ON POLYNOMIAL SOLUTIONS OF EQUATIONS OF ASSOCIATIVITY

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MIRAMARE - TRIESTE
September 1997
1 Introduction

We shall study a particular case of a problem arising originally in physical papers on two dimensional field theory, that is the problem of finding a quasihomogeneous function $F = F(t)$, $t = (t_1, t_2, \ldots, t_n)$, whose third derivatives,

$$c_{\alpha\beta\gamma}(t) := \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

satisfy the following conditions:

1) Normalization :

$$(\eta_{\alpha\beta}) := (c_{1\alpha\beta}(t))$$

is a constant nondegenerate matrix ;

2) Associativity : Let $c_{\alpha\beta\gamma} := \eta^{\epsilon\delta} c_{\epsilon\alpha\beta}(t)$ (summation over repeated indices is assumed) and let $e_1, e_2, \ldots, e_n$ denote the elements of a basis in n-dimensional space $V$. We endow $V$ with a product defined for every fixed $t$, by the following relations:

$$e_\alpha \cdot e_\beta = c_{\alpha\beta\gamma}(t)e_\gamma, \quad 1 \leq \alpha, \beta \leq n$$

Then we require the product just defined to be associative, so $A_t := (V, \cdot)_t$ becomes an associative algebra for every value of $t$.

The last condition give rise to the following system of nonlinear partial differential equations for $F$ :

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\mu\lambda} \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\delta \partial t^\mu} = \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\delta \partial t^\mu} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda}$$

Quasihomogeneity condition imposed to $F$ at the beginning can be set in a more precise way as :

$$\sum_\alpha d_\alpha t^\alpha \partial_\alpha F(t) = d_F F(t)$$

for some numbers $d_1, \ldots, d_n, d_F$ and every $t \in V$. We require the quasihomogeneity condition to be satisfied up to addition of a polynomial of degree two.

We also assume the extra condition $\eta_{11} = 0$. Then $(\eta_{\alpha\beta})$ can be reduced by a linear change of coordinates (possibly, with complex coefficients) to the antidiagonal form

$$(\eta_{\alpha\beta}) = (\delta_{\alpha+\beta, n+1})$$

In the new coordinates, say $\tau^\alpha$, $F$ takes the form

$$F(\tau) = \frac{1}{2}(\tau^1)^2 \tau^\alpha + \frac{1}{2} \tau^1 \sum_{\alpha=2}^{n-1} \tau^\alpha \tau^{n-\alpha+1} + f(\tau^2, \ldots, \tau^n)$$  \hspace{1cm} (1)$$

for some function $f$ of $\tau^2, \ldots, \tau^n$. Moreover $F$ remains quasihomogeneous and the numbers $d_\alpha$ in the quasihomogeneity condition satisfy the relations

$$d_\alpha + d_{n-\alpha+1} = d_F - d_1, \quad 1 \leq \alpha \leq n$$

We consider here only the case of positive degrees, i.e. $d_\alpha > 0$, $1 \leq \alpha \leq n$. Since the degrees $d_1, \ldots, d_n, d_F$ are completely determined up to a nonzero factor we may assume they have been normalized in such a way that $d_1 = 1$. Then $0 < d_\alpha < 1$, $1 \leq \alpha \leq n$.
We shall characterize all polynomial solutions of the preceding problem of the form
(1) for $n = 4$. It is convenient first to introduce the following notion of equivalence. We
shall say that two solutions of our problem of the form (1), say $F_1$, $F_2$ are equivalent if
one of the following relations is satisfied

\[ i) \quad F_1(t, x, y, z) = F_2\left(\frac{t}{c}, c'x, \frac{c}{c'}y, c^2z\right); \]
\[ ii) \quad F_1(t, x, y, z) = F_2\left(\frac{t}{c}, c'y, \frac{c}{c'}x, c^2z\right); \]

for some numbers $c$, $c'$ different from zero.

Now we state our theorem in terms of the equivalence relation just defined.

**Theorem**: For $n = 4$, every polynomial solution of our problem of the form (1) must
be equivalent to one of the following:

1) $F(t, x, y, z) = \frac{1}{2} t^2 z + t x y + a x^{2l} z + x^{l+1} y$

2) $F(t, x, y, z) = \frac{1}{2} t^2 z + t x y + x^l$

3) $F(t, x, y, z) = \frac{1}{2} t^2 z + t x y + x^l z$

4) $F(t, x, y, z) = \frac{1}{2} t^2 z + t x y + x^3 z + y^3 z + 6 x y z^3 + \frac{54}{35} z^7$;

5) $F(t, x, y, z) = \frac{1}{2} t^2 z + t x y + x^4 z + y^3 z + 12 x^2 y z^3 + \frac{36}{5} y^2 z^5 + \frac{144}{7} x^2 z^7 + \frac{1728}{143} z^{13}$;

6) $F(t, x, y, z) = \frac{1}{2} t^2 z + t x y + x^4 + y^3 + 36 x^2 y z + 108 y^2 z^2 + 864 x^2 z^3 + \frac{93312}{5} z^6$;

7) $F(t, x, y, z) = \frac{1}{2} t^2 z + t x y + x^3 y + y^3 + \frac{9}{4} x^4 z + 9 x y^2 z + 27 x^2 y z^2 + 27 x^3 z^3 + 1458 y^2 z^3 + \frac{13122}{7} z^9$;

8) $F(t, x, y, z) = \frac{1}{2} t^2 z + t x y + x^5 z + y^3 z + 20 x^3 y z^3 + 36 x y^2 z^5 + 120 x^4 z^7 + 240 x^2 y z^9 + \frac{13122}{7} z^9$;
Moreover only types (4)--(8) give rise to semisimple algebras. This means that for a
generic point \( t = (t, x, y, z) \) the corresponding algebra \( A_t \) has no nontrivial nilpotents.

It was proved in [1] that for each Coxeter group in the \( n \)-dimensional real space
there exists a semisimple polynomial Frobenius Manifold satisfying the condition that all
degrees \( d_\alpha \) are positive. It was also conjectured in the same paper that these are all
possibilities.

For \( n = 4 \) this conjecture is a Corollary of our Theorem. We set this as follow.

**Corollary**: All 4-dimensional semisimple polynomial Frobenius Manifolds with degrees
positive are in a one–one correspondence with the finite Coxeter groups in 4–dimensional
real space.

## 2 Proof of the theorem

We look for those polynomials \( F(t) \), \( t = (t, x, y, z) \), of the form :

\[
F(t) = \frac{1}{2} t^2 z + t x y + f(x, y, z)
\]

such that \( f \) satisfies the following system of partial differential equations:

1) \(- 2 f_{xyz} - f_{xyy} f_{xyy} + f_{yyf_{xxx}} = 0\)
2) \(- f_{xxx} - f_{xyy} f_{xxx} + f_{yyf_{xxx}} = 0\)
3) \(- 2 f_{xyz} f_{xxx} + f_{yyy} f_{xxx} + f_{yyf_{xxx}} = 0\)
4) \(- f_{yyy} f_{xxx} + f_{yyy} f_{xxx} = 0\)
5) \(f_{xxx} - f_{xyy}^2 + f_{xxx} f_{xyy} - f_{yyf_{xxx}} + f_{yyf_{xxx}} = 0\)
6) \(f_{yyy} f_{xxx} - 2 f_{yyf_{xxx}} f_{xyy} = 0\)

and

\[
L_E F = d_F F + \text{polynomial of degree at most two}
\]

where

\[
L_E = d_0 t \partial_t + d_1 x \partial_x + d_2 y \partial_y + d_3 z \partial_z
\]

\( d_0 = 1, \; 0 < d_\alpha < 1, \; \alpha = 1, 2, 3, \) \( d_1 + d_2 = 1 + d_3, \) \( d_F = 2 + d_3 \)

The last condition is equivalent to the following one :

\[
d_1 x \partial_x f + d_2 y \partial_y f + d_3 z \partial_z f = (2 + d_3) f + \text{polynomial of degree at most two}
\]
We start looking for those \( f \) which are analytic at the origin, i.e.

\[
f = \sum_{p, q, r \geq 0} a_{pqr} x^p y^q z^r
\]

It turns out that \( p, q, r \) must satisfy

\[
d_1 p + d_2 q + d_3 r = 2 + d_3 = 1 + d_1 + d_2
\]

whenever \( p + q + r \geq 3 \) and \( a_{pqr} \neq 0 \). We consider several cases:

1) If \( p = 0 \). Then

\[
d_2 q + d_3 r = 1 + d_1 + d_2
\]

\[
r = \left( \frac{1 + d_1 + d_2}{d_3} \right) - \frac{d_2}{d_3} q
\]

2) If \( p = 1 \). Then

\[
d_2 q + d_3 r = 1 + d_2
\]

\[
r = \left( \frac{1 + d_2}{d_3} \right) - \frac{d_2}{d_3} q
\]

3) If \( p > 1 \). Then we must have \( q = 0 \) or \( q = 1 \). In fact we have

\[
d_3 r = 1 - (p - 1) d_1 - (q - 1) d_2
\]

\[
= 1 - (p - q) d_1 - (q - 1)(d_1 + d_2)
\]

\[
= 1 - (q - p) d_1 - (p - 1)(d_1 + d_2)
\]

If \( q \geq p \) then

\[
1 - (q - p) d_1 - (p - 1)(d_1 + d_2) < 0
\]

So \( q < p \) and

\[
1 - (p - q) d_1 - (q - 1)(d_1 + d_2) \geq 0 \quad \Rightarrow \quad q < 2
\]

i.e. \( q = 0 \) or \( q = 1 \). Now, when \( q = 0 \)

\[
d_1 p + d_3 r = 1 + d_1 + d_2
\]

\[
r = \left( \frac{1 + d_1 + d_2}{d_3} \right) - \frac{d_1}{d_3} p
\]

When \( q = 1 \)

\[
d_1 p + d_3 r = 1 + d_1
\]

\[
r = \left( \frac{1 + d_1}{d_3} \right) - \frac{d_1}{d_3} p
\]
We shall study in more detail the problem of finding the integer solutions of the following system:

\[
\begin{align*}
(i) & \quad \mu_1 = \left(\frac{1 + d_1 + d_2}{d_3}\right) - \frac{d_2}{d_3} \nu_1 \\
(ii) & \quad \mu_2 = \left(\frac{1 + d_2}{d_3}\right) - \frac{d_2}{d_3} \nu_2 \\
(iii) & \quad \mu_3 = \left(\frac{1 + d_1 + d_2}{d_3}\right) - \frac{d_1}{d_3} \nu_3 \\
(iv) & \quad \mu_4 = \left(\frac{1 + d_1}{d_3}\right) - \frac{d_1}{d_3} \nu_4
\end{align*}
\]

First we note the following property: \( \mu_2, \nu_2 \in \mathbb{Z} \) is a solution of (ii) iff \( \mu_1 = 2\mu_2 + 1, \nu_1 = 2\nu_2 - 2 \) is a solution of (i). In fact let \( \mu_2 = \left(\frac{1 + d_1 + d_2}{d_3}\right) - \frac{d_2}{d_3} \nu_2, \alpha = \frac{d_1 + d_2}{d_3}, \beta = \frac{d_3}{d_3} \) then

\[
\mu_2 = (2\alpha - \beta - 1) - (\alpha - \beta)\nu_2 \quad \Leftrightarrow \\
\mu_2 + 1 = \alpha - (\alpha - \beta)(\nu_2 - 1) \quad \Leftrightarrow \\
2\mu_2 + 1 = (2\alpha - 1) - (\alpha - \beta)(2\nu_2 - 2) \quad \Leftrightarrow \\
2\mu_2 + 1 = \left(\frac{1 + d_1 + d_2}{d_3}\right) - \frac{d_2}{d_3}(2\nu_2 - 2)
\]

Analogously \( \mu_4, \nu_4 \in \mathbb{Z} \) is a solution of (iv) iff \( \mu_3 = 2\mu_4 + 1, \nu_3 = 2\nu_4 - 2 \) is a solution of (iii).

Secondly, we have the following: if (i) has a solution \( \mu_1, \nu_1 \in \mathbb{Z} \) with \( \nu_1 \neq 2 \) then

\[
\frac{d_1}{d_3} \in \mathbb{Q} \quad \text{iff} \quad \frac{d_2}{d_3} \in \mathbb{Q}
\]

(The same is true if we consider (iii) instead of (i).) To see this let \( \mu_1, \nu_1 \) be a solution of (i) with \( \nu_1 \neq 2 \). Then

\[
\mu_1 = \left(\frac{1 + d_1 + d_2}{d_3}\right) - \frac{d_2}{d_3} \nu_1 \\
= 2 \frac{d_1}{d_3} - (\nu_1 - 2) \frac{d_2}{d_3} - 1 \\
\Rightarrow 2 \frac{d_1}{d_3} - (\nu_1 - 2) \frac{d_2}{d_3} \in \mathbb{Z}
\]

and the result follows.

From now on let \( \rho \) denote the minimum non-negative value of \( \nu_1 \) (if there is any) among all possible integer solutions of (i) with \( \mu_1 \geq 0 \).
Again, we consider several cases.

First let \( \rho = 0 \). Then \( \frac{1 + d_1 + d_2}{d_3} = m \in \mathbb{Z}_+ \). If \( \frac{d_1}{d_3}, \frac{d_2}{d_3} \notin \mathbb{Q} \) then \( \mu_1 = m, \nu_1 = 0 \) and \( \mu_3 = m, \nu_3 = 0 \) are the unique solutions of (i) and (iii) respectively. Moreover (ii) and (iv) have a solution iff \( m \) is odd. If this is the case then \( \mu_2 = \frac{m-1}{2}, \nu_2 = 1 \) and \( \mu_4 = \frac{m-1}{2}, \nu_4 = 1 \) are the solutions of (ii) and (iv) respectively. Then \( f \) has one of the following forms:

\[
f = a z^{2k}, \quad k > 1
\]
or

\[
f = a z^{2k+1} + bx y z^k, \quad k > 1
\]

It’s easy to check that only \( f \equiv 0 \) is a solution of \((*)\).

Now let \( \frac{d_1}{d_3} = \frac{s}{l}, s, l \in \mathbb{N} \) such that \( \gcd(s, l) = 1 \). Then from

\[
\begin{cases}
\frac{1 + d_1 + d_2}{d_3} = m \\
d_1 + d_2 = 1 + d_3
\end{cases}
\]

we get \( \frac{d_2}{d_3} = \frac{m+1}{2} - \frac{s}{l} \) and \((**)\) takes the form:

\[
\begin{align*}
(i) \quad \mu_1 &= m - \left( \frac{m+1}{2} - \frac{s}{l} \right) \nu_1 \\
(ii) \quad \mu_2 &= \left( m - \frac{s}{l} \right) - \left( \frac{m+1}{2} - \frac{s}{l} \right) \nu_2 \\
(iii) \quad \mu_3 &= m - \frac{s}{l} \nu_3 \\
(iv) \quad \mu_4 &= \left( \frac{m-1}{2} + \frac{s}{l} \right) - \frac{s}{l} \nu_4
\end{align*}
\]

with \( m, s \) and \( l \) satisfying the inequality

\[
\frac{m-1}{2} > \frac{s}{l} > 1
\]

It follows (iii) has general solution :

\[
\begin{cases}
\mu_3 = m - sh \\
\nu_3 = lh, \\
h \in \mathbb{Z}
\end{cases}
\]

Consequently (iv) has a solution iff \( l \) is even or \( m \) is odd. When \( m \) is odd we find

\[
\begin{cases}
\mu_4 = \frac{m-1}{2} - sh \\
\nu_4 = 1 + lh, \\
h \in \mathbb{Z}
\end{cases}
\]
and if both $m$ and $l$ are even we have

$$
\begin{aligned}
\mu_4 &= \frac{m - s - 1}{2} - sh \\
\nu_4 &= \left( \frac{l}{2} + 1 \right) + lh, \quad h \in \mathbb{Z}
\end{aligned}
$$

On the other hand we have the following situation for (i) and (ii). If $m$ is even and $l$ is odd then $\gcd(l(m + 1) - 2s, 2l) = 1$ and we find

$$
\begin{aligned}
\mu_1 &= m + (l(m + 1) - 2s)h \\
\nu_1 &= -2lh, \quad h \in \mathbb{Z}
\end{aligned}
$$

$$
\begin{aligned}
\mu_2 &= \frac{m - 1}{2} + \frac{l(m + 1) - 2s}{2}(2h + 1) \\
\nu_2 &= (1 - l) - 2lh, \quad h \in \mathbb{Z}
\end{aligned}
$$

If $m$ is even and $l$ is even then $\gcd\left(\frac{l(m+1)}{2} - s, \frac{l}{2}\right) = 1$. Two possibilities arise, the first one when $\frac{l}{2}$ is odd. Then

$$
\begin{aligned}
\mu_1 &= m + \left( \frac{l(m+1) - 2s}{4} \right)h \\
\nu_1 &= -\frac{l}{2}h, \quad h \in \mathbb{Z}
\end{aligned}
$$

and (ii) has no solution. The second one arises when $\frac{l}{2}$ is even. Then we have

$$
\begin{aligned}
\mu_1 &= m + \left( \frac{l(m+1) - 2s}{2} \right)h \\
\nu_1 &= -lh, \quad h \in \mathbb{Z}
\end{aligned}
$$

$$
\begin{aligned}
\mu_2 &= \frac{m - 1}{2} + \left( \frac{l(m+1) - 2s}{4} \right)(2h + 1) \\
\nu_2 &= 1 - \frac{l}{2}(2h + 1), \quad h \in \mathbb{Z}
\end{aligned}
$$
Finally when \( m \) is odd we find

\[
\begin{cases}
  \mu_1 = m - \left( \frac{l(m + 1)}{2} - s \right) h \\
  \nu_1 = lh, & \quad h \in \mathbb{Z}
\end{cases}
\]

\[
\begin{cases}
  \mu_2 = \frac{m - 1}{2} - \left( \frac{l(m + 1)}{2} - s \right) h \\
  \nu_2 = 1 + lh, & \quad h \in \mathbb{Z}
\end{cases}
\]

We get the following possibilities for our function \( f \):

\[
f = \sum_i a_i \ y^{2i} \ z^{m-(l(m+1)-2s)i} +
\]

\[
\sum_i b_i \ x^i \ y^{1+2i} \ z^{\frac{m-1}{2} - \left( \frac{l(m+1)-2s}{2} \right)(2i-1)} + \tag{1}
\]

\[
\sum_i \alpha_i \ x^i \ z^{m-2s} +
\]

with \( m \) even and \( l \) odd;

\[
f = \sum_i a_i \ y^{i} \ z^{m-(l(m+1)-2s)i} +
\]

\[
\sum_i \alpha_i \ x^i \ z^{m-2s} + \tag{2}
\]

\[
\sum_i \beta_i \ x^{1+\frac{i}{2}(2i+1)} \ y \ z^{\frac{m-s-1}{2} - s i} +
\]

with \( m \) even, \( l \) even and \( \frac{l}{2} \) odd;

\[
f = \sum_i a_i \ y^{i} \ z^{m-(l(m+1)-2s)i} +
\]

\[
\sum_i b_i \ x^i \ y^{1+\frac{i}{2}(2i-1)} \ z^{\frac{m-1}{2} - \left( \frac{l(m+1)-2s}{2} \right)(2i-1)} + \	ag{3}
\]

\[
\sum_i \alpha_i \ x^i \ z^{m-2s} +
\]

\[
\sum_i \beta_i \ x^{1+\frac{i}{2}(2i+1)} \ y \ z^{\frac{m-s-1}{2} - s i} +
\]
with $m$ even, $l$ even and $\frac{l}{2}$ even;

$$f = \sum a_i y^i z^{m - \left(\frac{l(m+1)-2s}{2}\right)i} +$$

$$\sum b_i x^i y^{1+i} z^{m-\frac{l}{2} - \left(\frac{l(m+1)-2s}{2}\right)i} +$$

$$\sum \alpha_i x^i z^{m-si} +$$

$$\sum \beta_i x^{1+si} y z^{\frac{m-1}{2} - si}$$

(4)

with $m$ odd. We assume in each summation only those coefficients corresponding to terms with non-negative integer powers of $x$, $y$, and $z$ are different from zero and throughout we shall use this convention in order to simplify our notations.

We shall consider a more general $f$, say

$$f = \sum a_i y^i z^{m - \left(\frac{l(m+1)-s}{2}\right)i} +$$

$$\sum b_i x^i y^{1+i} z^{m-\frac{l}{2} - \left(\frac{l(m+1)-s}{2}\right)i} +$$

$$\sum \alpha_i x^i z^{m-\frac{s}{2}i} +$$

$$\sum \beta_i x^{1+si} y z^{\frac{m-1}{2} - \frac{s}{2}i}$$

where $m$, $s$, and $l$ must satisfy

$$m - 1 > \frac{s}{l} > 2$$

It is convenient to consider several cases.

I) $m \geq s$, $l > 2$. Then

$$f = \sum \alpha_i x^i z^{m-\frac{s}{2}i} +$$

$$\sum \beta_i x^{1+si} y z^{\frac{m-1}{2} - \frac{s}{2}i}$$

II) $2m > s > m$, $l > 2$. Then

$$f = a_0 z^m + a_1 y^l z^{m - \left(\frac{l(m+1)-s}{2}\right)} + b_0 x y z^{\frac{m-1}{2}} + \alpha_1 x^{l} z^{m-\frac{s}{2}}$$
III) \( s > 2m, l > 2 \). Then

\[
 f = \sum_i a_i y^{l_i} z^{m - \left(\frac{l(m+1)-s}{2}\right)i_1} + \\
 \sum_i b_i x^i y^{1+i_i} z^{\frac{m-1}{2} - \left(\frac{l(m+1)-s}{2}\right)i_1}
\]

IV) \( m \geq s, l = 2 \). Then

\[
 f = a_1 y^2 z^\frac{s}{2} - 1 + \sum_i \alpha_i x^{2i} z^{m-\frac{s}{2}i} + \\
 \sum_i \beta_i x^{1+2i} y z^{\frac{m-1}{2} - \frac{s}{2}i}
\]

V) \( 2m > s > m, l = 2 \). Then

\[
 f = \sum_i a_i y^{2i} z^{m - \left(\frac{2(m+1)-s}{2}\right)i_1} + \\
 \sum_i b_i x^{1+2i} y z^{\frac{m-1}{2} - \left(\frac{2(m+1)-s}{2}\right)i_1} + \alpha_1 x^2 z^{m-\frac{s}{2}}
\]

(Since \( m - 1 > \frac{s}{2} \) we have \( 2m > s \).)

When \( l = 1 \) only \( m > s \) is possible and we shall consider the following cases:

VI) \( \frac{m+1}{2} \geq s \). Then

\[
 f = a_1 y z^{\frac{m+s-1}{2}} + a_2 y^2 z^{s-1} + a_3 y^3 z^{\frac{3s-m+3}{2}} + \\
 b_1 x y^{2} z^{\frac{s}{2} - 1} + \\
 \sum_i \alpha_i x^i z^{m-\frac{s}{2}i} + \\
 \sum_i \beta_i x^{1+i} y z^{\frac{m-1}{2} - \frac{s}{2}i}
\]

\( \frac{m+1}{2} \geq s \). Then
VII) $\frac{m+1}{2} < s$ . Then

$$f = \sum_i a_i y^i z^{m-(\frac{m+1}{2})i} + \sum_{i \geq 0} b_i x y^{1+i} z^{\frac{m-1}{2}-(\frac{m+1}{2})i} +$$

$$\alpha_1 x z^{m-\frac{s}{2}} + \alpha_2 x^2 z^{m-s} + \alpha_3 x^3 z^{m-\frac{3s}{2}} + \beta_1 x^2 y z^{\frac{m-s}{2}}$$

Now we study each case separately. In all what follows we shall refer the n-th partial differential equation in (*) as (*n).

I) Let $s | (m-1)$ . Then (1) implies $\beta_0 = \cdots = \beta_{\kappa-1} = 0$ where $\kappa = \frac{m-1}{s}$ . It follows that $m - \frac{s}{2}(2\kappa) = 1$ . Since $\frac{s}{2} > l$ , (5) implies $\alpha_0 = \cdots = \alpha_{2\kappa-1} = 0$ so

$$f = \alpha_{2\kappa} x^{2\kappa} z + \beta_{\kappa} x^{\kappa l+1} y$$

and $\kappa l = \frac{m-1}{s} l > 2$ . We get

$$f = \alpha x^{2l} z + \beta x^{l+1} y, \quad l > 2$$

(for every $l > 2$ we may take $s > 2l$ arbitrary and $m = s + 1$ )

Let $s \nmid (m-1)$ . Then (1) implies

$$f = \sum_i \alpha_i x^{\kappa i} z^{m-\frac{s}{2}i}$$

and (2), (5) imply $f_{xx} = 0$ and $f_{zz} = 0$ respectively. Since $\frac{s}{2} > l$ , $f$ is non trivial iff $s | 2m$ or $s | 2(m-1)$ . In the first case

$$f = \alpha_{\kappa} x^{\kappa l}, \quad \kappa = \frac{2m}{s} > 1$$

and in the second one

$$f = \alpha_{\kappa} x^{\kappa l} z, \quad \kappa = \frac{2(m-1)}{s} > 1$$

II) Let $l \neq 3$ , then $a_1 = 0$ , (1) implies $b_0 = 0$ and (5) is satisfied only if $a_0 = 0$ . Thus

$$f = \alpha_{1} x^{l} z^{m-\frac{s}{2}}$$

and we get the following solutions :

$$f = \alpha x^{l}, \quad l > 3$$

(for every $l > 3$ take $m > l$ arbitrary and $s = 2m$ )

$$f = \alpha x^l z, \quad l > 3$$
(for every $l > 3$ take $m > l + 1$ arbitrary and $s = 2(m - 1)$)

Now let $l = 3$. If $b_0 \neq 0$ then after plugging $f$ into (*) the following system of non-linear equations results:

$$36a_1 \alpha_1 + (1 - m) b_0 = 0$$

$$(1 - m)(m - 3) b_0 + 72(s - m - 3) a_1 \alpha_1 = 0$$

$$(s - 2m)(s - 2m + 2) \alpha_1 = 0$$

$$(s - m - 5)(s - m - 3) a_1 \alpha_1 = 0$$

$$(m - 1)(2s - 3m - 3) \alpha_1 b_0 = 0$$

$$(m - 3)(m - 1) b_0 + 72(s - 2m) a_1 \alpha_1 = 0$$

$$(s - m - 5)(s - m - 3) \alpha_1 = 0$$

$$(s - m - 7)(s - m - 5)(s - m - 3) a_1 = 0$$

$$(m - 5)(m - 3)(m - 1) b_0 + 72(s - 2m)(s - m - 3) a_1 \alpha_1 = 0$$

$$(m - 1)^2 b_0^2 - 4m(m - 1)(m - 2) a_0 = 0$$

$$(s - 2m)(s - 2m + 2)(s - 2m + 4) \alpha_1 = 0$$

$$(s - 2m)(s - 2m + 2) a_1 \alpha_1 = 0$$

$$(m - 1)(3m - 2s + 3) a_1 b_0 = 0$$

This system has a solution iff $m = 7$ and $s = 12$ ($m$ and $s$ must satisfy the inequality $m - 1 > \frac{s}{3} > 2$). In this case we may find the solution in terms of $a_1$ and $\alpha_1$. We have

$$b_0 = 6a_1 \alpha_1,$$

$$a_0 = \frac{54}{35} a_1^2 \alpha_1^2$$

and

$$f = \alpha_1 x^3 z + a_1 y^3 z + 6a_1 \alpha_1 x y z^3 + \frac{54}{35} a_1^2 \alpha_1^2 z^7$$

It is easy to see that this solution fits pattern (4) for $m = 7$, $l = 1$ and $s = 2$.

On the other hand, if $b_0 = 0$ we get from (*) that $a_1 \alpha_1 = 0$ so

$$f = a_0 z^m + a_1 y^3 z^{m-3}$$

or

$$f = a_0 z^m + \alpha_1 x^3 z^{m-\frac{5}{2}}$$

In both cases (*) implies $a_0 = 0$ and we should add four new solutions to our list, they are:

$$f = \alpha x^3 z, \quad f = \alpha x^3, \quad f = a y^3 z \quad \text{and} \quad f = a y^3$$
III) We make $\delta = l(m + 1) - s$. Then we can proceed exactly as we did in case (I). It is easy to see that every solution is contained in one of the following families:

$$f = ay^{2l-1}z + by^{l+1}x, \quad l > 2$$

$$f = ay^l, \quad l > 2$$

$$f = ay^{l+z}, \quad l > 2$$

By the symmetry of our system of partial differential equations in the variables $x$ and $y$ we know from (I) and (II) that each member of these families is a solution of our system.

IV) Since $f_{yyy} = 0$, (\*) takes the form

$$f_{yzz} f_{xy} = 0$$

This implies $a_1 = 0$. Now we continue as in (I). When $s | (m-1)$ we get a new solution if we take $s > 4$ arbitrary and $m = s + 1$:

$$f = \alpha x z^4 + \beta x^3 y$$

When $s \not| (m-1)$ nothing new arise.

V) Just make $\delta = 2(m + 1) - s$ and proceed as in the case before. Then we may apply the same argument used in (III) but now with the enlarged family:

$$f = ay^{2l-1}z + by^{l+1}x, \quad l > 2$$

$$f = ay^l, \quad l > 2$$

$$f = ay^{l+z}, \quad l > 2$$

VI) $\frac{m+1}{2} \geq s$. Let $a_3 = 0$. Then $f_{yyy} = 0$ and (\*) becomes:

$$2 \left( \frac{s}{2} - 1 \right) b_1 y z_{2s-2} +$$

$$\sum_i (i+1) \left[ (i+2)b_1 \beta_{i+1} + \left( \frac{m-1}{2} - \frac{s}{2} \right) \beta_i \right] x^i z\frac{m-1}{2} - \frac{s}{2} i = 0$$

It follows that $b_1 = 0$. Moreover, if $s \not| (m-1)$ then $\beta_i = 0$ for every $i$. Otherwise i.e. if $s | (m-1)$ then $\beta_i = 0$ for every $i$ but $i = \frac{m-s}{s}$. In any case (\*) implies $a_1 = a_2 = 0$ and we may continue as in (I). All solutions we find in this case are contained in the preceding ones.

Now let $a_3 \neq 0$, then $f$ must be of the form

$$f = a_1 y z^{\frac{m+1}{2}} + a_2 y^2 z^{s-1} + a_3 y^3 z^{\frac{3s}{2} - \frac{m+3}{2}} +$$

$$b_1 x y z^{\frac{s}{2} - 1} +$$

$$\sum_{i=0}^{5} \alpha_i x^i z^{m-\frac{s}{2} i} +$$

$$\sum_{i=0}^{2} \beta_i x^{1+i} y z^\frac{m-1}{2} - \frac{s}{2} i$$
We make $\sigma = \frac{m-1}{2} - s$ and plug our $f$ into (*) . Then we get the following system of non-linear equations in the coefficients of $f$, $s$ and $\sigma$:

$$
60 a_3 \alpha_5 - \sigma \beta_2 = 0 \\
77 a_3 \alpha_4 - 6 b_1 \beta_2 - (2\sigma + s) \beta_1 = 0 \\
18 a_3 \beta_2 + (2 - s) b_1 = 0 \\
18 a_3 \alpha_3 - 2 b_1 \beta_1 - (\sigma + s) \beta_0 = 0 \\

\sigma (1 + 2\sigma) \alpha_4 - 10 (s - \sigma - 2) \alpha_3 b_1 = 0 \\
(2 + 4\sigma - s)(s - 4\sigma) \alpha_5 = 0 \\
72 (s - 2\sigma - 4) a_3 \beta_2 + (2 - s)(s - 4) b_1 = 0 \\
(4\sigma + 2s) b_1 \beta_1 - 12 (s - 1) a_2 \beta_2 - 18 (s - 2\sigma - 4) a_3 \alpha_3 - (1 - \sigma - s)(\sigma + s) \beta_0 = 0 \\
48 (s - 1) a_2 \alpha_3 - 16 (1 + 2\sigma + s) \alpha_2 b_1 - (4\sigma + 3s)(2 + 4\sigma + 3s) \alpha_1 = 0 \\
3 \sigma (1 - \sigma) \beta_2 + 180 (s - 2\sigma - 4) a_3 \alpha_5 = 0 \\
480 (s - 1) a_2 \alpha_5 + 96 (s - 2\sigma - 3) \alpha_4 b_1 - 3 (4\sigma + s)(2 + 4\sigma + s) \alpha_3 = 0 \\
144 (s - 2\sigma - 4) a_3 \alpha_4 + 12 (s - 2\sigma - 2) b_1 \beta_2 + + (2\sigma + s)(2 - 2\sigma - s) \beta_1 = 0 \\
12 (\sigma + 1) a_3 b_1 - 24 (s - 1) a_2 \alpha_4 + (2\sigma + s)(1 + 2\sigma + s) \alpha_2 = 0 \\

48 \sigma (\sigma + 2) \alpha_4 \beta_2 + 15 (16\sigma + 8\sigma^2 + 10s - 5s^2) \alpha_5 \beta_1 = 0 \\
(16\sigma + 8\sigma^2 + 2s - s^2) \alpha_5 \beta_2 = 0 \\
(s - 2\sigma - 6)(s - 2\sigma - 4) a_3 \beta_2 = 0 \\
(2 - s)(4 + 8\sigma + 3s) b_1 \beta_1 + 9 (s - 2\sigma - 6)(s - 2\sigma - 4) a_5 \alpha_3 + 24 (s - 2)(s - 1) a_4 \beta_2 = 0 \\
4 (\sigma + 1)(\sigma + s) \beta_0 \beta_1 + + 8 (s - 2)(1 + 2\sigma + s) \alpha_2 b_1 -
\[-24 (s - 2)(s - 1) a_2 a_3 - 3 (3s + 2\sigma - 2)(3s + 2\sigma) a_1 \beta_2 = 0\]

\[8 (\sigma + s)(1 + 2\sigma + s) a_2 \beta_0 - (4\sigma + 3s)(2 + 4\sigma + 3s) a_1 \beta_1 - 3 (2\sigma + 3s - 2)(2\sigma + 3s) a_1 a_3 = 0\]

\[6 \sigma(\sigma + 2) \beta_2^3 - 5 (s - 2)(5s - 8\sigma - 16) a_5 b_1 = 0\]

\[3 \beta_0 - 5 (3s + 2\sigma + 3s - 2)(3s + 2\sigma) a_1 \alpha_1 = 0\]

\[(s - 2\sigma - 4)(s - 2\sigma - 6) a_3 a_5 = 0\]

\[3 (3s^2 - 6s - 16\sigma - 8\sigma^2) \beta_1 \beta_2 + 24 (s - 2)(s - 4\sigma - 6) a_1 b_1 + 240 (s - 1)(s - 2) a_2 a_5 = 0\]

\[(6\sigma + 6\sigma^2 + 4s + 6s + s^2) a_3 \beta_1 - 4(\sigma + s)(1 + 2\sigma + s) a_2 \beta_2 - 8(\sigma + s)(s - \sigma - 2) a_4 \beta_0 - 5 (3s + 2\sigma)(3s + 2\sigma - 2) a_1 a_5 = 0\]

\[3 (s - 2)(3s - 8\sigma - 12) b_1 \beta_2 + 36 (s - 2\sigma - 4)(s - 2\sigma - 6) a_3 a_4 = 0\]

\[3 (s - 2)(s + 8\sigma + 8) a_3 b_1 + (2\sigma + s)(2\sigma + s + 2) \beta_1^2 - 12 (s - 1)(\sigma + s) \beta_0 \beta_2 - 48 (s - 1)(s - 2) a_2 a_4 = 0\]

\[36 (\sigma + 1)(\sigma + s) a_3 \beta_0 - 3 (3s + 4\sigma)(3s + 4\sigma + 2) a_1 \beta_2 - 12 (3s + 2\sigma)(3s + 2\sigma - 2) a_1 a_4 = 0\]

\[\sigma(\sigma - 1) \beta_2 + 60 (s - 4\sigma - 2) a_3 a_5 = 0\]

\[(s - 2\sigma - 4)(s - 2\sigma - 6) a_3 = 0\]

\[288 (2\sigma + 1) a_3 a_4 - 24 (s - 2) b_1 \beta_2 - (s + 2\sigma)(s + 2\sigma - 2) \beta_1 = 0\]

\[24 (\sigma + 1) a_3 \beta_1 - 2 (s - 1)(s - 2) a_2 = 0\]

\[48 (s + 2\sigma + 1) a_2 a_3 - 16 (s - 1) a_2 \beta_1 - (3s + 2\sigma)(3s + 2\sigma - 2) a_1 = 0\]

\[36 (s - 4\sigma - 4) a_3 \beta_2 + (s - 2)(s - 4) b_1 = 0\]

\[12 (s - 1) a_2 \beta_2 - 18 (s + 4\sigma + 2) a_3 a_3 + 2 (s - 2) b_1 \beta_1 + (s + \sigma)(s + \sigma - 1) \beta_0 = 0\]
\[\begin{align*}
6\sigma(\sigma + 2)\beta_2^2 - 4\sigma(2\sigma - 1)(2\sigma + 1)\alpha_4 - \\
&-5(s - 4\sigma - 2)(5s - 4\sigma - 8)\alpha_3 b_1 = 0
\end{align*}\]
\[
(s - 4\sigma)(s - 4\sigma - 2)(s - 4\sigma + 2)\alpha_5 = 0
\]
\[
(s - 2\sigma - 4)(s - 2\sigma - 6)(s - 2\sigma - 8)\alpha_3 = 0
\]
\[
3(4 + 2\sigma - s)(6 + 6\sigma + s)\alpha_3 \beta_1 + s(2 - s) b_1^2 + \\
+2(s - 1)(s - 2)(s - 3)\alpha_2 = 0
\]
\[
16(\sigma + 1)(\sigma + s)\beta_0 b_1 + \\
+48(4 + 2\sigma - s)(1 + 2\sigma + s)\alpha_2 \alpha_3 + \\
+16(s - 1)(s - 2\sigma - 4)\alpha_2 \beta_1 + \\
+(3s + 2\sigma)(3s + 2\sigma - 2)(3s + 2\sigma - 4)\alpha_1 = 0
\]
\[
8(1 - s)(s + 2\sigma + 1)\alpha_2 - 2(\sigma + s)^2\beta_2^2 + \\
+(3s + 2\sigma)(3s + 2\sigma - 2)\alpha_1 \beta_1 + \\
+(3s + 4\sigma)(3s + 4\sigma + 2)\alpha_1 b_1 + \\
+4(\sigma + s)(2\sigma + 2s - 1)(2\sigma + 2s + 1)\alpha_0 = 0
\]
\[
\sigma(\sigma - 1)(\sigma - 2)\beta_2 + 30(s - 4\sigma - 2)(s - 2\sigma - 4)\alpha_3 \alpha_5 = 0
\]
\[
32(2\sigma + 1)(6 + 4\sigma - 3s)\alpha_4 b_1 + \\
+4(3s^2 - 6s - 8\sigma^2 - 16\sigma)\beta_1 \beta_2 + \\
+160(s - 1)(s - 4\sigma - 2)\alpha_2 \alpha_5 + \\
(4\sigma + s - 2)(4\sigma + s)(4\sigma + s + 2)\alpha_3 = 0
\]
\[
288(2\sigma + 1)(2\sigma - s + 4)\alpha_3 \alpha_4 + \\
+24(8 + 6\sigma + 2\sigma^2 - 6s - 4\sigma s + s^2) b_1 \beta_2 + \\
+(s + 2\sigma)(s + 2\sigma - 2)(s + 2\sigma - 4)\beta_1 = 0
\]
\[
48(2\sigma + 1)(1 - s)\alpha_2 \alpha_4 + \\
+3(4 + 4\sigma - s)(2 + 4\sigma + s)\alpha_3 b_1 - \\
-(2\sigma + s)(2\sigma + s + 2)\beta_2^2 + 6(s - 1)(\sigma + s)\beta_0 \beta_2 + \\
+2(2\sigma + s - 1)(2\sigma + s)(2\sigma + s + 1)\alpha_2 = 0
\]
\[
36(s - 6\sigma - 6)(s - 2\sigma - 4)\alpha_3 \beta_2 + (s - 2)(s - 4)(s - 6) b_1 = 0
\]
\[
9(4 + 2\sigma - s)(2 + 4\sigma + s)\alpha_3 \alpha_3 + \\
+(8 + 8\sigma + 4\sigma^2 - 2s - 2\sigma s - s^2) b_1 \beta_1 + \\
+12(s - 1)(s - \sigma - 2)\alpha_2 \beta_2 + \\
+(\sigma + s)(\sigma + s - 1)(\sigma + s - 2)\beta_0 = 0
\]
\[
16(\sigma + 1)(\sigma + s)\beta_0 \beta_1 + \\
+48(s - 1)(2 + 4\sigma + s)\alpha_2 \alpha_3 - \\
-16(1 + 2\sigma + s)(2 + 4\sigma + s)\alpha_2 b_1 - \\
-12(3s + 2\sigma - 2)(3s + 2\sigma)\alpha_1 \beta_2 - \\
-(3s + 4\sigma - 2)(3s + 4\sigma)(3s + 4\sigma + 2)\alpha_1 = 0
\]
\[(s - 4\sigma)(s - 4\sigma - 2) a_3 \alpha_5 = 0\]

\[48\sigma(2\sigma + 1) a_3 \alpha_4 + 2\sigma(5 + \sigma - 3s) b_1 \beta_2 = 0\]

\[(\sigma + 1)(\sigma + s) a_3 \beta_0 = 0\]

\[(2\sigma + 2\sigma^2 + 4s + 2\sigma s - s^2) a_3 b_1 = 0\]

\[8(1 - s)(\sigma + s) a_2 \beta_0 + (3s + 2\sigma)(3s + 2\sigma - 2) a_1 b_1 +
\quad +3(3s + 4\sigma)(3s + 4\sigma + 2) a_1 a_3 = 0\]

\[\sigma(3 + 3\sigma - s) a_3 \beta_2 = 0\]

\[24\sigma(1 - s) a_2 \beta_2 + (s + 2\sigma)(6 + 2\sigma - 3s) b_1 \beta_1 +
\quad +9(4\sigma + s)(4\sigma + s + 2) a_3 a_3 = 0\]

\[3(6 + 6\sigma - s)(2\sigma + s) a_3 \beta_1 + s(2 - s) b_1^2 = 0\]

\[2(\sigma + 1)(\sigma + s) \beta_0 b_1 + 4(1 - s)(2\sigma + s) a_2 \beta_1 +
\quad +12(2\sigma + s)(2\sigma + s + 1) a_2 a_3 = 0\]

Let \(\alpha_5 = 0\), since we are assuming \(\alpha_3 \neq 0\) we must have \(s = 2\sigma + 4\) or \(s = 2\sigma + 6\).

Let \(s = 2\sigma + 6\). We know \(\sigma \geq -1\).

If \(\sigma = -1\) then \(s = 4\) and for this choice we find after solving the resulting system in the coefficients of \(f\) with respect to \(a_3\) and \(a_3\):

\[\alpha_0 = \frac{54}{35} a_3^2 \alpha_3^2,\]

\[\beta_0 = 6 a_3 \alpha_3,\]

and the other coefficients are 0. Thus

\[f = \alpha_3 x^3 z + a_3 y^3 z + 6 a_3 \alpha_3 x y z^3 + \frac{54}{35} a_3^2 \alpha_3^2 z^7\]

We have already met this solution in (II).

If \(\sigma = -\frac{1}{2}\) then \(s = 5\) and all coefficients but \(a_3\) must vanish so we get

\[f = a_3 y^3 z\]

This solution is contained in the preceding one.

If \(\sigma = 0\) then \(s = 6\) and solving the system in the coefficients of \(f\) with respect to
We have \( a_3 \) and \( \alpha_4 \) we get

\[
\begin{align*}
    a_1 &= \alpha_1 = \alpha_3 = b_1 = \beta_0 = \beta_2 = 0, \\
    a_2 &= \frac{36}{5} a_3^2 \alpha_4, \\
    \alpha_0 &= \frac{1728}{143} a_3^4 \alpha_4^3, \\
    \alpha_2 &= \frac{144}{7} a_3^2 \alpha_4^2, \\
    \beta_1 &= 12 a_3 \alpha_4
\end{align*}
\]

and so

\[
f = \alpha_4 x^4 z + a_3 y^3 z + 12 a_3 \alpha_4 x^2 y z^3 + \frac{36}{5} a_3^2 \alpha_4 y^2 z^5 + \\
    + \frac{144}{7} a_3^2 \alpha_4^2 x^2 z^7 + \frac{1728}{143} a_3^4 \alpha_4^3 z^{13}
\]

This \( f \) fits pattern (4) with \( m = 13, s = 3 \) and \( l = 1 \).

Now let \( s = 2 \sigma + 4 \). Since \( s > 2 \), \( \sigma \) must be bigger than \(-1\). So we start considering the case \( \sigma = -\frac{1}{2} \). In this case \( s = 3 \) and we get

\[
\begin{align*}
    a_1 &= \alpha_1 = \alpha_3 = b_1 = \beta_0 = \beta_2 = 0, \\
    a_2 &= 108 a_3^2 \alpha_4, \\
    \alpha_0 &= \frac{93312}{5} a_3^4 \alpha_4^3, \\
    \alpha_2 &= 864 a_3^2 \alpha_4^2, \\
    \beta_1 &= 36 a_3 \alpha_4
\end{align*}
\]

\( a_3 \) and \( \alpha_4 \) free and

\[
f = \alpha_4 x^4 + a_3 y^3 + 36 a_3 \alpha_4 x^2 y z + 108 a_3^2 \alpha_4 y^2 z^2 + \\
    + 864 a_3^2 \alpha_4^2 x^2 z^3 + \frac{93312}{5} a_3^4 \alpha_4^3 z^6
\]

This solution fits pattern (2) for \( m = 6, s = 3 \) and \( l = 2 \).
If $\sigma = 0$ and $s = 2\sigma + 4$ then $s = 4$ and this time we get the solution in terms of $a_3$ and $\beta_2$. We have

$$a_1 = a_1 = \alpha_5 = \beta_0 = 0,$$

$$a_2 = 54 a_3^2 \beta_2^2,$$

$$\alpha_0 = \frac{13122}{7} a_3^4 \beta_2^6,$$

$$\alpha_2 = \frac{1458}{5} a_3^2 \beta_2^4,$$

$$\alpha_3 = 27 a_3 \beta_2^3,$$

$$\alpha_4 = \frac{9}{4} \beta_2^2,$$

$$b_1 = 9 a_3 \beta_2,$$

$$\beta_1 = 27 a_3 \beta_2^2$$

and

$$f = \beta_2 x^3 y + a_3 y^3 + \frac{9}{4} \beta_2^2 x^4 z + 9 a_3 \beta_2 x y^2 z +$$

$$+ 27 a_3 \beta_2^2 x^2 y z^2 + 27 a_3 \beta_2^3 x^3 z^3 + 54 a_3^2 \beta_2^2 y^2 z^3 +$$

$$+ \frac{1458}{5} a_3^2 \beta_2^4 x^2 z^5 + \frac{13122}{7} a_3^4 \beta_2^6 z^9$$

which fits pattern (4) with $m = 9$, $s = 2$ and $l = 1$.

If we take $\sigma > 0$ we find that all coefficients but $a_3$ must vanish and two possibilities for $f$ arise:

$$f = a_3 y^3 z \quad \text{or} \quad f = a_3 y^3$$

It remains to consider the case on which neither $a_3$ nor $\alpha_5$ equal 0. This is possible
only if $\sigma = 3$ and $s = 12$. Then we find

$$a_1 = a_2 = 0,$$

$$a_2 = 1152 a_3 a_5^2,$$

$$a_0 = \frac{14155776}{899} a_3^{10} a_5^6,$$

$$a_2 = \frac{92160}{19} a_3^6 a_5^4,$$

$$\alpha_3 = 960 a_3^4 a_5^3,$$

$$\alpha_4 = 120 a_3^2 a_5^2,$$

$$b_1 = 36 a_3^2 a_5,$$

$$\beta_1 = 240 a_3^3 a_5^2,$$

$$\beta_2 = 20 a_3 a_5$$

where $a_3, a_5$ can take any non-zero value. We have for $f$ the following expression:

$$f = \alpha_5 x^5 z + a_3 y^3 z + 20 a_3 \alpha_5 x^3 y z^3 + 36 a_3^2 \alpha_5 x y^2 z^5 +$$

$$+ 120 a_3^2 \alpha_5 x^4 z^7 + 240 a_3^3 \alpha_5 x^2 y z^9 +$$

$$+ \frac{92160}{11} \alpha_3^4 \alpha_5^2 y^2 z^{11} + 960 \alpha_3^4 \alpha_5^3 x^3 z^{13} +$$

$$+ \frac{14155776}{899} a_3^{10} a_5^6 \cdot z^{31}$$

This $f$ fits pattern (4) when $m = 31, l = 1$ and $s = 6$.

VII) We make $\delta = m + 1 - s$. Then our $f$ turns into that of (VI) with $x$ and $y$ interchanged. Conditions $m - 1 > s > 2$ and $\frac{m+1}{2} < s$ transform into $m - 1 > \delta > 2$ and $\frac{m+1}{2} > \delta$ respectively.

Let’s consider now the case $\rho = 1$. Then $\left(\frac{1+\delta_1}{\delta_3}\right) \in \mathbb{Z}_+$. If $\frac{\delta_2}{\delta_3} = \lambda \notin \mathbb{Q}$ then $\frac{\delta_1}{\delta_3} \notin \mathbb{Q}$ and we find $\mu_1 = m, \nu_1 = 1; \mu_4 = m, \nu_4 = 0; \mu_3 = 2m + 1, \nu_3 = -2$ are the unique solutions of (i),(iv) and (iii) respectively. Since $\nu_1$ is odd (ii) has no solution. In this case $\lambda, m$ must satisfy the inequality $1 < \lambda < m - 1$. So we get

$$f = a y z^m, \quad m > 2$$
and $f$ is solution of (**) only if $a = 0$.

Now let $\frac{d^2}{d^3} = \frac{s}{l}$. Then (**) takes the form:

\[
\begin{align*}
(i) & \quad \mu_1 = \left( m + \frac{s}{l} \right) - \frac{s}{l} \nu_1 \\
(ii) & \quad \mu_2 = \left( m - \frac{1}{2} + \frac{3s}{2l} \right) - \frac{s}{l} \nu_2 \\
(iii) & \quad \mu_3 = \left( m + \frac{s}{l} \right) - \left( m + \frac{1}{2} - \frac{s}{2l} \right) \nu_3 \\
(iv) & \quad \mu_4 = m - \left( m + \frac{1}{2} - \frac{s}{2l} \right) \nu_4
\end{align*}
\]

with $m$, $s$ and $l$ satisfying the inequality

\[1 < \frac{s}{l} < m - 1\]

We distinguish two cases. If $l$ is odd and $m + s$ is odd then

\[
\begin{align*}
\mu_1 &= m - s h \\
\nu_1 &= 1 + lh, \\
\mu_2 &= \frac{m - s - 1}{2} - s h \\
\nu_2 &= \frac{l + 3}{2} + lh, \\
\mu_3 &= 2m + 1 - \left( \frac{l(m + 1) - s}{2} \right) h \\
\nu_3 &= -2 + lh, \\
\mu_4 &= m - \left( \frac{l(m + 1) - s}{2} \right) h \\
\nu_4 &= lh
\end{align*}
\]
Otherwise we have

\[
\begin{align*}
\mu_1 &= m - sh \\
\nu_1 &= 1 + lh,
\end{align*}
\]

\[h \in \mathbb{Z}\]

\[
\begin{align*}
\mu_3 &= 2m + 1 - (l(m + 1) - s)h \\
\nu_3 &= -2 + 2lh,
\end{align*}
\]

\[h \in \mathbb{Z}\]

\[
\begin{align*}
\mu_4 &= m - (l(m + 1) - s)h \\
\nu_4 &= 2lh,
\end{align*}
\]

\[h \in \mathbb{Z}\]

and (ii) has no solution. We find the following possibilities for \( f \) :

\[
f = \sum_i a_i y^{l + li} z^{m - si +}
\]

\[
\sum_i b_i x y^{\frac{l+3}{2} + li} z^{\frac{m-s-1}{2} - si +}
\]

\[
\sum_i \alpha_i x^{-2+li} z^{2m+1-(\frac{l(m+1)-s}{2})i+}
\]

\[
\sum_i \beta_i x^{2li} y z^{-\left(\frac{l(m+1)-s}{2}\right)i}
\]

where both \( l \) and \( m + s \) are odd and

\[
f = \sum_i a_i y^{l + li} z^{m - si +}
\]

\[
\sum_i \alpha_i x^{-2+2li} z^{2m+1-(l(m+1)-s)i+}
\]

\[
\sum_i \beta_i x^{2li} y z^{-\left(l(m+1)-s\right)i}
\]

where \( l \) or \( m + s \) is even.

As we did when \( \rho = 0 \) we shall divide our analysis in several cases easier to deal with. They are :

I) \( m \geq s, \ l > 2 \). Then we have

\[
f = \sum_i a_i y^{l + li} z^{m - si +}
\]

\[
\sum_i b_i x y^{\frac{l+3}{2} + li} z^{\frac{m-s-1}{2} - si}
\]

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II) \( m \geq s, l = 2 \). Then
\[
f = \sum_i a_i y^{1+2i} z^{m-si} + \alpha_2 x^2 z^{s-1}
\]

III) \( m < s \). Then
\[
f = \sum_i a_i x^{-2+li} z^{2m+1-(\frac{(m+1)-s}{2})i} + \sum_i \beta_i x^{li} y z^{-(\frac{(m+1)-s}{2})i}
\]

Note that since we are assuming \( \rho = 1 \), \( l \) must be bigger than one.

As for \( \rho = 0 \) we shall consider each case separately.

I) Let \( 2s \mid (m - s - 1) \) and \( l \neq 3 \). Then (1) implies \( b_i = 0 \) for every \( i \) but \( i = \frac{m-s-1}{2s} \). Let us denote this value by \( \kappa \). It follows that \( m - s(1 + 2\kappa) = 1 \) and since \( s > 2 \), (5) implies
\[
f = a_{2\kappa+1} y^{l(2\kappa+1)+1} z + b_{\kappa} x y^{1+3\kappa} + 1 \kappa
\]

If \( 2s \mid (m - s - 1) \) and \( l = 3 \) then p.d.e. (1) implies \( b_i = 0 \) except when \( i = \kappa \) or \( i = -1 \) so
\[
f = \sum_i a_i y^{1+ki} z^{m-si} + \sum_i b_{-1} x z^{\frac{m-s-1}{2} + 1} + b_{\kappa} x y^{1+3+1\kappa}
\]

Then (2) implies \( b_{-1} = 0 \) and using (5) we get
\[
f = a_{2\kappa+1} y^{3(2\kappa+1)+1} z + b_{\kappa} x y^{3(\kappa+1)}
\]

Now let \( 2s \nmid (m - s - 1) \) and \( l \neq 3 \). Then from (1) and (4) it follows that
\[
f = a_{\kappa} y^{1+\kappa},
\]
for \( \kappa = \frac{m-s}{s} \in \mathbb{Z} \) or
\[
f = a_{\kappa} y^{1+\kappa z},
\]
for \( \kappa = \frac{m-1}{s} \in \mathbb{Z} \).

When \( l = 3 \) and \( 2s \nmid (m - s - 1) \) we proceed in a similar way, this time using (1), (2) and (5).

II) Since \( s > l = 2 \), p.d.e. (2) implies \( \alpha_2 = 0 \) and (4) implies
\[
f = a_{\kappa} y^{1+2\kappa},
\]
for \( \kappa = \frac{m}{s} \in \mathbb{Z} \) or
\[
f = a_{\kappa} y^{1+2\kappa z},
\]
for \( \kappa = \frac{m-1}{s} \in \mathbb{Z} \).
III) As in case (I) we consider two possibilities. First, if \((l(m+1) - s) \mid 2m\) then from (*)1, (*)4 and (*)5 we get
\[
f = \alpha_2 x^{2+2\kappa}z + \beta_\kappa x^\kappa y
\]
where \(\kappa = \frac{2m}{l(m+1) - s}\).

If \((l(m+1) - s) \nmid 2m\) then (*)1 and (*)4 imply
\[
f = \sum_i \alpha_i x^{-2+i}z^{2m+1-\left(\frac{(m+1)-s}{2}\right)i}
\]
and using (*)2 and (*)5 we get the following possibilities for \(f\) :

for \(\kappa = \frac{4m}{l(m+1) - s} \in \mathbb{Z}\) and
\[
f = \alpha_\kappa x^{\kappa-2}z
\]
for \(\kappa = \frac{4m+2}{l(m+1) - s} \in \mathbb{Z}\).

Let \(\rho = 2\). Then \(2l_1 d_3 \in \mathbb{Z}\). If \(2l_1 d_3\) is even then \(\frac{d_1}{d_3} \in \mathbb{Z}\) and (iii) has no solution. Otherwise \(\left(\frac{1+d_1 +d_2}{d_3}\right) \in \mathbb{Z}\) and \(\mu_1 = \left(\frac{1+d_1 +d_2}{d_3}\right)\), \(\nu_1 = 0\) is a solution of (i) which is impossible since we are assuming \(\rho = 2\). Thus (iii) has no solution and consequently (iv) has no solution. Let \(\frac{d_1}{d_3} = m\) and \(\frac{d_2}{d_3} = \lambda \notin \mathbb{Q}\). Then we must have \(\lambda > 1\), \(m > 1\) and (i) and (ii) take the form :

\begin{align*}
(i) & \quad \mu_1 = (2m + 2\lambda - 1) - \lambda \nu_1 \\
(ii) & \quad \mu_2 = (m + 2\lambda - 1) - \lambda \nu_2
\end{align*}

Since \(\lambda \notin \mathbb{Q}\) (i) has unique solution \(\mu_1 = 2m - 1\), \(\nu_1 = 2\) and (ii) has unique solution \(\mu_2 = m - 1\), \(\nu_2 = 2\). Then we get
\[
f = ay^2z^{2m-1} + bx^2z^{m-1}, \quad m > 1
\]
It follows from (*) that \(a = b = 0\).

Let \(\frac{d_2}{d_3} = \frac{s}{l} \in \mathbb{Q}\), \(gcd(s,l) = 1\). Then \(l < s\), \(m > 1\) and the general solutions of (i) and (ii) are given by
\[
\begin{align*}
\mu_1 &= (2m - 1) + sh \\
\nu_1 &= 2 - lh \\
\mu_2 &= (m - 1) + sh \\
\nu_2 &= 2 - lh
\end{align*}
\]
\(h \in \mathbb{Z}\)

Thus we get
\[
f = \sum_i a_i y^{2+li}z^{(2m-1)-si} + \\
\sum_i b_i x y^{2+li}z^{(m-1)-si},
\]

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where \( 2 < l < s \) and \( m > 1 \).

Now let’s suppose \( \frac{d_2}{d_3} \) is odd. Then (iii) has solution if and only if \( \left( \frac{1 + d_2}{d_3} \right) \in \mathbb{Z} \). Let \( d_3 = m > 2 \), \( m \) odd. If \( \frac{d_2}{d_3} = \lambda \notin \mathbb{Q} \), (i) and (ii) take the form:

\[
\begin{align*}
  i) &\quad \mu_1 = \left( m + 2\lambda - 1 \right) - \lambda \nu_1 \\
  ii) &\quad \mu_2 = \left( \frac{m}{2} + 2\lambda - 1 \right) - \lambda \nu_2,
\end{align*}
\]

this time with \( \lambda > 1 \) and \( m > 2 \). Clearly, in this case \( \left( \frac{1 + d_2}{d_3} \right) = \left( \frac{m}{2} + 2\lambda - 1 \right) \notin \mathbb{Z} \). Moreover \( \mu_1 = m - 1, \nu_1 = 2 \) is the unique solution of (i) and (ii) has no integer solution. We get

\[ f = ay^2z^{m-1}, \]

with \( m \) odd, \( m > 2 \) and \( f \) is a solution of \( (\ast) \) only when \( a = 0 \).

If \( \frac{d_2}{d_3} = \frac{l}{l} \in \mathbb{Q}, \gcd(s, l) = 1 \) then \( \left( \frac{1 + d_2}{d_3} \right) = \left( \frac{m}{2} + \frac{2s}{l} - 1 \right) \in \mathbb{Z} \) if and only if \( l = 4 \). So for \( l \neq 4 \) we have \( l < s, m > 2 \) and

\[
\begin{align*}
  i) &\quad \mu_1 = \left( m + \frac{2s}{l} - 1 \right) - \frac{s}{l} \nu_1 \\
  ii) &\quad \mu_2 = \left( \frac{m}{2} + \frac{2s}{l} - 1 \right) - \frac{s}{l} \nu_2
\end{align*}
\]

If \( l \) is odd

\[
\begin{align*}
\begin{cases}
\mu_1 = (m - 1) - sh \\
\nu_1 = 2 + lh,
\end{cases} \quad h \in \mathbb{Z}
\end{align*}
\]

and (ii) has no solution. If \( l \) is even we find

\[
\begin{align*}
\begin{cases}
\mu_1 = (m - 1) - sh \\
\nu_1 = 2 + lh,
\end{cases} \quad h \in \mathbb{Z}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\mu_2 = \left( \frac{m-s}{2} - 1 \right) - sh \\
\nu_2 = 2 + \frac{l}{2} + lh,
\end{cases} \quad h \in \mathbb{Z}
\end{align*}
\]
When \( l = 4 \), \((**)\) takes the form

\[
\begin{align*}
(i) & \quad \mu_1 = \left( m + \frac{s}{2} - 1 \right) - \frac{s}{4} \nu_1 \\
(ii) & \quad \mu_2 = \left( \frac{m}{2} + \frac{s}{2} - 1 \right) - \frac{s}{4} \nu_2 \\
(iii) & \quad \mu_3 = \left( m + \frac{s}{2} - 1 \right) - \frac{m}{2} \nu_3 \\
(iv) & \quad \mu_4 = \left( \frac{m}{2} + \frac{s}{2} - 1 \right) - \frac{m}{2} \nu_4
\end{align*}
\]

with \( m > 2 \), \( s > 4 \) and \( s \) odd and we find that \( \mu_3 = \frac{m+s}{2} - 1 \), \( \nu_3 = 1 \) is a solution of \((iii)\). Thus we add only the following possibilities for \( f \):

\[
f = \sum_i a_i y^{2+i} z^{(m-1)-si}
\]

with \( l \) odd, \( 2 < l < s \) and \( m \) odd, \( m > 2 \);

\[
f = \sum_i a_i y^{(m-1)-si} +
\]

\[
\sum_i b_i x y^{(m+2i/3) + i} z^{(m-3s)/2 - 1 - si}
\]

with \( l \) even, \( 2 < l < s \) and \( m \) odd \( m > 2 \).

Let \( \rho = 3 \). If \( \mu_3 = \xi \), \( \nu_3 = \eta \) is a solution of \((iii)\) such that both \( \eta \) and \( \xi \) are non-negative then \( \eta = 1 \) or \( \eta \geq 6 \). In fact in this case \( 2 \frac{d_2}{d_3} - \frac{d_2}{d_3} \in \mathbb{Z}_+ \) and using this we can discriminate the other possibilities for \( \eta \). Since we have already seen the case \( \rho = 1 \), we may assume only \( \eta \geq 6 \) is possible. Obviously, for \( \eta = 1 \) it is enough to interchange \( x \) and \( y \) in all solutions we found when \( \rho = 1 \).

We show \( \eta \geq 6 \), \( \xi \geq 0 \) is not possible. In fact \( 1 + d_1 - 2d_2 \geq 0 \) and \( 1 - (\eta-1)d_1 + d_2 \geq 0 \) implies:

\[
2 + d_1 + d_2 = 3 + d_3 \geq 2d_2 + 2d_3 + (\eta - 3)d_1 \\
= 2 + 2d_3 + 1 + d_3 - d_2 + (\eta - 4)d_1 \\
= 3 + 3d_3 + ((\eta - 4)d_1 - d_2) > 3 + d_3
\]

which is a contradiction. Consequently, if \((iv)\) has a solution \( \mu_4 = \zeta \), \( \nu_4 = \omega \) with \( \omega \geq 0 \), \( \zeta \geq 0 \) then \( \omega = 0 \) or \( \omega > 3 \). If \( \omega > 3 \) we have \( 1 \geq (\omega - 1)d_1 \geq 3d_1 \) and so

\[
2 + d_1 \geq 2d_2 + 3d_1 = 2 + 2d_3 + d_1 > 2 + d_1
\]

a contradiction. So we have to take into account only equations \((i)\) and \((ii)\). It’s easy to see that the same is also valid if there does not exist an integer solution of \((iii)\) satisfying \( \mu_3 \geq 0 \) and \( \nu_3 \geq 0 \) simultaneously.
Let \( \rho = 4 \). Then we have \( 2 \frac{d_3}{d_5} - 2 \frac{d_3}{d_5} \in \mathbb{Z}_+ \). Let \( \frac{d_3}{d_5} = \lambda \) and \( 2 \frac{d_3}{d_5} - 2 \frac{d_3}{d_5} = m \). Then our system takes the form:

\[
\begin{align*}
(i) \quad \mu_1 &= (4\lambda - m - 1) - \left( \lambda - \frac{m}{2} \right) \nu_1 \\
(ii) \quad \mu_2 &= (3\lambda - m - 1) - \left( \lambda - \frac{m}{2} \right) \nu_2 \\
(iii) \quad \mu_3 &= (4\lambda - m - 1) - \lambda \nu_3 \\
(iv) \quad \mu_4 &= \left( 3\lambda - \frac{m}{2} - 1 \right) - \lambda \nu_4
\end{align*}
\]

This time \( m \) and \( \lambda \) must satisfy the following inequalities:

\[ m > 0 \quad \text{and} \quad \frac{m}{2} + 1 < \lambda \]

If there exists an integer solution of (iii), say \( \mu_3 = \xi, \nu_3 = \eta \), satisfying \( \xi \geq 0 \) and \( \eta \geq 0 \), we must have \( \eta < 4 \). Then solutions can be obtained from those we got when \( \rho = 0, 1, 2 \) or 3. Otherwise, i.e. if the set of integer solutions of equation (iii) satisfying \( \mu_3 \geq 0 \) and \( \nu_3 \geq 0 \) is empty, we can disregard equations (iii) and (iv).

Let \( \rho > 4 \) and \( \xi, \eta \) as in the case before, then \( \eta \leq 4 \). Otherwise we get a contradiction. In fact from \( (\frac{1+d_1+d_2}{d_5}) - \frac{d_5}{d_5} \eta \geq 0 \) and \( (\frac{1+d_1+d_2}{d_5}) - \frac{d_5}{d_5} \rho \geq 0 \) it follows that

\[ 2 + d_1 + d_2 = 3 + d_3 \geq (\eta - 1)d_1 + (\rho - 1)d_2 \]

If we assume \( \rho > 4, \eta > 4 \) the right-hand side of the last inequality is bigger than \( 3(d_1 + d_2) = 3 + 3d_3 \) which is impossible. Once more, we don’t need to study these cases explicitly since they are contained somehow in the cases before. On the other hand, if there does not exist such a solution \( \xi, \eta \), only equations (i) and (ii) are to be taken into account.

Summarizing the last results, when \( \rho > 2 \) it is enough to take into account only equations (i) and (ii).

Now, let \( \rho > 2, \frac{d_3}{d_5} = \lambda \) and \( (\frac{1+d_1+(\rho-1)d_2}{d_3}) = m \in \mathbb{Z}, m \geq 0 \). Then (i) and (ii) take the form:

\[
\begin{align*}
i) \quad \mu_1 &= (\rho \lambda + m) - \lambda \nu_1 \\
(ii) \quad \mu_2 &= \left( \frac{(\rho + 2)\lambda + m - 1}{2} \right) - \lambda \nu_2
\end{align*}
\]

with \( 1 < \lambda \).

If \( \lambda \notin \mathbb{Q} \) then \( \mu_1 = m, \nu_1 = \rho \) is the unique solution of (i). On the other hand, (ii) has a solution iff \( \rho \) is even and \( m \) is odd. In this case the unique solution of (ii) will be given by \( \mu_2 = \frac{m-1}{2}, \nu_2 = 1 + \frac{\rho}{2} \). Thus we have the following possibilities for \( f \):

\[
f = a y^\rho z^m + b x^{1+\frac{\rho}{2}} z^{\frac{m-1}{2}}
\]
when $\rho$ is even and $m$ is odd and

$$f = ay^\rho z^m$$

otherwise.

Since $f$ must satisfy the system of partial differential equations at the beginning we find that

$$f = ay^\rho z + bx y^{1+\frac{\rho}{2}} ,$$

with $\rho$ even, $\rho > 2$ or

$$f = ay^\rho z^m ,$$

with $m < 2$.

Now let $\lambda = \frac{s}{l} \in \mathbb{Q}$, $gcf(s, l) = 1$. Then (i) has general solution:

$$\begin{cases}
\mu_1 = m - sh \\
\nu_1 = \rho + lh ,
\end{cases} \quad h \in \mathbb{Z}$$

and (ii) has general solution

$$\begin{cases}
\mu_2 = \left( \frac{m-1}{2} \right) - \frac{s}{2} h \\
\nu_2 = \left( 1 + \frac{\rho}{2} \right) + \frac{l}{2} h ,
\end{cases} \quad h \in H$$

where $H$ stands for the set of those integers $h$ such that $(\frac{m-1}{2}) - \frac{s}{2} h \in \mathbb{Z}$ and $(1 + \frac{\rho}{2}) + \frac{l}{2} h \in \mathbb{Z}$. Then

$$f = \sum_i a_i y^{\mu_1+li} z^{m-si} +$$

$$\sum_i b_i x y^{(1+\frac{\rho}{2}) + \frac{l}{2} i} z^{(\frac{m-1}{2}) - \frac{s}{2} i}$$

In order to take into account also the case $\rho = 2$, we replace restriction $\rho > 2$ by $\rho \geq 2$. In fact it is easy to see that all possibilities we found for $f$ when $\rho = 2$ are contained in

$$f = \sum_i a_i y^{2+li} z^{m-si} +$$

$$\sum_i b_i x y^{2+\frac{l}{2} i} z^{(\frac{m-1}{2}) - \frac{s}{2} i} , \quad m \geq 0$$

We have two cases basically. First, let $s | (m-1)$ and $\kappa = \frac{m-1}{s}$. From (.*1) it follows that

$$f = \sum_i a_i y^{\rho+li} z^{m-si} + b_\kappa x y^{(1+\frac{\rho}{2}) + \frac{l}{2} \kappa}$$
and then using \((* . 5)\) we get

\[
f = a_\kappa y^{\rho + i \kappa} z + b_\kappa x y^{(1 + \frac{\rho}{2}) + \frac{1}{2} \kappa}
\]

On the other hand, if \( s \nmid (m - 1) \) then from \((* . 1)\) we get that

\[
f = \sum_i a_i y^{q + i} z^{m - s i}
\]

and using \((* . 4)\) we find that only when \( s \mid m \) we have non-trivial solutions, say

\[
f = a_\kappa y^{\rho + i \kappa}
\]

with \( \kappa = \frac{m}{s} \).

Finally, suppose we have \( \nu_1 < 0 \) whenever \( \mu_1 \geq 0 \), for every integer solution of \((i)\). By the symmetry of \((*)\) in \( x \) and \( y \) it is enough to look at the case in which \((iii)\) has the analogous property. Then using the first property of \((***)\) we saw at the beginning, we conclude that

\[
f = bxz^\frac{1 + d_1}{\alpha_3} + \beta yz^\frac{1 + d_1}{\alpha_3}
\]

and from \((* . 2)\) and \((* . 4)\) we get \( b = \beta = 0 \).

Thus, the whole list of functions \( f \) which are solutions of our problem is:

1) \( f = ax^{2l}z + bx^{l+1}y \), \( l \geq 2 \)
2) \( f = ax^l \), \( l > 2 \)
3) \( f = ax^l z \), \( l > 2 \)
4) \( f = \alpha x^3 z + a y^3 z + 6a \alpha x y z^3 + \frac{54}{35} a^2 \alpha^2 z^7 \)
5) \( f = \alpha x^4 z + a y^3 z + 12a \alpha x^2 y z^3 + \frac{36}{5} a^2 \alpha y^2 z^5 + \)
\[+ \frac{144}{7} a^2 \alpha^2 x^2 z^7 + \frac{1728}{143} a^4 \alpha^3 z^{13} \]
6) \( f = \alpha x^4 + a y^3 + 36a \alpha x^2 y z + 108a^2 \alpha y^2 z^2 + \)
\[+ 864 a^2 \alpha^2 x^2 z^3 + \frac{93312}{5} a^4 \alpha^3 z^6 \]
7) \( f = \beta x^3 y + a y^3 + \frac{9}{4} \beta^2 x^4 z + 9a \beta x y^2 z + \)
\[+ 27a^2 \beta^2 x^2 y z^2 + 27a \beta^3 x^3 z^3 + 54a^2 \beta^2 y^2 z^3 + \]
where $a$, $b$, $\alpha$ and $\beta$ are arbitrary. The first part of our theorem follows immediately from this result.

Now we shall see that only solutions (4)-(8) and (12)-(15) give rise to semisimple algebras. In fact it will be enough to consider solutions (1) through (8). It is not too difficult to check that for cases (1)-(3) the matrix $(E, \cdot)$ defined by

$$(E, \cdot)^\gamma_\beta = t c^\gamma_0 + d_1 x c^\gamma_1 + d_2 y c^\gamma_2 + d_3 z c^\gamma_3$$

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where $c_{\alpha \beta} = \eta^{\varepsilon \delta} \partial_\varepsilon \partial_\delta \partial_\beta F$, $\varepsilon, \alpha, \beta \in \{t, x, y, z\}$, $(\eta^{\varepsilon \delta}) = (\eta_{\varepsilon \delta})^{-1}$ and $\eta_{\varepsilon \delta} = \partial_\varepsilon \partial_\delta \partial_\eta F$, $\gamma, \epsilon \in \{t, x, y, z\}$, has multiple eigenvalues for every choice of $(t, x, y, z)$.

For cases (6) and (7) just take $(t, x, y, z) = (0, 0, 1, 0)$. Then $(E, \cdot)_\beta^\gamma$ has distinct eigenvalues.

We look at (4), (5) and (8) separately.

In the case (4) we make $t = z = 0$, $x = \sqrt[3]{X}$ and $y = \sqrt[3]{Y}$ in $(E, \cdot)_\beta^\gamma$ and the characteristic polynomial takes the form:

$$P(\lambda) = \lambda^4 - 12(L + K)\lambda^2 + 36(L - K)^2$$

where $L = a d_2^2 Y$ and $K = \alpha d_1^2 X$. We may choose $X$ and $Y$ such that all roots of $P$ are different.

For (5) we proceed in a similar way. After making $t = z = 0$ and substitutions $x = \sqrt[3]{X}$, $y = \sqrt[3]{Y}$ in $(E, \cdot)_\beta^\gamma$ the characteristic polynomial takes the same form as in the case before, this time with $L = a d_2^2 Y$ and $K = 2\alpha d_1^2 X$.

It remains (8). Here we make again $t = z = 0$ and then substitutions $x = \sqrt[3]{X}$ and $y = \sqrt[3]{Y}$. We find the characteristic polynomial of $(E, \cdot)_\beta^\gamma$ is

$$P(\lambda) = \lambda^4 - 4(L + K)\lambda^2 + 4(L - K)^2$$

with $L = 3a d_2^2 Y$ and $K = 10\alpha d_1^2 X$. Then a suitable choice of $X$ and $Y$ leads to $P$ without multiple roots.

References