
<td>3.883(-3)</td>
</tr>
<tr>
<td>4.8</td>
<td>4.039</td>
<td>2.630(-1)</td>
<td>4.039(-1)</td>
<td>-3.542(-1)</td>
<td>4.088(-3)</td>
</tr>
<tr>
<td>4.6</td>
<td>3.782</td>
<td>2.78(-1)</td>
<td>3.974(-1)</td>
<td>-3.560(-1)</td>
<td>5.693(-3)</td>
</tr>
<tr>
<td>4.4</td>
<td>3.551</td>
<td>2.959(-1)</td>
<td>3.909(-1)</td>
<td>-3.579(-1)</td>
<td>6.957(-3)</td>
</tr>
<tr>
<td>4.2</td>
<td>3.347</td>
<td>3.146(-1)</td>
<td>3.84(-1)</td>
<td>-3.598(-1)</td>
<td>8.556(-3)</td>
</tr>
<tr>
<td>4.0</td>
<td>3.160</td>
<td>3.351(-1)</td>
<td>3.782(-1)</td>
<td>-3.619(-1)</td>
<td>1.059(-2)</td>
</tr>
<tr>
<td>3.8</td>
<td>3.007</td>
<td>3.57(-1)</td>
<td>3.720(-1)</td>
<td>-3.640(-1)</td>
<td>1.319(-2)</td>
</tr>
<tr>
<td>3.6</td>
<td>2.870</td>
<td>3.82(-1)</td>
<td>3.66(-1)</td>
<td>-3.662(-1)</td>
<td>1.653(-2)</td>
</tr>
<tr>
<td>3.4</td>
<td>2.757</td>
<td>4.08(-1)</td>
<td>3.60(-1)</td>
<td>-3.684(-1)</td>
<td>2.083(-2)</td>
</tr>
<tr>
<td>3.2</td>
<td>2.664</td>
<td>4.37(-1)</td>
<td>3.55(-1)</td>
<td>-3.706(-1)</td>
<td>2.537(-2)</td>
</tr>
<tr>
<td>3.0</td>
<td>2.589</td>
<td>4.68(-1)</td>
<td>3.51(-1)</td>
<td>-3.726(-1)</td>
<td>3.047(-2)</td>
</tr>
<tr>
<td>2.9</td>
<td>2.525</td>
<td>4.84(-1)</td>
<td>3.491(-1)</td>
<td>-3.735(-1)</td>
<td>3.573(-2)</td>
</tr>
<tr>
<td>2.8</td>
<td>2.475</td>
<td>5.10(-1)</td>
<td>3.475(-1)</td>
<td>-3.744(-1)</td>
<td>4.229(-2)</td>
</tr>
<tr>
<td>2.7</td>
<td>2.435</td>
<td>5.17(-1)</td>
<td>3.461(-1)</td>
<td>-3.751(-1)</td>
<td>4.780(-2)</td>
</tr>
<tr>
<td>2.6</td>
<td>2.395</td>
<td>5.33(-1)</td>
<td>3.451(-1)</td>
<td>-3.756(-1)</td>
<td>5.36(-2)</td>
</tr>
<tr>
<td>2.5</td>
<td>2.362</td>
<td>5.50(-1)</td>
<td>3.44(-1)</td>
<td>-3.759(-1)</td>
<td>6.064(-2)</td>
</tr>
<tr>
<td>2.3</td>
<td>2.262</td>
<td>5.79(-1)</td>
<td>3.450(-1)</td>
<td>-3.775(-1)</td>
<td>7.387(-2)</td>
</tr>
</tbody>
</table>

Consequently, as the separation is decreased, the system reaches first a state where secular instability sets in and next comes to the Roche limit state. This behavior is also seen in models of LRSl for sequences whose mass ratios are the same as those of ours.

3.2 Irrotational Roche–Riemann type binary systems

As mentioned in Introduction, neutron stars may be composed of inviscid fluids (Kochanek 1992; Bildsten & Cutler 1992). For incompressible fluids, the vorticity \(\zeta_0\) is conserved during
Table 3. Equilibrium sequence of Roche type binary systems with $M_+/M_- = 0.1$

<table>
<thead>
<tr>
<th>$\bar{d}$</th>
<th>$d_{eq}$</th>
<th>$\bar{\Omega}$</th>
<th>$\bar{J}$</th>
<th>$\bar{E}$</th>
<th>$\bar{dE}/dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.0</td>
<td>4.630</td>
<td>3.655(-1)</td>
<td>1.407(-1)</td>
<td>-3.315(-2)</td>
<td>2.239(-4)</td>
</tr>
<tr>
<td>9.6</td>
<td>4.661</td>
<td>3.173(-1)</td>
<td>1.382(-1)</td>
<td>-3.269(-2)</td>
<td>2.699(-4)</td>
</tr>
<tr>
<td>9.2</td>
<td>4.293</td>
<td>2.302(-1)</td>
<td>1.356(-1)</td>
<td>-3.326(-2)</td>
<td>3.271(-4)</td>
</tr>
<tr>
<td>8.8</td>
<td>4.127</td>
<td>2.442(-1)</td>
<td>1.330(-1)</td>
<td>-3.388(-2)</td>
<td>3.988(-4)</td>
</tr>
<tr>
<td>8.4</td>
<td>3.963</td>
<td>2.596(-1)</td>
<td>1.304(-1)</td>
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</tr>
<tr>
<td>8.0</td>
<td>3.800</td>
<td>2.764(-1)</td>
<td>1.278(-1)</td>
<td>-3.524(-2)</td>
<td>6.032(-4)</td>
</tr>
<tr>
<td>7.6</td>
<td>3.641</td>
<td>2.939(-1)</td>
<td>1.252(-1)</td>
<td>-3.599(-2)</td>
<td>7.485(-4)</td>
</tr>
<tr>
<td>7.2</td>
<td>3.485</td>
<td>3.150(-1)</td>
<td>1.226(-1)</td>
<td>-3.678(-2)</td>
<td>9.539(-4)</td>
</tr>
<tr>
<td>6.8</td>
<td>3.332</td>
<td>3.370(-1)</td>
<td>1.200(-1)</td>
<td>-3.757(-2)</td>
<td>1.170(-3)</td>
</tr>
<tr>
<td>6.4</td>
<td>3.185</td>
<td>3.608(-1)</td>
<td>1.175(-1)</td>
<td>-3.835(-2)</td>
<td>1.473(-3)</td>
</tr>
<tr>
<td>6.0</td>
<td>3.043</td>
<td>3.865(-1)</td>
<td>1.150(-1)</td>
<td>-3.913(-2)</td>
<td>1.857(-3)</td>
</tr>
<tr>
<td>5.6</td>
<td>2.909</td>
<td>4.139(-1)</td>
<td>1.127(-1)</td>
<td>-4.037(-2)</td>
<td>2.341(-3)</td>
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</tr>
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<td>0.969(-1)</td>
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</tr>
<tr>
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<td>0.954(-1)</td>
<td>-5.219(-2)</td>
<td>1.325(-3)</td>
</tr>
<tr>
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<td>0.939(-1)</td>
<td>-5.313(-2)</td>
<td>1.448(-3)</td>
</tr>
<tr>
<td>0.4</td>
<td>1.721</td>
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<td>0.924(-1)</td>
<td>-5.405(-2)</td>
<td>1.570(-3)</td>
</tr>
</tbody>
</table>

Evolution due to the gravitational radiation reaction (Miller 1974). Furthermore the orbital angular velocity of the star at a large separation is much smaller than that at a separation of the stellar size. Since $\zeta_0$ is of the order of the orbital angular velocity at large distances, the constant in equation (13) can be regarded very small and can be approximated to zero. In other words, almost all realistic neutron star binary systems can be approximated by irrotational configurations as far as viscosity is small.

We show our results for irrotational Roche-Riemann sequences with $M_+/M_- = 1, 0.5$ and 0.1 in Tables 4, 5 and 6, respectively. In Figures 3 and 4, our results and those of LRS1 are displayed for the sequences with $M_+/M_- = 1$ and 0.1. For all the sequences, the turning points of $\bar{J}$ and $\bar{E}$ against the separation exist as seen in Figures 3 and 4, which is qualitatively the same as that of the ellipsoidal approximation. It implies that there is a point where the dynamical instability will set in on the sequence between large separation states and the Roche-Riemann limit of the sequence. Here the Roche-Riemann limit is defined as the generalization of the Roche limit to fluid stars with internal flows. Therefore, it is probable that an irrotational star around a massive point source will fall onto the point mass due to dynamical instability as the separation becomes smaller. This behavior is the same as that obtained by the ellipsoidal approximation. Under the ellipsoidal approximation, there always exists a dynamical instability point for any mass ratio for Roche-Riemann...
Figure 1. Physical quantities of a sequence of Roche type binary systems with equal mass. (a) Total angular momentum as a function of a binary separation, (b) Total energy as a function of a binary separation, (c) Orbital angular velocity as a function of the total angular momentum. Dashed and solid curves denote the sequence of LRS1 and our sequence, respectively. See text about the normalization factors for each quantity.

type binary systems before reaching the Roche–Riemann limit as the separation decreases (LRS1). The quantitative difference of the physical values between these two results near the turning point of the sequence amounts roughly to $3 \sim 5\%$ as seen from Tables and Figures. In particular, the separation of the minimum of $J$ and $E$ increases by $3.5\%$ and $2.5\%$ for the sequences with $M_+ / M_- = 1$ and $0.1$, respectively, and the orbital frequency decreases by $5\%$ and $4\%$, respectively.
3.3 Irrotational Darwin–Riemann type binary systems

In LRS2 they calculated physical quantities for irrotational Darwin–Riemann sequences by assuming ellipsoidal configurations. As shown in the previous sections, exact computations have given qualitatively the same results for models even with smaller separations. Thus we have further computed stationary solutions of two fluid stars with $\zeta_0 = 0$. These types of binary systems can be regarded as models of viscosity-free NS–NS binary systems just before coalescence. Our numerically exact results for an irrotational Darwin–Riemann sequence with equal mass are tabulated in Table 7. In Figures 5(a)–(c) we show our results and those of LRS2. Two results agree with each other very well even near the turning point of the
Table 4. Equilibrium sequence of irrotational Roche—Riemann type binary systems with equal mass

<table>
<thead>
<tr>
<th>$\dot{a}$</th>
<th>$\dot{a}_d$</th>
<th>$\dot{\Omega}$</th>
<th>$\dot{J}$</th>
<th>$\dot{E}$</th>
<th>$dE/dt$</th>
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</thead>
<tbody>
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<tr>
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</tr>
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<tr>
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</table>

Table 5. Equilibrium sequence of irrotational Roche—Riemann type binary systems with $M_+/M_- = 0.5$

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<tr>
<th>$\dot{a}$</th>
<th>$\dot{a}_d$</th>
<th>$\dot{\Omega}$</th>
<th>$\dot{J}$</th>
<th>$\dot{E}$</th>
<th>$dE/dt$</th>
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<td>4.934(-2)</td>
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</table>

sequence to within 0.5%. Two fluid components of our present computation come to contact each other at the point of the smallest separation in Figure 5 and Table 7.

The turning point of $\dot{J}$ or $\dot{E}$ corresponds to the dynamical instability point of the sequence just as the Roche—Riemann binary systems. The dynamical instability point appears before the two stars come to contact as the separation decreases. This suggests that the binary configuration becomes dynamically unstable when the separation decreases due to gravitational radiation emission before these two fluid components contact each other. This behavior of the sequence is the same as that obtained in LRS2. The separation of the sequence where $J$ or $E$ attains its minimum is larger by 6% and the corresponding frequency is 10% smaller than those of LRS’s result. These differences from those of LRS are larger...
Table 6. Equilibrium sequence of irrotational Roche–Riemann type binary systems with $M_+/M_- = 0.1$

<table>
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<th>$\dot{E}$</th>
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</table>

than those for the point mass-fluid star binary systems. It is because, for the fluid star-fluid star binary system, both components are subject to tidal effect from the other companion star.

In Table 8, we tabulate a sequence of irrotational fluid-fluid binary configurations with $M_+/M_- = 0.5$. Physical quantities of this sequence are also plotted in Figure 6. By considering that the mass of neutron stars ranges from the minimum mass of $\sim 0.7M_\odot$ to the maximum mass of $\sim 1.4M_\odot$, it is reasonable to take the minimum of the mass ratio
Figure 3. Same as Figure 1 but for quantities of a sequence of irrotational Roche–Riemann type binary systems with equal mass.

to $M_+/M_- = 0.5$. For any mass ratio, our results coincide qualitatively with those of LRS again, except for the sequences of the mass ratios around $M_+/M_- = 0.1$. Since the primary star deforms considerably for models on such sequences, results of our computations become less accurate as far as we employ the grid numbers mentioned before.

4 DISCUSSION AND CONCLUSION

Since we have succeeded in developing a highly accurate numerical code, we can investigate fully deformed configurations for various types of incompressible fluid binary systems without any approximation. Among them we have concentrated on sequences of the Roche, the irrotational Roche–Riemann and the irrotational Darwin–Riemann type configurations in
this paper because these sequences are physically important for the evolution of BH–NS or NS–NS binary systems due to gravitational radiation emission. Concerning general binary systems for which the vorticity seen from the inertial frame does not vanish and the internal motion exists in the rotating frame, such as Roche–Riemann or Darwin–Riemann binaries, we have computed several sequences. However, if the initial spin of the fluid star is rather small compared with the final orbital angular velocity, the spin does not affect the final results appreciably. Therefore we did not show our results for such sequences in this paper.

Differences between our results and those of LRS become relatively large when the two component stars are coming closer and the mass ratio of two components differs from unity. It is because stars deform considerably around the Roche lobe overflowing stage which cannot be expressed by the ellipsoidal approximation used by LRS. However the qualitative

Figure 4. Same as Figure 3 but for $M_j/M_\pm = 0.1$. 
Figure 5. Same as Figure 1 but for quantities of a sequence of irrotational Darwin-Riemann type binary systems with equal mass.

character of the equilibrium sequences are the same. In other words, the dynamical and/or secular instability point appears before the models reach the Roche lobe overflowing stage, as shown in LRS1 and LRS3. Our results show that generally the separation at the onset of the instability is larger and correspondingly the orbital frequency at that point is smaller compared to the results of the ellipsoidal approximation by the amount of $2 \sim 10\%$.

In order to clarify the final fate of binary systems, it would be better to define four critical distances as follows (see e.g. Lai, Rasio & Shapiro 1993b). (i) $r_R$: the Roche (Darwin)–Riemann limit below which no equilibrium states exist, i.e. the minimum separation of binary stars; (ii) $r_C$: the contact limit where two stars contact each other (note that this limit exists only for a fluid–fluid star type binary system with equal mass and the equilibrium
Table 8. Equilibrium sequence of irrotational Darwin–Riemann type binary systems with $M_+/M_- = 0.5$

<table>
<thead>
<tr>
<th>$d$</th>
<th>$d_2$</th>
<th>$d_1$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$dE/,dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>4.20</td>
<td>2.20</td>
<td>3.60</td>
<td>4.00</td>
<td>2.266(-3)</td>
</tr>
<tr>
<td>4.8</td>
<td>4.25</td>
<td>2.35</td>
<td>3.65</td>
<td>4.05</td>
<td>2.390(-3)</td>
</tr>
<tr>
<td>4.6</td>
<td>4.30</td>
<td>2.40</td>
<td>3.70</td>
<td>4.10</td>
<td>2.490(-3)</td>
</tr>
<tr>
<td>4.4</td>
<td>4.35</td>
<td>2.45</td>
<td>3.75</td>
<td>4.15</td>
<td>2.570(-3)</td>
</tr>
<tr>
<td>4.2</td>
<td>4.40</td>
<td>2.50</td>
<td>3.80</td>
<td>4.20</td>
<td>2.630(-3)</td>
</tr>
<tr>
<td>4.0</td>
<td>4.45</td>
<td>2.55</td>
<td>3.85</td>
<td>4.25</td>
<td>2.670(-3)</td>
</tr>
<tr>
<td>3.8</td>
<td>4.50</td>
<td>2.60</td>
<td>3.90</td>
<td>4.30</td>
<td>2.700(-3)</td>
</tr>
<tr>
<td>3.6</td>
<td>4.55</td>
<td>2.65</td>
<td>3.95</td>
<td>4.35</td>
<td>2.720(-3)</td>
</tr>
<tr>
<td>3.4</td>
<td>4.60</td>
<td>2.70</td>
<td>4.00</td>
<td>4.40</td>
<td>2.740(-3)</td>
</tr>
<tr>
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<td>2.75</td>
<td>4.05</td>
<td>4.45</td>
<td>2.760(-3)</td>
</tr>
<tr>
<td>3.0</td>
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<td>2.80</td>
<td>4.10</td>
<td>4.50</td>
<td>2.780(-3)</td>
</tr>
<tr>
<td>2.8</td>
<td>4.75</td>
<td>2.85</td>
<td>4.15</td>
<td>4.55</td>
<td>2.800(-3)</td>
</tr>
<tr>
<td>2.75</td>
<td>4.80</td>
<td>2.90</td>
<td>4.20</td>
<td>4.60</td>
<td>2.820(-3)</td>
</tr>
<tr>
<td>2.65</td>
<td>4.85</td>
<td>2.95</td>
<td>4.25</td>
<td>4.65</td>
<td>2.840(-3)</td>
</tr>
<tr>
<td>2.55</td>
<td>4.90</td>
<td>3.00</td>
<td>4.30</td>
<td>4.70</td>
<td>2.860(-3)</td>
</tr>
<tr>
<td>2.45</td>
<td>4.95</td>
<td>3.05</td>
<td>4.35</td>
<td>4.75</td>
<td>2.880(-3)</td>
</tr>
<tr>
<td>2.35</td>
<td>5.00</td>
<td>3.10</td>
<td>4.40</td>
<td>4.80</td>
<td>2.900(-3)</td>
</tr>
</tbody>
</table>

(iii) $r_m$: the hydrodynamical stability limit below which stars fall down onto each other dynamically due to the tidal potential; and (iv) $r_{GR}$: the general relativistic stability limit where no stable orbits exist due to the effect of strong gravity. Although the last critical distance $r_{GR}$ does not appear in our present treatment, the binary stars fall down onto each other when two stars come within $r_{GR} \sim 6G(M_+/\,c^2)$. Concerning a typical BH–NS binary system whose masses are around $\sim 10 M_\odot$ (BH) and $\sim 1 M_\odot$ (NS), we should consider the general relativistic effect of BH at the final stage of evolution. Taniguchi & Nakamura (1996) have computed the incompressible ellipsoidal model of an irrotational neutron star around a black hole which is represented by using the improved pseudo-Newtonian potential. In their paper they studied models for which relativistic effect dominates, i.e. the models with $r_R < r_m < r_{GR}$ as typical BH–NS systems. It implies that such binary systems become dynamically unstable due to the general relativistic effect but not due to the tidal effect and that the dynamical instability occurs before the star deforms significantly.

Since the point of dynamical instability always appears prior to the Roche (Darwin)–Riemann overflowing state as shown by LRS as far as the ellipsoidal approximation is concerned, a possibility of occurrence of the Roche (Darwin)–Riemann overflow has not been considered seriously. Therefore their conclusion is simple, i.e. that binary systems necessarily result in dynamical coalescence, although general relativity, tidal effect and compressibility may change the results. On the other hand, for realistic BH–NS binary systems, there may
exist configurations for which the Roche–Riemann overflow will occur prior to dynamically falling as an alternative final fate, i.e. models with $r_m < r_R$. In order to show whether such a situation can be realized, we should take the compressibility of the fluid component into consideration because we have shown that there always exist dynamical instability points on the incompressible BH-NS binary sequences. We can expect that the effect of compressibility leads the fluid star to reach the Roche–Riemann overflow state before getting to a dynamically unstable state. It is opposite to the effect of strong gravity of BH. This effect of compressibility will be discussed in the subsequent paper (Uryū & Eriguchi 1997).

If the equilibrium sequence of a binary system cannot reach a point where dynamical instability sets in, we need to consider the problem whether this Roche–Riemann overflow proceeds violently or not. It will certainly depend on the compressibility and the gravitational
field of the black hole. Roughly speaking, for the fluids with stiff equations of state, if the star loses mass, the radius of the star is likely to shrink. Thus the mass overflow will stop. However, since there is gravitational wave emission, the system will evolve to the Roche–Riemann overflow stage again. In this case the mass overflow will continue on the time scale of gravitational wave emission. On the other hand, if the equation of state is soft, the radius of a mass losing star will expand. It implies that the mass overflow will continue violently. Such a mass transfer between two stars has been discussed by many authors (see Kochanek 1992; Bildsten & Cutler 1992 and references therein). When we consider the black hole, this situation may be analogous to the runaway mass overflow for self-gravitating axisymmetric toroidal configurations (see e.g. Nishida et al. 1996; Masuda & Eriguchi 1997).

Although we have only considered incompressible fluids in this paper, the compressibility would make Roche (Darwin)–Riemann overflow more probable. Consequently the final fate of real binary systems with non-equal masses could be determined from the competition between the occurrence of the state of the marginally stable orbit due to general relativity and the occurrence of the state of Roche (Darwin)–Riemann overflow. For compressible gases, we have recently succeeded in developing a new computational method for irrotational gaseous binary systems with equal mass (Uryū & Eriguchi 1997). We will be able to investigate more realistic situations in the near future as long as Newtonian gravity is concerned, and the results presented here will be extensively used to calibrate the results computed by the new computational code. Concerning the distance $r_{GR}$, we will be able to implement the general relativistic effect in our treatment of the Roche–Riemann type binary systems by using the pseudo-Newtonian potential as was done by Taniguchi & Nakamura (1996). After such computations, the nature of the final fate will be finally clarified.

One of us (KU) would like to thank Profs. Dennis W. Sciama and John C. Miller and Dr. Antonio Lanza for their warm hospitality at ICTP and SISSA. He would also like to thank Prof. Marek Abramowicz and Dr. Vladimir Karas for their encouragements. A part of the numerical computation was carried out at the Astronomical Data Analysis Center of the National Astronomical Observatory, Japan.
In this section, we describe the basic equations used in our actual computations. To clarify the discussion hereafter, we call the stellar surface $R(\theta, \varphi)$ and Fourier coefficients $a_m$ as the physical variables or the variables and the other quantities, $\zeta, \Omega, d$ and $C$ as the physical parameters or the constants. They are the unknowns which appear in our formulation as will be seen below.

Because of the incompressibility and the choice of the coordinate mentioned in section 2, we can integrate the gravitational potential of the fluid and express it in terms of the stellar surface $R(\theta, \varphi)$. It means that the value of the gravitational potential of the incompressible fluid is determined only from the shape of the stellar surface $R(\theta, \varphi)$. Consequently we can reduce the physical variables in the Bernoulli equation (2) to the stellar surface $R(\theta, \varphi)$, the Fourier coefficients $a_m$ and other constants. Therefore we will describe each term of this Bernoulli equation on the surface.

The first term of the LHS of equation (2) on the surface is calculated after substituting equation (11) into (1) as follows:

$$\frac{1}{2}(u_x^2 + u_y^2) = \frac{1}{2} \zeta^2 \left\{ \frac{1}{4} [R(\theta, \varphi) \sin \theta]^2 \right.$$

$$+ \sum_{m=1}^{\infty} m a_m [R(\theta, \varphi) \sin \theta]^m \cos m \varphi$$

$$+ \left[ \sum_{m=1}^{\infty} m a_m [R(\theta, \varphi) \sin \theta]^{m-1} \sin m \varphi \right]^2$$

$$+ \left[ \sum_{m=1}^{\infty} m a_m [R(\theta, \varphi) \sin \theta]^{m-1} \cos m \varphi \right]^2 \right\}.$$  \hspace{1cm} (A1)

The second term of the LHS of equation (2) on the surface is directly expressed from equation (11) as

$$(\zeta + 2\Omega) \Psi = -(\zeta + 2\Omega) \zeta \left\{ \frac{1}{4} [R(\theta, \varphi) \sin \theta]^2 \right.$$

$$+ \sum_{m=0}^{\infty} a_m [R(\theta, \varphi) \sin \theta]^m \cos m \varphi \right\}.$$ \hspace{1cm} (A2)

For the fluid star-fluid star binary system, we need to add subscripts $\pm$ in the two equations shown above. The third term of the LHS of equation (2) on the surface is calculated as follows for the fluid star-fluid star binary:

$$-\frac{1}{2} \Omega^2 (X_\pm^2 + Y_\pm^2) = -\frac{1}{2} \Omega^2 \left\{ [R_\pm(\theta_\pm, \varphi_\pm) \sin \theta_\pm]^2 \right.$$

$$\pm 2d_\pm R_\pm(\theta_\pm, \varphi_\pm) \sin \theta_\pm \cos \varphi_\pm + d_\pm^2 \right\}.$$ \hspace{1cm} (A3)

The fourth term of the LHS vanishes identically on the stellar surface.
The fifth term of the LHS of equation (2) which is the gravitational potential term is divided into two parts: 1) the contribution from its own component \( \phi_s \) and 2) that from the other \( \phi_e \), i.e. the relation \( \phi = \phi_s + \phi_e \) holds. These gravitational potentials are expressed in the integral form of the Poisson equation (see equation (6)). Since the Green's function can be expanded in terms of the Legendre functions, we can analytically integrate the potentials along the radial direction (see e.g. Eriguchi, Hachisu & Sugimoto 1982; Eriguchi & Hachisu 1983; Eriguchi & Hachisu 1985). After the integration, the gravitational potentials on the stellar surface can be expressed as follows. The self-gravity of its own component becomes

\[
\phi_s(\theta, \varphi) = -G \rho \int_0^\pi \sin \theta' d\theta' \int_0^{2\pi} d\varphi' \times \sum_{n=0}^{\infty} f_n[R(\theta, \varphi), R(\theta', \varphi')] P_n(\cos \beta), \tag{A4}
\]

where \( f_n[R(\theta, \varphi), R(\theta', \varphi')] \) is defined as

\[
\begin{align*}
\frac{1}{n + 3} & \frac{R(\theta', \varphi')^{n+3}}{R(\theta, \varphi)^{n+1}} & \text{for } R(\theta', \varphi') \leq R(\theta, \varphi), \\
\left( \frac{1}{n + 3} + \frac{1}{n - 2} \right) & R(\theta, \varphi)^2 - \frac{1}{n - 2} \frac{R(\theta, \varphi)^n}{R(\theta', \varphi')^{n-2}} & \text{for } R(\theta, \varphi) \leq R(\theta', \varphi') \text{ and } n \neq 2, \\
R(\theta, \varphi)^2 \left[ \frac{1}{5} + \ln \frac{R(\theta, \varphi)}{R(\theta', \varphi')} \right] & \text{for } R(\theta, \varphi) \leq R(\theta', \varphi') \text{ and } n = 2.
\end{align*}
\tag{A5}
\]

Here \( P_n \) is the Legendre function and \( \cos \beta \) is defined as

\[
\cos \beta = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi'). \tag{A6}
\]

In the above expression we omit subscripts \( \pm \) which should be used to distinguish the primary and the secondary stars.

The external gravitational potential from the other component can be expressed as follows:
\[
\phi_{\pm}(\theta_{\pm}, \varphi_{\pm}) = -G\rho \int_0^\pi \sin \theta_{\mp} \, d\theta_{\mp} \int_0^{2\pi} d\varphi_{\mp} \\
\times \sum_{n=0}^{\infty} \frac{1}{n + 3} \frac{R_{\mp}(\theta_{\pm}, \varphi_{\pm})^{n+3}}{D_{\pm}^{n+1}} P_n(\cos \gamma_{\pm}),
\]  
(A7)

where \( D_{\pm} \) and \( \cos \gamma_{\pm} \) are defined as

\[
D_{\pm} = \left\{ \left( d_+ + d_- \right)^2 + R_{\pm}(\theta_{\pm}, \varphi_{\pm}) \right\}^{1/2},
\]

(A8)

and

\[
\cos \gamma = \left\{ \left( d_+ + d_- \right) R_{\mp}(\theta_{\mp}, \varphi_{\mp}) \right\} \sin \theta_{\mp} \cos \varphi_{\mp} \\
+ R_{\pm}(\theta_{\pm}, \varphi_{\pm}) R_{\mp}(\theta_{\mp}, \varphi_{\mp}) \cos \beta \right\} / R_{\pm}(\theta_{\mp}, \varphi_{\mp}) D_{\pm},
\]

(A9)

respectively. Because of the symmetry of the \( x-z \) and the \( x-y \) planes, we can impose the following condition for the shape of the star:

\[
R(\theta, \varphi) = R(\pi - \theta, \varphi) = R(\theta, 2\pi - \varphi).
\]

(A10)

Consequently we can reduce the integral region to a quarter of the whole space,

\[
\int_0^\pi d\theta' \int_0^{2\pi} d\varphi' \quad \rightarrow \quad \int_0^{\pi/2} d\theta' \int_0^\pi d\varphi'.
\]

(A11)

Accordingly, the Legendre polynomials in the gravitational potentials should be rearranged. For \( P_n(\cos \beta) \) in \( \phi_3 \), we should replace it as follows:

\[
P_n(\cos \beta) \quad \rightarrow \quad P_n(\cos \beta_a) + P_n(\cos \beta_b) \\
+ P_n(\cos \beta_c) + P_n(\cos \beta_d),
\]

(A12)

where

\[
\cos \beta_a = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi'), \\
\cos \beta_b = -\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi'), \\
\cos \beta_c = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi + \varphi'), \\
\cos \beta_d = -\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi + \varphi').
\]

(A13)

The quantity \( P_n(\cos \gamma) \) in the external potential is rearranged in the same way.

As we mentioned in section 2, the Bernoulli equation is discretized on the surface grid points. The number of the discretized Bernoulli equation is the same as the number of the discretized surface \( R(\theta_i, \varphi_i) \). Therefore it is convenient to regard the Bernoulli equation as the equation to determine the surface.

We have another physical variable \( a_m \) in our formulation. As mentioned in section 2, they are determined from the boundary condition of the stream function \( \Psi \). By using equa-
tion (11), equation (12) can be expressed as follows:

\[
\frac{1}{4} R(\pi/2, \varphi)^2 + \sum_{m=0}^{\infty} a_m R(\pi/2, \varphi)^m \cos m\varphi = 0. \tag{A14}
\]

This equation is also discretized on the grid points of \( \varphi_i \). Since we include the coefficients up to \( 0 \leq m \leq N_\varphi/2 \), if we adopt the grid points with even integers of \( i_\varphi \) on the equatorial plane, the number of the equations and the number of the coefficients coincide. As an alternative way, it is possible to use all the grid points on the equatorial surface of the star and employ the least square method to determine \( a_m \).

**APPENDIX B: CONDITIONS FOR PARAMETERS**

In this section we describe the scheme to determine the physical parameters or the constants. The choice of the parameters is essential to construct a successful numerical code which can be used to get converged equilibrium solutions. In our actual computations, we use nondimensional variables as follows:

\[
\begin{align*}
\tilde{R} &= \frac{R}{R_0}, & \tilde{u}_i &= \frac{u_i}{\sqrt{G\rho R_0}}, & \tilde{\zeta} &= \frac{\zeta}{\sqrt{G\rho}}, \\
\tilde{\Omega} &= \frac{\Omega}{\sqrt{G\rho}}, & \tilde{\Psi} &= \frac{\Psi}{\sqrt{G\rho R_0^2}}, & \tilde{p} &= \frac{p}{p_c}, \\
\tilde{C} &= \frac{C}{G\rho R_0^2}, & \tilde{\phi} &= \frac{\phi}{G\rho R_0^2}, & \tilde{\beta} &= \frac{p_c}{G\rho^2 R_0^2}, \\
\tilde{a}_m &= \frac{a_m}{R_0^{m-2}}, & \tilde{X} &= \frac{X}{R_0}, & \tilde{Y} &= \frac{Y}{R_0},
\end{align*}
\]

where \( R_0 \) is the half of the geometrical width of the primary star along the \( x \) axis, i.e. \( R_0 = (R_{cut}^- - R_{cut}^+) / 2 \) and \( p_c \) is the pressure at the coordinate center of the each component.

Using these nondimensional variables, we can rewrite the Bernoulli equation (2) as follows:

\[
\frac{1}{2} \left( \tilde{u}_x^2 + \tilde{u}_y^2 \right) + (\tilde{\zeta} + 2\tilde{\Omega}) \tilde{\Psi} - \frac{1}{2} \tilde{\Omega}^2 (\tilde{X}^2 + \tilde{Y}^2) + \tilde{\beta} \tilde{p} + \tilde{\phi} = \tilde{C}. \tag{B2}
\]

We note that as we have shown in the previous section we only need to handle the equations on the stellar surface where the pressure vanishes. Therefore the parameter \( \tilde{\beta} \) does not appear in the equation but can be determined afterwards.

For the fluid star-fluid star binary system, there are seven physical parameters, i.e.

\[
\tilde{\zeta}_\pm, \quad \tilde{\Omega}, \quad \tilde{a}_\pm, \quad \tilde{C}_\pm. \tag{B3}
\]
Therefore we should impose seven conditions so as to determine these parameters. As for the quantities $\zeta_\pm$, we can specify some simple relations of $\zeta$ and $\Omega$ as

$$F_\pm(\zeta_\pm, \Omega) = 0.$$  \hfill (B4)

For the circulation preserving sequences, this relation should be

$$\tilde{\zeta}_\pm + \tilde{\Omega} = \tilde{\zeta}_{0\pm},$$  \hfill (B5)

where $\tilde{\zeta}_{0\pm}$ are given constants. For the synchronously rotating sequences, we can choose the relation

$$\tilde{\zeta}_\pm = 0.$$  \hfill (B6)

As for the distances from the rotational axis to the origins of the spherical coordinate systems of each component $d_\pm$, two conditions are required. The total separation $d_+ + d_-$ and the mass ratio $M_+/M_-$ are specified for this purpose. Since the origins of the spherical coordinates of two component stars are chosen to be located at the geometrical centers of the two stars on the $x$ axis, we can impose the following three conditions:

$$\tilde{R}_+ (\pi/2, 0) = 1, \quad \tilde{R}_+ (\pi/2, \pi) = 1,$$

$$\tilde{R}_- (\pi/2, 0) = \tilde{R}_- (\pi/2, \pi).$$  \hfill (B7)

It should be noted that the physically meaningful conditions to specify the parameters are four, in other words they are: 1) two conditions for $\zeta_\pm$, 2) one condition for the total separation and 3) one condition for the mass ratio of the binary system.

For the point source-fluid star binary system, the physical constants related to the secondary point source are different from those of the fluid-fluid binary system. In this case parameters $\zeta_-$ and $C_-$ appear no more but we need to include one parameter which characterizes the gravity of the secondary star. In this paper we choose the mass of the secondary as a parameter:

$$\phi_s = \frac{GM_-}{D_+}.$$  \hfill (B8)

Therefore the total number of parameters becomes six for this case:

$$\tilde{\zeta}_+, \quad \tilde{\Omega}, \quad d_\pm, \quad C_+, \quad M_-$$  \hfill (B9)

Consequently we should specify six relations for them, five of which are the same as the previous case: we need not use the relation for $\tilde{\zeta}_-$ and the relation for the radius of the secondary star. The sixth relation is obtained from the requirement that the motion of the point mass should be Keplerian subjected to the gravitational force of the primary star. It can be written by using the nondimensional variables as follows:
\[ \tilde{d}_n \tilde{\Omega}^2 = \sum_{n=0}^{\infty} \frac{I_n}{(d_+ + d_-)^{n+2}} \]  

where
\[ I_n \equiv \frac{n+1}{n+3} \int_0^\pi \sin \theta_+ d\theta_+ \int_0^{2\pi} d\varphi_+ \times \tilde{R}_+(\theta_+, \varphi_+)^{n+3} P_n(\cos \gamma). \]

Therefore, the physically meaningful conditions for the parameters are three: 1) the relation for \( \zeta_+ \), 2) one condition for the total separation and 3) one condition for the mass ratio of the binary system.

These equations for the parameters are solved simultaneously with the physical variables mentioned before by using the Newton-Raphson scheme. This method is so powerful that we can obtain converged solutions after only 5 or 6 iteration cycles.

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