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IC/97/129
ISAS/EP/97/106

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

NON-INTEGRABLE ASPECTS OF THE MULTI-FREQUENCY SINE-GORDON MODEL[1]

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MIRAMARE – TRIESTE

September 1997

[1]Work done under partial support of the EC TMR Programme *Integrability, non–perturbative effects and symmetry in Quantum Field Theories*, Grant FMRX-CT96-0012.

ABSTRACT

We consider the two-dimensional quantum field theory of a scalar field self-interacting via two periodic terms of frequencies α and β. Looking at the theory as a perturbed Sine-Gordon model, we use Form Factor Perturbation Theory to analyse the evolution of the spectrum of particle excitations. We show how, within this formalism, the non-locality of the perturbation with respect to the solitons is responsible for their confinement in the perturbed theory. The effects of the frequency ratio α/β being a rational or irrational number and the occurrence of massless flows from the gaussian to the Ising fixed point are also discussed. A generalisation of the Ashkin-Teller model and the massive Schwinger model are presented as examples of application of the formalism.

1 Introduction

For its attractive features, both at classical and quantum level, the $(1+1)$-dimensional Sine-Gordon (SG) model has always played a prominent role in the investigation of theoretical non-perturbative aspects of quantum field theory (QFT) [1—7] as well as in the concrete discussion of innumerable physical systems: equations of Sine-Gordon type are in fact relevant in quasi one-dimensional charge density waves of strongly correlated electrons and in statistical mechanics aspects of one–dimensional quantum chains [?], in self-induced trasparency effects of non–linear optics [?, ?] or spin wave propagation in the quantum liquid phase of ^{3}He [?], just to mention a few. The lagrangian of the model may be written as

$$\mathcal{L}_{SG} = \frac{1}{2}(\partial_\mu \varphi)^2 + \mu \cos(\beta\varphi) \ . \tag{1.1}$$

In many of the physical applications, the mapping onto the pure SG model is, however, only an approximation and therefore is expected to hold only in some special regions of the physical parameter space. A more complete and refined description of several physical effects often requires that a lagrangian with two or more periodic interaction terms is considered. A simple example is provided by a system of two Ising models coupled both by thermal and magnetic operators, system which may be considered as a generalization of the Ashkin-Teller model[1]. Other examples, arising from different physical contexts, are given by the aforementioned cases of spin waves propagating in anisotropic magnetic liquids or ultra-short optical pulses propagating in resonant degenerate medium [?]. The purpose of this paper is to analyse the quantum field theory aspects of this kind of generalisation of the SG model and in particular the case in which two periodic interactions are present, i.e. to study the QFT defined by the lagrangian[2]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \mu \cos\beta\varphi + \lambda \cos(\alpha\varphi + \delta) \ . \tag{1.2}$$

Apart from the practical motivations we have just mentioned, there are also some theoretical issues which make the analysis of (??) an interesting problem. Let us try to describe them.

The addition of a new interaction term to the lagrangian (??) spoils one of the crucial properties of the SG model, i.e. its integrability. As a consequence, the resulting two-frequency model (??) is not exactly solvable and perturbation theory is basically the only available tool for a quantitative study. Of all the different perturbative approaches that could be applied to the problem at hand, the one where the new periodic term is viewed as perturbing a pure SG model is probably the least conventional but the most

[1]This model will be discussed in some details in Section 5 of this paper.

[2]Meaning and restrictions on the parameters entering eq. (??) will be the subject of Section 2.

convenient. Perturbation of integrable models was discussed in Ref. [?]: its theory relies on the fact that the matrix elements of local operators between asymptotic states of the unperturbed (integrable) theory are exactly computable. The corrections to the masses of the particles of the unperturbed theory and to the scattering amplitudes can be computed in terms of the matrix elements of the perturbing operator in much the same way as in quantum mechanics. Since the matrix elements of the local operators are usually known as Form Factors, we refer to this perturbative approach as Form Factor Perturbation Theory (FFPT). The Form Factors of the SG model were computed in Refs. [?, ?, ?] and consequently the technical information required for these perturbative calculations is available. However, our interest is not to perform specific perturbative computations, but rather to illustrate some general features of the perturbed model and the use of the FFPT to describe it.

We are particularly interested in the spectrum of the physical excitations. The invariance of the Lagrangian (??) under the discrete transformation $\varphi \longrightarrow \varphi + 2\pi q/\beta$, $q \in \mathbb{Z}$ implies that the elementary excitations of the pure SG model are solitons interpolationg between adjacent degenerate minima of the potential. Extra interactions will generally remove the degeneracy among the different classical vacua and therefore these topological excitations are expected to disappear from the spectrum of asymptotic states of the perturbed theory. Therefore it is essential to ascertain whether the FFPT accounts for such a dramatic evolution in the particle spectrum of the theory. As we will show, the answer is indeed affirmative and the phenomenon of confinement of the original topological excitations emerges in the FFPT as a consequence of the mutual non-locality between the perturbing operator and the field which interpolates the confined particle. Notably, the non–local mechanism responsible for the confinement of the topological excitations proves to be a general feature of many two-dimensional theories: it causes, for instance, the disappearence of the kinks from the spectrum of the low-temperature phase of the Ising model as soon as an external magnetic field is turned on [?, ?].

The role of the quantum vacuum of the theory is particularly relevant for the analysis below. In the pure SG model, the quantum vacuum is selected from the infinite number of classical equivalent vacua through the spontaneous symmetry breaking mechanism. At the operational level, this involves selecting arbitrarily one of the classical minima as the quantum vacuum and constructing the space of asymptotic states as excitations above it. Physical quantities such as scattering amplitudes, Form Factors, etc. are then computed as matrix elements on this space of states. In the perturbed theory, the degeneracy of the classical vacua is explicitely broken and the quantum vacuum is uniquely defined. A fundamental requirement for the application of the FFPT is to ensure that, when the perturbation is adiabatically switched off, the vacuum matches the one previously chosen for the computations in the unperturbed theory. This "adiabatic condition" corresponds to the general requirement that the vacuum of a quantum field theory in the presence of a spontaneously broken symmetry must be identified by taking the limit in which a symmetry breaking perturbation is made to vanish.

The vacuum of the perturbed theory is however unique and the above considerations apply, only if the periods of the two interaction terms are incommensurate. In the case of rational frequency ratio, the survival of a subset of degenerate classical vacua in the perturbed potential must be given appropriate consideration. It is already clear from this observation that the theory will exhibit an extremely irregular behaviour as a function of the frequency ratio.

In the non–integrable model (??), the two periodic interactions clearly play a symmetric role and each of the cosine interactions can be viewed as a deformation of the integrable theory defined by the other: as discussed in more detail in Section 3, there is a dimensionless variable

$$\eta \equiv \lambda\mu^{-(1-\Delta_\alpha)/(1-\Delta_\beta)} = \lambda\mu^{-(8\pi-\alpha^2)/(8\pi-\beta^2)} \ , \tag{1.3}$$

which characterises the two perturbative regimes of the model (??). The first regime is obtained in the limit $\eta \to 0$, while the other is reached for $\eta \to \infty$, the latter by just swapping the role played by the two operators. The possibility to control the changes of the spectrum in both perturbative limits allows one to deduce interesting information about its evolution in the intermediate, non-perturbative region. In certain cases, there may be a topological excitation in one limit which is no longer present in the other. When this is the case, the very nature of topological excitations requires that a change in the vacuum structure of the theory takes place somewhere in the non-perturbative region, namely that a phase transition occurs[3]. Lines of phase transition are then expected to appear in the multi–frequency SG model for particular values of the parameters: they generally correspond to Renormalisation Group (RG) trajectories along which the system flows from the gaussian fixed point with central charge $C = 1$ to the Ising fixed point with central charge $C = 1/2$.

The paper is organised as follows. The next section reviews several results of the pure SG model which will be used later. The two-frequency model is introduced in Section 3: some restrictions on the parameters of the theory dictated by conformal perturbation theory and the aforementioned adiabatic condition will be discussed. The Form Factor Perturbation Theory, the relation between non-locality and soliton confinement, the evolution of the particle spectrum and the occurrence of phase transition are analysed in

[3]This phenomenon was discussed in these terms long ago by S. Coleman in his classic study of the massive Schwinger model [?].

Section 4. The system of two–layer Ising model, alias the generalized Ashkin-Teller model, and the massive Schwinger model are presented as examples of specific applications of the formalism in Section 5. Summary of the results and conclusions are in Section 6. The paper has also two appendices: the first contains a derivation of the soliton-antisoliton form factors of exponential operators at the reflectionless points of SG model while the second presents an analysis of soliton confinement within the semiclassical approximation.

2 Few Notions of the Sine–Gordon Model

In this section we briefly review some basic features of the pure SG model and establish several notations which will be used in the following. The SG model is a relativistic theory in $(1+1)$ dimensions for a scalar bosonic field $\varphi(x)$ with the lagrangian given in eq. (??). At the classical level, the model possesses an infinite degeneracy of the vacuum states which can be labelled by a relative integer q: finite energy configurations must necessarely approach one of the vacuum values $\frac{2\pi}{\beta}q_-$ for $x \to -\infty$ and another one $\frac{2\pi}{\beta}q_+$ for $x \to +\infty$. The difference $Q \equiv q_+ - q_-$ plays a role of topological charge which characterises inequivalent sectors of the theory. As well known, the model is completly solvable both at the classical and the quantum level. At the classical level, all solutions of the equation of motion can be computed by means of the inverse scattering method, as reviewed for instance in [?]. Building blocks for expressing all the classical solutions are the static soliton and antisoliton configurations with $Q = \pm 1$: their explicit expressions are given by

$$\varphi_{s,\bar{s}}(x) = \frac{4}{\beta}\arctan[\exp(\pm\beta\sqrt{\mu}\,x)] \ , \tag{2.1}$$

where $+$ refers to the soliton and $-$ to the antisoliton. Multi–solitons are then obtained by the non–linear superposition principle provided by the Bäcklund transformations. For the purposes of this paper, it may be convenient to think of soliton configurations as step functions with jumps $\pm 2\pi/\beta$ (Figure 1).

At the quantum level, a normal ordering of the cosine interaction term in the lagrangian (??) is sufficient to get rid of the only diverging diagrams (tadpoles) in the theory. From the conformal perturbation theory viewpoint, the SG model can be considered as a deformation of the gaussian fixed point action

$$\mathcal{A}_{Gaussian} = \int d^2x \, \frac{1}{2} \left(\partial_\nu \varphi\right)^2 \quad , \tag{2.2}$$

(with central charge $C = 1$) by the operator $\cos\beta\varphi$ with conformal dimension $\Delta_\beta = \beta^2/8\pi$. The latter is relevant for $\beta^2 < 8\pi$, and in this range the theory presents a massive phase[4]. The exact particle spectrum of the theory [?, ?] consists of a pair of soliton and

[4]The limitation $\beta^2 < 8\pi$ will be always assumed to hold in the following.

antisoliton excitations with equal mass M and a number $N = \left[\frac{\pi}{\xi}\right]$ of soliton–antisoliton bound states B_n (breathers) with masses

$$m_k = 2M \sin\left(\frac{k\xi}{2}\right) \quad , \quad k = 1, 2, \ldots, < \frac{\pi}{\xi} \quad , \tag{2.3}$$

where

$$\xi = \frac{\pi\beta^2}{8\pi - \beta^2} \quad . \tag{2.4}$$

The elementary field φ entering the lagrangian (??) may be regarded as associated to the lightest breather B_1. Note, however, that the k-th bound state crosses the soliton–antisoliton threshold when $\xi = \pi/k$, and for $\xi \geq \pi/k$ it leaves the physical spectrum. Hence, for $\xi \geq \pi$, the spectrum consists of soliton and antisoliton only and does not contain any particle associated to the field φ.

An additional fundamental result about SG model is its exact equivalence with the massive Thirring model [?], namely with the theory of a self-interacting Dirac fermion defined by the Lagrangian

$$\mathcal{L}_{MTM} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2 \ . \tag{2.5}$$

The correspondence between the two models is specified by the bosonisation rules and by the following relation among the couplings of the two models

$$g = \pi\frac{4\pi - \beta^2}{\beta^2} \quad . \tag{2.6}$$

The exact S–matrix relative to factorizable and elastic scattering processes of the SG model has been determined in [?]. We refer the reader to this paper for all details about this subject. The matrix elements of local operators between the asymptotic states of the SG model have also been computed by using the so called Form Factor bootstrap approach [?, ?]. Recently, Lukyanov used a technique based on free field representation to compute the Form Factors of the exponential operators $e^{i\alpha\varphi}$ [?]. Since these Form Factors are of particular interest in the present paper, we show in Appendix A how they can be easily computed in the ordinary bootstrap framework, at least for the special values of the coupling $\xi = \pi/k$ where the S-matrix is reflectionless and the technical problem gets considerably simplified.

3 The Two-frequency Sine-Gordon Model

In this section we will derive and illustrate some limitations on the real parameters α, β and δ entering the quantum action

$$\mathcal{A} = \int d^2x \left[\frac{1}{2}(\partial_\mu\varphi)^2 + \mu\cos\beta\varphi + \lambda\cos(\alpha\varphi + \delta)\right] \quad . \tag{3.1}$$

We can always assume that α, β, μ and λ are positive, since this situation can be arranged by using appropriate redefinitions of φ and δ.

There are at least two ways of viewing the above theory, both of which provide some useful insight to its physical content. The first regards (??) as a perturbation of the gaussian action (??) by means of the two scaling operators $\mathcal{O}_1 = \cos\beta\varphi$ and $\mathcal{O}_2 = \cos(\alpha\varphi+\delta)$, with conformal dimensions $\Delta_\beta = \beta^2/8\pi$ and $\Delta_\alpha = \alpha^2/8\pi$, respectively. The coupling constants μ and λ have scaling dimensions $(mass)^{2(1-\Delta_\beta)}$ and $(mass)^{2(1-\Delta_\alpha)}$, respectively and therefore the parameter η which was defined in (??) forms a dimensionless combination which labels the different RG flows originating from the gaussian fixed point. The relevant nature of both operators then implies

$$\begin{aligned} &\alpha^2 < 8\pi \;\; ; \\ &\beta^2 < 8\pi \;\; . \end{aligned} \tag{3.2}$$

An additional constraint on the two parameters α and β derives from the renormalizability of the action (??), an aspect which can be efficiently studied by conformal perturbation theory [?]. To illustrate this point, let us briefly discuss the pattern of renormalisation of a scaling operator $\Phi(x)$ in a theory obtained by perturbing a conformal action[5] by a relevant operator ϕ of conformal dimension Δ

$$\mathcal{A}_g \;=\; \mathcal{A}_{CFT} + g\int d^2x\,\phi(x) \;\; . \tag{3.3}$$

Let X denote a generic product of operators and consider the usual perturbative expansion of the correlator

$$\langle X\,\Phi(0)\rangle \;=\; \langle X\,\Phi(0)\rangle_{CFT} + g\int_{\epsilon<|x|<R} d^2x \langle X\,\Phi(0)\,\phi(x)\rangle_{CFT} + O(g^2) \;\; . \tag{3.4}$$

ϵ and R are ultraviolet and infrared cutoffs, respectively, and the correlators on the right hand side of the above equation are computed at the conformal point. The integral in Eq. (??) is ultraviolet divergent only if the conformal OPE

$$\phi(x)\,\Phi(0) \;=\; \sum_k C^k_{\phi\Phi} |x|^{2(\Delta_k-\Delta_\Phi-\Delta)}\,A_k(0) \tag{3.5}$$

contains operators A_k with conformal dimension Δ_k such that

$$\gamma_k \equiv \Delta_k - \Delta_\Phi - \Delta + 1 \leq 0 \;\; . \tag{3.6}$$

If this is the case, a first order UV finite correlator $\langle X\,\Phi'(0)\rangle$ is obtained by defining the renormalised operator

$$\Phi' \;\equiv\; \Phi + g\sum_k b_k \epsilon^{2\gamma_k} A_k + O(g^2) \;, \tag{3.7}$$

[5] We consider for simplicity the case of perturbation by a single scaling operator, but the following considerations are easily extended to more general situations.

where $b_k = -\pi C^k_{\phi\Phi}/\gamma_k$ and the sum runs only over the operators A_k satisfying the condition (??). The last equation shows that the renormalisation of UV divergences induces a mixing of the operator $\Phi(x)$ with a finite number of operators having smaller conformal dimensions.

The above general considerations are easily applicable to perturbations of the gaussian action, with the spectrum of primary operators spanned by the exponentials $e^{i\alpha\varphi(x)}$ of conformal dimension $\Delta_\alpha = \alpha^2/8\pi$ and their OPE given by

$$e^{i\alpha_1\varphi(x)}\, e^{i\alpha_2\varphi(0)} = |x|^{\alpha_1\alpha_2/2\pi}\, e^{i(\alpha_1+\alpha_2)\varphi(0)} \ . \tag{3.8}$$

Using eq. (??), it is easy to see that the two perturbing terms in (??) do not mix with other operators (so that no extra terms are generated in the action (??)) if the parameters α and β satisfy the inequality

$$\alpha\beta < 4\pi \ . \tag{3.9}$$

If the above condition is not fulfilled, the action (??) results unstable under renormalization, i.e. additional counterterms $e^{\pm i(\alpha-\beta)\varphi}$ should already be included at the first order in g so as to have a consistent theory[6].

Let us now consider the second approach to the QFT (??) – the one we will mostly adopt in this paper – which consists of viewing it as a deformation of a pure Sine-Gordon action. Although the two interaction terms play a completely symmetric role, in the following we will assume $\lambda\cos(\alpha\varphi + \delta)$ to be the perturbing term and derive some restrictions on the parameter δ.

When $\lambda = 0$, the pure SG potential has a periodicity $2\pi/\beta$ and therefore exhibits an infinite number of classical degenerate vacua placed at $\varphi = 2\pi q/\beta$. At the quantum level, however, spontaneous symmetry breaking selects a unique quantum vacuum, which by convention we choose to be at the origin, $\varphi = 0$. Let us now switch the perturbation on. Our analysis begins by considering the case when the frequency ratio

$$\omega \equiv \frac{\alpha}{\beta} \tag{3.10}$$

is a rational number, m/n, with m and n coprime positive integers. In this case, the new potential acquires a periodicity $2\pi n/\beta$. For λ very small, the perturbed potential presents, within the first period, n relative minima at $\varphi_k = 2\pi k/\beta + O(\lambda)$, $k = 0, 1, \ldots, n-1$ (Figure 2). Let the absolute minimum of the perturbed potential be located nearby $\varphi_{\tilde{k}} = 2\pi\tilde{k}/\beta$. According to the adiabatic condition, we have then to shift

$$\varphi(x) \longrightarrow \varphi - 2\pi\tilde{k}/\beta \ , \tag{3.11}$$

[6] In the particular case $\alpha = \beta$ this simply leads to the introduction of an inessential additive constant in the action (??).

in order to move the vacuum back near to the origin: the action (??) is intended to precisely address to this situation. Hence, the only values of δ which lead to inequivalent quantum field theories – those which do not differ by a simple shift in the location of the vacuum – are those given by

$$|\delta| \leq \frac{\pi}{n} \ . \tag{3.12}$$

These are in fact the values for which the absolute minimum of the perturbed potential remains located close to $\varphi = 0$. At the perturbative level in λ, eq. (??) is simply derived from the requirement that the variation of the potential at the minimum close to $\varphi = 0$ is larger (in absolute value) than at the other $(n-1)$ minima. When the bound (??) is saturated, one of the $(n-1)$ minima and the one close to the origin degenerate (Figure 3).

Let us now consider the case in which the ratio ω of the two frequencies is an irrational number. Since any irrational number can be approximated with arbitrary precision by the sequence of rational approximants provided by its continued fraction expansion, this case may be viewed as a particular limit of the previous situation. Let

$$\frac{m_i}{n_i} \equiv \frac{1}{r_1 + \frac{1}{r_2 + \frac{\cdots}{r_{i-1} + \frac{1}{r_i}}}} \tag{3.13}$$

be the i-th approximation of the irrational number ω. Both m_i and n_i become larger by increasing the index i of the sequence[7]. Since the restriction (??) only involves the denominator n_i of the rational number, the range of values of δ which give rise to inequivalent theories shrinks consequently to zero in the irrational case.

In conclusion, the family of RG trajectories in the $\lambda\mu$–plane which are labelled by the dimensionless parameter η acquires, somehow, a "fractal" nature as a function of the frequency ratio ω: while no further parametric dependence is present for irrational values of ω, on the contrary a multiplicity of trajectories – labelled by the inequivalent values of δ (??) – results for each rational value of the frequency ratio.

The varied behaviour of the theory (??) – depending on whether ω is a rational or an irrational number – can be understood in a number of ways. In the presence of an irrational frequency ratio, the perturbed potential has a very irregular shape: it is no longer periodic and possesses an infinite number of inequivalent relative minima. The values of the potential at these minima form a continuous spectrum above the absolute minimum value $V_{\text{min}} = -\mu - \lambda$. Suppose that the absolute minimum lies at $\varphi = \varphi'$ for a given value of δ (e.g. $\varphi' = 0$ for $\delta = 0$). An infinitesimal variation of the parameter δ induces a corresponding variation of the potential at φ'. Since the values of the potential

[7] A well known example of this statement is provided by the sequence F_i/F_{i+1} of the Fibonacci numbers which converges to the golden ratio $(\sqrt{5}-1)/2$.

at the minima form a continuous spectrum, the absolute minimum will no longer be located near φ' but will jump to another point φ'' which is generally far from φ'. Hence, a variation of δ simply amounts to a shift of the vacuum from φ' to φ''. This in turn means that δ is an irrelevant parameter in the case of irrational frequency ratio and can be safely set to zero.

The above considerations can also be rephrased in more geometrical terms, as follows. Let us introduce an additional field $\zeta(x)$ and write the potential of the lagrangian (??) as

$$V(\varphi,\zeta) \;=\; -\mu\cos\beta\varphi - \lambda\cos(\alpha\zeta+\delta) \;\; . \tag{3.14}$$

The original theory is of course recovered by the identification

$$\zeta \equiv \varphi \;\; . \tag{3.15}$$

Independently from the nature of the ratio ω, the potential $V(\varphi,\zeta)$ presents a double periodicity $\left(\frac{2\pi}{\beta},\frac{2\pi}{\alpha}\right)$ as a function of the two variables φ and ζ. Hence, this function is naturally defined on a compact torus $\mathcal{T}=\left\{(\varphi,\zeta): -\frac{\pi}{\beta}\leq\varphi\leq\frac{\pi}{\beta}\,;\,-\frac{\pi}{\alpha}\leq\zeta\leq\frac{\pi}{\alpha}\right\}$, and in this domain has an absolute minimum located at $P^{\star}=(0,-\frac{\delta}{\alpha})$. To this point, the rational or irrational nature of ω has played no role. Only by enforcing the identification (??) does the difference between the two cases emerge. Note in fact that on the torus $\mathcal{T}$ eq.(??) defines a family of straight lines which all have slope 1 (Figure 4). These straight lines may be viewed as the motion of a geometrical point bouncing on the boundaries of the torus and reappearing on the opposite sides via its periodicity. When ω is a rational number m/n, the motion of this point repeats over along the finite number $n+m-1$ of these diagonal lines (Figure 4.a). The motion may pass through the minimum $P^{\star}$ or remain at a finite distance from it, which depends on the value of δ. It is however clear that in the rational case all values $|\delta|<\frac{\pi}{n}$ are inequivalent and that when δ exceeds this bound we can go back to the case already considered by simply redefining the fundamental domain $\mathcal{T}$ of the torus. In the irrational case, on the contrary, the motion of the geometrical point is ergodic and therefore spans the entire torus $\mathcal{T}$ (Figure 4.b). Hence, it passes arbitrarly close to the minimum $P^{\star}$, *regardless* on the value of δ. Consequently, the theory with $\delta\neq 0$ is physically indistinguishable from the one with $\delta=0$, the latter value only reflects the choice of $\varphi=0$ as the vacuum state.

4 Non-locality, Soliton Confinement and Phase Transitions

In this section we will consider the quantum theory (??) as a perturbed Sine-Gordon model with the purpose of investigating some features of the evolution of the particle spectrum of the theory as a function of the parameter η defined in (??).

4.1 Form Factor Perturbation Theory and Non–locality

Perturbation theory around integrable models was discussed in Ref. [?], where it was shown how the corrections to the mass spectrum and scattering amplitudes can be expressed in terms of the exactly computable Form Factors of the perturbing operator on the asymptotic states of the unperturbed theory. For the purposes of this paper we only need to point out some of their properties and few basic results of Ref. [?].

Let $\Psi(x)$ be the scalar operator which perturbs an integrable action,

$$\mathcal{A} = \mathcal{A}_{\rm int} + \lambda \int d^2x\, \Psi(x) \ . \tag{4.1}$$

One of the first effects of moving away from integrability is a change in the spectrum of the theory: the first order correction to the mass of a particle a belonging to the spectrum of the unperturbed theory is given by

$$\delta m_a^2 \simeq 2\lambda F^{\Psi}_{a\bar{a}}(i\pi) \ , \tag{4.2}$$

where the particle-antiparticle Form Factor of the operator $\Psi(x)$, defined as the matrix element[8]

$$F^{\Psi}_{a\bar{a}}(\theta) \equiv \langle 0|\Psi(0)|a(\theta_1)\bar{a}(\theta_2)\rangle \ , \tag{4.3}$$

is introduced.

In integrable theories Form Factors of a generic scalar operator $\mathcal{O}(x)$ satisfy manageable functional equations in virtue of the simple form assumed by the unitarity and crossing symmetry equations [?, ?]. For the two–particle case, we have

$$F^{\mathcal{O}}_{a\bar{a}}(\theta) = S^{b\bar{b}}_{a\bar{a}}(\theta) F^{\mathcal{O}}_{b\bar{b}}(-\theta) \ , \tag{4.4}$$

$$F^{\mathcal{O}}_{a\bar{a}}(\theta + 2i\pi) = e^{-2i\pi\gamma_{\mathcal{O},a}} F^{\mathcal{O}}_{\bar{a}a}(-\theta) \ . \tag{4.5}$$

The first of these equations expresses the fact that in an integrable theory the two-particle threshold is the only unitarity branch point in the plane of the Mandelstam variable $s = (p_a + p_{\bar{a}})^2 = 4m_a^2 \cosh^2(\theta/2)$, the discontinuity across the cut being determined by the two-body scattering amplitude $S^{b\bar{b}}_{a\bar{a}}$.

In the second equation the explicit phase factor $e^{-2i\pi\gamma_{\mathcal{O},a}}$ is inserted to take into account a possible semi-locality of the operator which interpolates the particle a (i.e.

[8] The standard parameterisation of two-dimensional on mass-shell momenta in terms of rapidities is adopted: $p_a^0 = m_a \cosh\theta_1$, $p_a^1 = m_a \sinh\theta_1$, $\theta \equiv \theta_1 - \theta_2$.

any operator φ_a such that $\langle 0|\varphi_a|a\rangle \neq 0$) with respect to the operator $\mathcal{O}(x)$ [9]. We must bear in mind that two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are said to be mutually non-local if the euclidean correlator $\langle \ldots \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots\rangle$ is not a single valued function of x_1 and x_2. In the particular case in which the correlator simply acquires a phase factor under the analytic continuation bringing x_2 to its original position along a path encircling x_1, then the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are said to be semi-local with respect to each other. Equation (??) can be regarded as a momentum space version of this statement for the operators $\mathcal{O}(x)$ and φ_a(x). When $\gamma_{\mathcal{O},a} = 0$, there is no crossing symmetric counterpart to the unitarity cut but when $\gamma_{\mathcal{O},a} \neq 0$, there is a non-locality discontinuity in the s-plane with $s = 0$ as branch point. In the rapidity parameterisation there is however no cut because the different Riemann sheets of the s-plane are mapped onto different sections of the θ-plane; the branch point $s = 0$ is mapped onto the points $\theta = \pm i\pi$ which become the locations of simple *annihilation* poles, whose residues are given by [?] (see also [?])

$$-i\,\mathrm{Res}_{\theta=\pm i\pi} F^{\mathcal{O}}_{a\bar{a}}(\theta) = (1 - e^{\mp 2i\pi\gamma_{\mathcal{O},a}})\langle 0|\mathcal{O}|0\rangle\ . \tag{4.6}$$

With the above information, let us now proceed to the perturbative analysis of the action (??). As before, in the following we will refer to the case $\eta \ll 1$, so that the role of the perturbing operator Ψ in (??) is played by $\cos(\alpha\varphi + \delta)$. It is understood that a similar analysis can be repeated in the opposite limit $\eta \longrightarrow \infty$, with $\mu\cos\beta\varphi$ assumed as the perturbation.

4.2 Fate of the solitons

The particle spectrum of the unperturbed theory at $\eta = 0$ consists of a soliton-antisoliton pair and – for $\beta^2 < 4\pi$ – a certain number of breathers.

Breathers are mutually local with respect to the elementary Sine-Gordon field φ, and consequently to the perturbing operator $\cos(\alpha\varphi + \delta)$. Thus, formula (??) can be safely used to explicitely compute the first order correction to the unperturbed masses (??)[10].

The situation is quite different for the solitonic sector of the Hilbert space. The soliton is an elementary excitation of the unperturbed theory which interpolates between two constant configurations of the field φ differing by $2\pi/\beta$. This means that $\varphi \longrightarrow \varphi + 2\pi/\beta$ across a soliton configuration. As a consequence, the exponential operator $e^{i\alpha\varphi}$ is semi-local with respect to the soliton with semi-locality index $\gamma_{\alpha,s} = \alpha/\beta$. According to equation (??), the soliton-antisoliton Form Factor of the perturbing operator $\Psi = \cos(\alpha\varphi + \delta)$ contains annihilation poles in $\theta = \pm i\pi$ whose residues are immediately

[9] Consistency of eq. (??) requires $\gamma_{\mathcal{O},\bar{a}} = -\gamma_{\mathcal{O},a}$.

[10] The Form Factors which are actually needed can be found in [?, ?].

computed as[11]

$$-i\,\mathrm{R}es_{\theta=\pm i\pi}F^{\Psi}_{s\bar{s}}(\theta) = [\cos\delta - \cos(\delta \mp 2\pi\alpha/\beta)]\ \langle 0|e^{i\alpha\varphi}|0\rangle\ . \tag{4.7}$$

The presence of these poles has a drastic consequence on the spectrum of the solitonic sector of the perturbed theory. Indeed, for generic values of α and δ, the non–zero residue (??) implies an infinite correction to the soliton and antisoliton masses. This divergence may be viewed as the technical manifestation of the fact that soliton and antisoliton no longer survive as asymptotic particles of the perturbed theory. Of course, this result could be anticipated by observing that for generic values of α, the lagrangian (??) loses its original $2\pi/\beta$-periodicity, so that solitons and antisolitons become unstable excitations. As discussed in more detail in Appendix B, the lifting of the degeneracy of the original minima induces a linear attractive potential between the soliton-antisoliton pairs which causes their collapse into a string of bound states.

Let us consider more closely the situation in which $\omega = m/n$, with m and n coprime integers, so that the lagrangian (??) retains a $2\pi n/\beta$-periodicity. As just argued, as soon as the perturbation $\lambda\cos(\alpha\varphi+\delta)$ is switched on, the solitons of the original theory become confined into states with zero topological charge[12] but in this case "packets" formed by n of the original solitons (or antisolitons) survive as stable excitations. These are nothing else but the topological excitations which interpolate between the degenerate minima of the perturbed potential[13] (see Figure 1.c and 2.b). Clearly, a complete space of topological excitations for the perturbed theory can be constructed in terms of the "n-soliton" and "n-antisoliton" states.

The above conclusion holds for generic values of δ. However, for specific values of this parameter, the single-soliton states of the unperturbed theory may remain stable and unconfined even in the presence of the perturbation. In the FFPT framework, this corresponds to the case in which the correction to the soliton mass is finite, namely when the residue on the annihilation pole (??) vanishes. Taking into account the condition (??) satisfied by δ in the rational case, the residue (??) vanishes when

$$|\delta| = \pi/n \tag{4.8}$$

provided m and n satisfy the condition

$$|kn - m| = 1 \ \text{ for some } k \in \mathbf{Z}\ . \tag{4.9}$$

[11]This result refers to the choice $\langle 0|\varphi|0\rangle = 0$ for the unperturbed theory. In this case the unperturbed theory is invariant under the reflection $\varphi \longrightarrow -\varphi$ and the equality $\langle 0|e^{i\alpha\varphi}|0\rangle = \langle 0|e^{-i\alpha\varphi}|0\rangle$ follows.

[12]This is a statement which holds or $n \neq 1$ and generic values of δ.

[13]Such packets, considered as a whole, are local with respect to the exponential operator $e^{i\alpha\varphi}$ since their semi-locality phase is $e^{\pm 2i\pi n\alpha/\beta} = 1$.

These are precisely the values of ω and δ for which the minimum at $\varphi = 0$ of the unperturbed potential becomes degenerate with one of its adjacent minima when the perturbation is switched on (Figure 3.b), and therefore the single-soliton excitations interpolating between the two minima remain stable. Note, however that it is generally impossible to simultaneously cancel both the annihilation poles at $\theta = i\pi$ and $\theta = -i\pi$; it can be argued that this corresponds to the fact that only one of the adjacent minima (the one located at $\varphi = 2\pi/\beta + O(\eta)$ or the one located at $\varphi = -2\pi/\beta + O(\eta)$) can become degenerate with the absolute minimun around $\varphi = 0$ (order η). Suppose that the degeneracy is realised between the central minimum and its right neighbour. The presence of the degeneracy implies that, for the values of the parameters we are considering, the n-soliton discussed above actually breaks into the sequence of two asymptotically stable excitations: a kink K_1 of mass M interpolating between $\varphi = 0$ and $\varphi = 2\pi/\beta$, and a kink K_{n-1} of mass $(n-1)M$ interpolating between $\varphi = 2\pi/\beta$ and $\varphi = 2\pi n/\beta$ (here and below $O(\eta)$ corrections to the masses and positions of the minima are understood). Obviously, this pattern is repeated with periodicity $2\pi n/\beta$.

More generally, it is not difficult to see that under the action of the perturbation, for $\delta = \varepsilon\pi/n$ ($\varepsilon = \pm 1$), and for any pair m,n ($m \neq 1$), the absolute minimum close to $\varphi = 0$ remains degenerate with the minimun located near to $\varphi = 2\pi j/\beta$, with j given by

$$j = \frac{kn - \varepsilon}{m} \in \{1, 2, \ldots, n-1\} \ . \tag{4.10}$$

There is always a unique integer k for which the above equation is satisfied. In this case, the n-soliton breaks into a kink K_j with mass jM and a kink K_{n-j} with mass $(n-j)M$.

The denomination of kinks we gave to these excitations requires some clarification. Generally, kinks are elementary excitations which interpolate between different vacua of a theory with spontaneously broken symmetry. When the structure of the vacua is sufficiently non-trivial, suitable ordering prescriptions must be assigned to construct acceptable multi-kink configurations. In this circumstance, kinks drastically differ from ordinary particles, in terms of which the space of multi-particle states can be constructed without any restriction. The situation we are dealing with is not of this kind. In fact, due to the simple vacuum structure (alternation of two different kinds of minima), any string of K_j and K_{n-j} is an allowed multi-kink configuration. Differently stated, for any sequence of K_j and K_{n-j} there exists an unique interpolation pattern among the degenerate vacua of the theory[14].

[14]Since we are dealing with excitations above the real vacuum of the theory, the interpolation starts from $\varphi = 0$. Moreover, the sequences $K_j K_j$ and $K_{n-j} K_{n-j}$ can only correspond to going back and forth between a couple of consecutive minima. Then, as long as the alternating minima remain inequivalent, there is no need to define antikink states.

It should be clear that the above considerations regarding the evolution of the particle spectrum of the theory (??) have a validity which goes beyond the perturbative framework in which they have been derived: the stable or instable character of an excitation under the action of the perturbation is a qualitative feature which can be extended away from the perturbative domain. For example, had we performed our perturbative analysis in the limit $\eta \longrightarrow \infty$ rather than in the limit $\eta \longrightarrow 0$, we would have established the existence of m-solitons rather than n-solitons. These are different names for an unique excitation which exists and is stable for any value of η. Simply, it is intuitively helpful to see this object as the bound state of n solitons of the $\eta = 0$ theory, in one perturbative limit, or m solitons of the $\eta = \infty$ theory, in the other.

4.3 Phase Transition

Let us apply the above considerations to the case $\delta = \pm\pi/n$ in which the n-solitons "decay" into the sequence of kinks K_j and K_{n-j}. By increasing η, the masses of these particles evolve preserving though their stability, and for η very large they can be reinterpreted perturbatively as K_l and K_{m-l} kinks, with l determined by eq. (??) but with m and n interchanged.

An exception to the above smooth evolution pattern arises when $m = 1$: in this case, the n-soliton – which may be seen as "sum" of two kinks K_1 and K_{n-1} – evolves for $\eta \longrightarrow \infty$ into a "1-soliton". The latter, up to a mass correction, is simply the soliton of the $\eta = \infty$ theory and therefore is completely stable (in this limit the perturbation is local with respect to all the excitations of the unperturbed theory for $m = 1$). Thus, a composite topological excitation has surprisingly transmuted into an elementary one along the way!

Such a transmutation necessarily requires the existence of an intermediate critical value η_c at which a phase transition takes place[15]. The phase transition presents the following pattern. For $\eta < \eta_c$, the n-soliton consists of the sequence of the two kinks K_1 and K_{n-1}. When $\eta \to \eta_c$ from below, the intermediate minimum which is in common between the two kinks approaches the one initially located at $\varphi = 0$ (Figure 5.a). The two minima become coincident at $\eta = \eta_c$ (Figure 5.b) and, at this value, K_1 disappears as a topological excitation. In particular, its mass (which is proportional to the potential barrier between the minima) shrinks from M at $\eta = 0$ to 0 at $\eta = \eta_c$. For $\eta > \eta_c$ the original composite topological excitation consists of a single massive kink, which interpolates between the vacua of the $\eta = \infty$ theory (Figure 5.c). However, the dynamical degree of freedom carried by K_1 do not disappear across the transition line: it is in fact

[15]One could imagine several critical thresholds. The simplest possibility is assumed here.

transferred to a particle with zero topological charge which is precisely massless at $\eta = \eta_c$ and becomes a breather at $\eta = \infty$[16]. This tranformation of a topological excitation into a non-topological one should not be surprising: we already remarked that, from the point of view of the construction of the space of asymptotic quantum states, K_1 is actually indistinguishable from an ordinary particle.

The presence of a massless particle in the spectrum of the $\eta = \eta_c$ theory amounts to say that the correlation length is infinite along the RG trajectory labelled by η_c. The correlation length cannot decrease in the RG flow towards the infrared limit, therefore such a trajectory must end into a new fixed point. Since the theory (??) is unitary, Zamolodchikov's C-theorem [?] ensures that such a fixed point is described by a CFT with central charge smaller than that of the ultraviolet fixed point, namely $C = 1$. On the other hand, it is known from Ref. [?] that the only unitary CFTs with $C < 1$ are those belonging to the unitary minimal series, with central charges $C_p = 1 - 6/[p(p+1)]$, $p = 3, 4, \ldots$. For η slightly below η_c, the mass of K_1 is much smaller than the mass of the other particles of the theory. Thus, the low energy limit is effectively described only by the kink K_1 interpolating among the vacuum and its adjacent degenerate minimum. Stated in a Landau-Ginzburg language, the action (??) can be effectively truncated to a spontaneously broken φ^4 theory, which is well known to describe the universality class of the Ising model. In fact, the one we described above is exactly the Ising phase transition with the parameter η playing the role of the temperature. Then we conclude that the RG trajectory labelled by η_c interpolates between the gaussian fixed point with $C = 1$ and the Ising fixed point with $C = C_3 = 1/2$ (Figure 6). A model which provides an explicit realization of this flow is discussed in the next Section.

5 Two examples

This section is intended to illustrate the wide applicability of the ideas discussed in this paper through a couple of examples referring to quite different physical situations. The first example is taken from classical statistical mechanics and deals with a deformation of the Ashkin-Teller model. The second one comes from quantum field theory and concerns the massive Schwinger model. Our interest is in exhibiting those features of the two models which are relevant for the discussion developed in this paper. The reader is referred to the original literature for comprehensive studies on both subjects (see Refs. [?, ?] for Ashkin-Teller and [?, ?] for the massive Schwinger model).

[16]Note that the existence of at least one breather in this limit is ensured, for $m = 1$, by the condition $\alpha\beta < 4\pi$ we derived in the previous section.

5.1 The Generalized Ashkin-Teller Model

The Ashkin-Teller model describes two planar Ising models interacting through a local four spin interaction. It is defined by the lattice Hamiltonian

$$H_{AT} = \sum_{(i,j)} [J(\sigma_1^i \sigma_1^j + \sigma_2^i \sigma_2^j) + K \sigma_1^i \sigma_1^j \sigma_2^i \sigma_2^j] \ , \tag{5.1}$$

where $\sigma_{1,2}^i = \pm 1$, the sum is over nearest neighbours and the same coupling J has been chosen for the two Ising models (isotropic case).

A scaling limit description of the model can be formulated in terms of the euclidean action of a two–layer Ising system (Figure 7)

$$\mathcal{A}_{AT} = \mathcal{A}_1^{Ising} + \mathcal{A}_2^{Ising} + \tau \int d^2x \, (\varepsilon_1(x) + \varepsilon_2(x)) + \rho \int d^2x \, \varepsilon_1(x) \varepsilon_2(x) \ , \tag{5.2}$$

where $\mathcal{A}_i^{Ising}$ and $\varepsilon_i(x)$ denote the fixed point action and the energy operator of the i-th Ising model, respectively. In this language, the scaling limit of the Ashkin-Teller model appears as a CFT with central charge $C = 1/2 + 1/2 = 1$ perturbed by the two operators $\mathcal{E} \equiv \varepsilon_1 + \varepsilon_2$ and $\varepsilon \equiv \varepsilon_1 \varepsilon_2$.

Let us keep at first the two Ising models at their critical temperature ($\tau = 0$). Since the operator ε is marginal ($\Delta_\varepsilon = 1/2 + 1/2$), its addition to the action of the two critical Ising models does not spoil criticality, but leads to a line of $C = 1$ fixed points parameterised by the values of the coupling ρ. It is known [?, ?] that, while the spin fields σ_1 and σ_2 retain along the critical line the conformal dimensions which they have at the decoupling point $\rho = 0$ (i.e. $\Delta_\sigma = 1/16$), the dimensionalities of the total energy operator $\mathcal{E} = \varepsilon_1 + \varepsilon_2$ and of the so called "polarisation" operator $P = \sigma_1 \sigma_2$ become instead ρ-dependent. Their ratio, however, remains the same as at the decoupling point

$$\frac{\Delta_P(\rho)}{\Delta_{\mathcal{E}}(\rho)} = \frac{\Delta_P(0)}{\Delta_{\mathcal{E}}(0)} = \frac{1/16 + 1/16}{1/2} = \frac{1}{4} \ . \tag{5.3}$$

Thus, the action (??) with $\tau \neq 0$ can be seen as a $C = 1$ CFT perturbed by the operator $\mathcal{E}$ with ρ-dependent conformal dimension, namely as a Sine-Gordon model (??) with the frequency β determined by the relation

$$\Delta_{\mathcal{E}}(\rho) = \beta^2 / 8\pi \ . \tag{5.4}$$

Note that, since $\Delta_{\mathcal{E}}(0) = 1/2$, the decoupling point corresponds to $\beta = \sqrt{4\pi}$. For this value of the coupling, the SG model is equivalent to a free Dirac fermion (see (??)), as expected from the well known equivalence between the thermal Ising model and a free Majorana fermion.

Consider now a generalisation of the Ashkin-Teller model defined by the action

$$\mathcal{A}_h = \mathcal{A}_{AT} + h \int d^2x \, \sigma_1(x)\sigma_2(x) \ , \tag{5.5}$$

with an additional spin-spin interaction between the two planar Ising models.

For $\tau = 0$, it can be shown that the action (??) defines an integrable massive field theory whose spectrum and scattering amplitudes have been determined in [?].

For $\tau \neq 0$, there is however the possibility to make a fine–tuning of the two parameters τ and h in such a way that a massless RG flow is obtained. The infrared fixed point can be easily identified by means of the following physical consideration. Let us consider first the strong–coupling limit $h \longrightarrow \infty$: in this regime, the last term in (??) forces the spins σ_1 and σ_2 to assume the same value on each site. Hence, the system results effectively reduced to a single Ising model. The reduced system is generally massive but, as any other Ising model, it may become critical provided the temperature of the two original layers is appropriately tuned to its critical value. It can be easily argued that the strong–coupling limit scenario just described should also occur for finite values of h: the coupling h tends to order the system (hence introducing a finite correlation length) but this tendency can be contrasted by a value of the temperature sufficiently high. In conclusion, for each value of ρ, there should be a critical line in the τh-plane along which the system flows from the $C = 1$ fixed point to the $C = 1/2$ Ising fixed point. For the symmetries of the problem, the approach to the Ising fixed point must occur along one of the irrelevant directions of the conformal family of the identity operator: in fact, the massless flow is along the line which separates the high– and the low–temperature phases, hence the conformal families of the magnetic and energy operators of the Ising model are both ruled out since the former is odd under the Z_2 spin symmetry while the latter is odd under the high–low temperature duality of the model. So, only the conformal family of the identity operator is left.

Let us finally rephrase the above discussion in the bosonic language. Since $\mathcal{E} \sim \cos\beta\varphi$, eq. (??) suggests the identification $P \sim \cos(\beta\varphi/2 + \delta_P)$, with δ_P to be fixed. The model (??) then corresponds to the two–frequency SG model (??), with the identification $\mu = \tau$, $\lambda = h$, $\alpha = \beta/2$, $\delta = \delta_P$, with β determined in terms of ρ through (??). We found in the previous section that, for $\alpha/\beta = 1/n$, a flow to the Ising fixed point takes place in the $\lambda\mu$-plane if $|\delta| = \pi/n$. This implies $|\delta_P| = \pi/2$, so that $P \sim \sin(\beta\varphi/2)$.

5.2 Massive Schwinger model

The massive Schwinger model describes quantum electrodynamics in two dimensions. It is defined by the Lagrangian

$$\mathcal{L}_{MSM} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m - e\gamma^\mu A_\mu)\psi \ . \tag{5.6}$$

If the "quark" mass m is set to zero, the theory reduces to the ordinary Schwinger model which is known to be equivalent to a free scalar field of mass $e/\sqrt{\pi}$ [?, ?]. On the other hand, if we set $e = 0$ in (??) we are left with a theory of free massive fermions which, as already remarked in section 2, is equivalent to a pure SG model with $\beta = \sqrt{4\pi}$. Putting this information all together, one obtains the following bosonised form of the theory (??) [?]

$$\mathcal{L}_{Bose} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{e^2}{2\pi}(\varphi+\gamma)^2 + cm^2\cos\sqrt{4\pi}\varphi \ , \tag{5.7}$$

where c is a positive constant whose precise value is immaterial for our considerations. The parameter γ accounts for the fact that, contrary to what happens in the two limiting cases $m = 0$ and $e = 0$, a shift of the field φ does not leave invariant the complete theory (??): it is in fact proportional to the Θ angle of the gauge lagrangian (??) which physically corresponds to the presence of a constant background electric field which breaks parity [?].

The bosonic lagrangian (??) can be seen as the specialisation of the theory (??) to the case $\beta = \sqrt{4\pi}$, $\lambda = (e/\sqrt{\pi}\alpha)^2$, $\delta = \alpha\gamma$, in the limit $\alpha \longrightarrow 0$. The analysis performed in the previous sections for the theory (??) is easily adapted to the present case. The requirement that the lagrangian (??) describes theories which do not differ simply for a shift of the vacuum leads to the restriction $|\gamma| \le \sqrt{\pi}/2$. Chosing the Θ angle to take values in the interval $(-\pi,\pi)$, then one has $\gamma = \Theta/\sqrt{4\pi}$. The potential term of the lagrangian (??) for a generic value of Θ and for the special value $\Theta = \pi$ is drawn in Figure 8.

Let us apply the perturbative arguments of Section 4 to the limit $e/m \longrightarrow 0$ of (??). The unperturbed theory is the SG at the free fermion point. The spectrum consists of soliton and antisoliton only, which in the fermionic description must be identified with the non-interacting quark and antiquark. The perturbing operator $\Psi \equiv (\varphi+\gamma)^2$ is non-local with respect to s and $\bar{s}$. Such a non-locality is of a less simple form than that of the exponential operators $e^{i\alpha\varphi}$, but is however easily deduced from the latter. In fact, taking the k-th derivative with respect to α of eqs. (??) and (??) and then setting $\alpha = 0$, one finds

$$\langle 0|\varphi^k(0)|s(\theta_1+2i\pi),\bar{s}(\theta_2)\rangle = \langle 0|(\varphi(0)-2\pi/\beta)^k|\bar{s}(\theta_2),s(\theta_1)\rangle \ , \tag{5.8}$$

$$-i\,\mathrm{Res}_{\theta_1-\theta_2=\pm i\pi}\langle 0|\varphi^k(0)|s(\theta_1),\bar{s}(\theta_2)\rangle = \langle 0|\varphi^k(0) - (\varphi(0)\mp 2\pi/\beta)^k|0\rangle \ . \tag{5.9}$$

With $\beta = \sqrt{4\pi}$ and $\langle 0|\varphi|0\rangle = 0$, the last equation gives

$$i\,\mathrm{Res}_{\theta=\pm i\pi} F^{\Psi}_{s\bar{s}}(\theta) = \pi \mp \Theta \ . \tag{5.10}$$

The interpretation of this result follows from the general discussion of Section 4. For $|\Theta| < \pi$ the soliton and antisoliton masses receive an infinite correction. In the fermionic description this amounts to say that quarks and antiquarks are confined by an arbitrary small electromagnetic interaction. Quark-antiquark pairs are bounded by a linear potential and give rise to a string of electrically neutral bound states. The confining potential and an estimate of the number of bound states as a function of energy are given by the formulae (??) and (??) with $A = e^2/2$ and $B = \Theta/\pi$.

For $|\Theta| = \pi$, however, the annihilation pole in the soliton-antisoliton form factor of the perturbing operator is (partially) cancelled and the mass correction is finite. Quark and antiquark survive unconfined in the perturbed theory as kinks interpolating between the vacuum located at $\varphi = \mathcal{O}(e^2)$ and a degenerate adjacent minimum (the latter lies at $\varphi = \pm\sqrt{\pi} + \mathcal{O}(e^2)$ depending on the sign of Θ) (Figure 8.b). Since no topological excitation is present in the opposite limit $e/m \longrightarrow \infty$, there must exist a critical value of the ratio e/m for which a confining phase transition takes place. The arguments of the previous section apply basically unchanged to the present situation and one concludes that along the critical trajectory in the em-plane the system flows from the $C = 1$ to the $C = 1/2$ fixed point.

6 Conclusion

The multi–frequency Sine–Gordon model is an interesting example of non–integrable QFT. In this paper we have exploited the Form Factor Perturbation Theory to study the evolution of the particle spectrum of this model as a function of the parameter η which labels its interpolation between two integrable SG models. The soliton sector is the one which results the most affected by the non–integrable dynamics: as discussed in Section 4 and also illustrated by the examples in Section 5, its evolution strongly depends on the values assumed by the frequency ratio ω and the phase–shift δ. In the FFPT, disappearing of soliton states and confinement of their multi–particle excitations originates from the non–locality of the perturbation operator with respect to the field which creates the one–soliton state. In the special cases $\omega = 1/n$ and $\delta = \pm 1/n$, a change of the vacuum state takes place by varying η, with the consequent presence of phase transition lines: these are generally associated to massless field theories which interpolate between the conformal field theory with $C = 1$ and the one associated to the Ising model, with $C = 1/2$.

It would be highly interesting to match the rich scenario of the multi–frequency SG model as coming from the analysis of its particle spectrum with a more detailed investigation of its dynamics, in particular the calculation of its scattering amplitudes. As shown in Ref. [?] for the double SG model, a remarkable pattern of resonances entering the collision events of the kinks already emerges at the classical level. A challenging open problem is therefore to see how the results of Ref. [?] can be translated at the quantum level and conversion amplitudes between different types of kinks, inelastic events and energy levels of the resonance states of the non–integrable model can be computed.

Acknowledgements. We wish to thank J.L. Cardy, M. Fabrizio, A. Neveu, P. Simonetti and Al.B. Zamolodchikov for helpful discussions and M. Halpern for a useful remark. The authors would also like to thank the the Institute of Theoretical Physics in Santa Barbara and the organizers of the program *Quantum Field Theory in Low Dimensions* for the warm hospitality during their staying at the institute, where this work was started.

Appendix A

At the special values of the coupling $\xi = \pi/n$, $n = 1, 2, \ldots$, soliton-antisoliton scattering in the Sine-Gordon model becomes reflectionless, and is described by the transmission amplitude

$$S(\theta) = -\prod_{k=1}^{n-1} \frac{\sinh \frac{1}{2}[\theta + i\pi(1-k/n)]}{\sinh \frac{1}{2}[\theta - i\pi(1-k/n)]} \ , \tag{A.1}$$

in which the simple poles located at $\theta = i\pi(1-k/n)$, $k = 1, 2, \ldots, n-1$, correspond to the breathers B_k appearing as bound states in $s\bar{s}$ scattering.

As discussed in Section 4, the soliton-antisoliton form factors of the exponential operators

$$F^{\alpha}_{s\bar{s}}(\theta_1 - \theta_2) \equiv \langle 0|e^{i\alpha\varphi(0)}|s(\theta_1)\bar{s}(\theta_2)\rangle \tag{A.2}$$

satisfy the unitarity and crossing equations

$$F^{\alpha}_{s\bar{s}}(\theta) = S(\theta) F^{\alpha}_{\bar{s}s}(-\theta) \ , \tag{A.3}$$

$$F^{\alpha}_{s\bar{s}}(\theta + 2i\pi) = e^{-2i\pi\alpha/\beta} F^{\alpha}_{\bar{s}s}(-\theta) \ . \tag{A.4}$$

Since we have chosen the vacuum of the theory to be located in $\varphi = 0$, the reflection $\varphi \longrightarrow -\varphi$ maps s in $\bar{s}$. The invariance of the theory under this reflection then requires

$$F^{\alpha}_{s\bar{s}}(\theta) = F^{-\alpha}_{\bar{s}s}(\theta) \ . \tag{A.5}$$

We parametrise the matrix element (??) as

$$F^{\alpha}_{s\bar{s}}(\theta) = e^{-\alpha\theta/\beta} \frac{P_\alpha(\theta)}{\sinh n\theta} F_0(\theta) \ , \tag{A.6}$$

where

$$F_0(\theta) = -i \sinh \frac{\theta}{2} \exp\left[\int_0^\infty \frac{dx}{x} \frac{\sinh \frac{x}{2}(1 - \xi/\pi)}{\sinh \frac{x\xi}{2\pi} \cosh \frac{x}{2}} \frac{\sin^2 \frac{x(i\pi - \theta)}{2\pi}}{\sinh x}\right] \tag{A.7}$$

satisfies

$$F_0(\theta) = (-1)^{n+1} S(\theta) F_0(\theta) \ , \tag{A.8}$$

$$F_0(\theta + 2i\pi) = F_0(\theta) \ . \tag{A.9}$$

The denominator $\sinh n\theta$ in (??) contains the kinematical pole in $\theta = i\pi$ associated to the residue equation

$$-i \, \mathrm{Res}_{\theta=i\pi} F^{\alpha}_{s\bar{s}}(\theta) = (1 - e^{-2i\pi\alpha/\beta}) \langle e^{i\alpha\varphi} \rangle \ , \tag{A.10}$$

as well as the bound state poles associated to the residue equations

$$-i \, \mathrm{Res}_{\theta=i\pi(1-k/n)} F^{\alpha}_{s\bar{s}}(\theta) = \Gamma_k \, F^{\alpha}_k \tag{A.11}$$

(here $\Gamma_k = [-i\,\mathrm{Res}_{\theta=i\pi(1-k/n)}S(\theta)]^{1/2}$ is the three-particle coupling between soliton, anti-soliton and k-th breather and $F_k^\alpha \equiv \langle 0|e^{i\alpha\varphi(0)}|B_k\rangle$). The pole in $\theta=0$ is cancelled by the zero contained in $F_0(\theta)$.

According to eqs. (??), (??) and (??), $P_\alpha(\theta)$ entering (??) must satisfy

$$P_\alpha(\theta) = (-1)^n P_{-\alpha}(-\theta)\ , \tag{A.12}$$

$$P_\alpha(\theta+2i\pi) = (-1)^{n+1} P_\alpha(\theta)\ , \tag{A.13}$$

and then can be written as

$$P_\alpha(\theta) = \begin{cases} \sum_{j=0}^{N}\left[a_j(\alpha)e^{j\theta} - a_j(-\alpha)e^{-j\theta}\right]\ , & n \text{ odd} \\ \sum_{j=0}^{M}\left[b_j(\alpha)e^{(j+1/2)\theta} + b_j(-\alpha)e^{-(j+1/2)\theta}\right]\ , & n \text{ even.} \end{cases} \tag{A.14}$$

The integers N and M can be fixed from the asymptotic behaviour of the known solutions for $\alpha=\pm\beta$ and $\alpha=\pm\beta/2$ [?]. One finds $N=(n-1)/2$ and $M=(n-2)/2$. It is convenient to define $\tilde{a}_0(\alpha)=a_0(\alpha)-a_0(-\alpha)$, $c_j(\alpha)=a_j(\alpha)/\tilde{a}_0(\alpha)$, $d_j(\alpha)=b_j(\alpha)/b_0(-\alpha)$ and to rewrite $P_\alpha(\theta)$ in the form

$$P_\alpha(\theta) = \begin{cases} \tilde{a}_0(\alpha)\left[1+\sum_{j=1}^{(n-1)/2}\left(c_j(\alpha)e^{j\theta}+c_j(-\alpha)e^{-j\theta}\right)\right]\ , & n \text{ odd} \\ b_0(\alpha)\sum_{j=0}^{(n-2)/2}\left[\frac{d_j(\alpha)}{d_0(\alpha)}e^{(j+1/2)\theta}+d_j(-\alpha)e^{-(j+1/2)\theta}\right]\ , & n \text{ even} \end{cases} \tag{A.15}$$

in which an inessential overall normalisation has been made explicit and (both for n odd and n even) $n-1$ α-dependent coefficients appear as the only remaining unknowns in the problem. They are fixed exploiting the following observation. The lightest breather B_1, being the particle interpolated by the elementary field φ, changes sign under the reflection $\varphi \longrightarrow -\varphi$. The k-th breather can be seen as a bound state of k breathers B_1, and then behaves as $B_k \longrightarrow (-1)^k B_k$ when $\varphi \longrightarrow -\varphi$. As a consequence we have

$$F_k^\alpha = (-1)^k F_k^{-\alpha}\ , \qquad k=1,2,\ldots,n-1. \tag{A.16}$$

Hence, remembering eq. (??) and defining $Q_\alpha(\theta)=e^{-\alpha\theta/\beta}P_\alpha(\theta)$, we obtain

$$Q_\alpha\left(i\frac{\pi k}{n}\right) = (-1)^k\, Q_\alpha\left(-i\frac{\pi k}{n}\right)\ , \qquad k=1,2,\ldots,n-1. \tag{A.17}$$

This is a set of $n-1$ linear equations which uniquely determine the coefficients in (??). The solution is

$$c_j(\alpha) = \prod_{m=1}^{j}\tan\frac{\pi}{2n}\left(m-\frac{n+1}{2}\right)\cot\frac{\pi}{n}\left[\frac{\alpha}{\beta}+\frac{1}{2}\left(\frac{n+1}{2}-m\right)\right]\ , \qquad j=1,\ldots,\frac{n-1}{2} \tag{A.18}$$

for n odd, and

$$d_j(\alpha) = \prod_{m=0}^{j}\tan\frac{\pi}{2n}\left(m-\frac{n}{2}\right)\cot\frac{\pi}{n}\left[\frac{\alpha}{\beta}+\frac{1}{2}\left(\frac{n}{2}-m\right)\right]\ , \qquad j=0,1,\ldots,\frac{n-2}{2} \tag{A.19}$$

for n even.

Appendix B

We discussed in Section 4 how the topological excitations of SG model can undergo a confinement phenomenon under the action of a perturbation. In this Appendix we give a more quantitative description of the confinement pattern within the semiclassical approximation.

Consider at first a soliton-antisoliton ($s\bar{s}$) state in the pure SG model (??). Classically, it is associated to a configuration of the field φ taking the vacuum value $\varphi = 0$ from minus spatial infinity to a point $x_1(t)$ (the position of the soliton at time t) where it switches to the value $\varphi = 2\pi/\beta$ of the right adjacent minimun of the potential; it keeps this value until the point $x_2(t)$ (location of the antisoliton) where it switches again to $\varphi = 0$. Similarly, the antisoliton-soliton ($\bar{s}s$) configuration corresponds to an interpolation from $\varphi = 0$ to the left adjacent minimum $\varphi = -2\pi/\beta$, and back (see Figure 9).

Let us add to the SG potential the perturbing term $V(\varphi) = -\lambda \cos(\alpha\varphi + \delta)$ which lifts the degeneracy among the vacuum at $\varphi = 0$ and its two adjacent minima[17]. The two configurations $s\bar{s}$ and $\bar{s}s$ described above are no longer stable since a surplus of energy equal, respectively, to $U_{s\bar{s}} = (x_2 - x_1)[V(2\pi/\beta) - V(0)]$ and $U_{\bar{s}s} = (x_1 - x_2)[V(-2\pi/\beta) - V(0)]$ is associated to the intermediate plateau. The tendency of the system is to shrink in order to minimise the energy of the configuration: hence, the new term gives rise to the following attractive potential between the soliton and the antisoliton

$$U(x) = \theta(x) U_{\bar{s}s} + \theta(-x) U_{s\bar{s}} = A(|x| - Bx) \ , \tag{B.1}$$

where $x \equiv x_1 - x_2$, $\theta(x)$ is the step function and

$$A = 2\lambda \sin^2 \frac{\pi\alpha}{\beta} \cos\delta \ , \qquad B = \cot \frac{\pi\alpha}{\beta} \tan\delta \ . \tag{B.2}$$

Such a linearly rising potential confines the soliton-antisoliton pair and gives rise to a discrete spectrum of bound states. A continuum spectrum corresponding to asymptotically free motion is recovered only in the limiting case $|B| = 1$.

The number of bound states $N(E)$ as a function of energy can be obtained by the semi–classical formula

$$N(E) = \frac{1}{2\pi} \oint p \, dx \ , \tag{B.3}$$

where the integral is computed along the classical orbit. In a non-relativistic approxima-

[17]For the reasons discussed in Section 3 the perturbation must be such that the central mininum remains the lowest among the three we are considering here. This amounts to require A positive and $|B| \leq 1$ in (??) below. Since the present considerations involve only a localised region of the potential, they lead to less severe restrictions on δ than those obtained in Section 3.

tion, the above integral computed with $p = [M(E - 2M - U)]^{1/2}$ gives

$$N(E) = \frac{4\sqrt{M}(E - 2M)^{3/2}}{3\pi A(1 - B^2)} \ , \tag{B.4}$$

where M is the soliton mass and the rest energy $2M$ of the soliton-antisoliton pair has been included. The number of stable bound states is $N(E_T)$, E_T being the value of the lowest threshold in the energy plane (Figure 10). E_T equals $4M$ for $\beta^2 \geq 4\pi$ and twice the mass m_1 of the lightest breather for $\beta^2 < 4\pi$. Comparing with the exact solution expressed in terms of Airy function of the Schrödinger equation with the potential (??), the semi–classical formula (??) provides excellent estimate of the energy levels (an error of 0.7% for the first energy level, which is further reduced at 0.1% for the second one). Therefore, for all practical purposes, we can consider formula (??) as an exact one.

The computation of the integral (??) with $p = [(E - U)^2 - M^2/4]^{1/2}$ required in the relativistic case cannot be performed exactly[18]. However, it can be checked numerically that the non-relativistic computation provides a reasonable estimate for $N(E_T)$.

The stable bound states originating from the threshold of soliton–antisoliton configurations may influence the large asymptotic behaviour of correlation functions. This is certanly the case for those fields which were coupled in the unperturbed system to the $s\bar{s}$ state. Let $\mathcal{O}$(x) be one of these fields. Neglecting states with higher number of particles and masses, the two-point function of the operator $\mathcal{O}(x)$ in the unperturbed system was given by

$$G(x) \ = \ \langle \mathcal{O}(x)\, \mathcal{O}(0) \rangle \sim \int_{(2M)^2}^{+\infty} \frac{d\mu^2}{2\pi}\, \rho(\mu^2) K_0(\mu \mid x \mid) \tag{B.5}$$

with $K_0(r)$ the Bessel function and

$$\rho(p^2) = \int \frac{dp_s}{(2\pi) 2E_s} \frac{dp_{\bar{s}}}{(2\pi) 2E_{\bar{s}}} \delta^2(p - p_s - p_{\bar{s}}) \mid \langle 0 \mid \mathcal{O}(0) \mid s\bar{s} \rangle \mid^2 \tag{B.6}$$

Hence, the leading behaviour for large $\mid x \mid$ in the unperturbed case is given by

$$G(x) \sim D \sqrt{\frac{M}{\pi}} \frac{e^{-2M|x|}}{\mid x \mid^{\frac{3}{2}+2\sigma}} \tag{B.7}$$

where D and σ are defined by the behaviour of the spectral function $\rho(\mu^2)$ near the threshold $2M$, $\rho(\mu^2) \sim D\,(\mu - 2M)^{2\sigma}$. When the branch cut associated to the threshold of the state $s\bar{s}$ breaks down into the infinite sequence of bound states – N_b of which stable and whose energies are obtained by inverting eq. (??) – the correlation function becomes

$$G(x) \ \sim \ \frac{4}{3} \sum_{n=1}^{N_b} \frac{1}{n^{1/3}}\, E_n\, \rho[(E_n)^2] K_0[E_n \mid x \mid] \ . \tag{B.8}$$

[18]It is easy to see though that in this case, $N(E) \sim E^2$ for large energy.

Hence, the leading asymptotic behaviour for $\mid x \mid \rightarrow \infty$ is determined by the lowest of them and we have

$$G(x) \sim \frac{4}{3}\sqrt{\frac{\pi}{4M}} E_1\, \rho[E_1^2] \frac{e^{-E_1|x|}}{x^{1/2}} \quad . \tag{B.9}$$

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Figure Caption

Figure 1 . Topological excitations: (a) soliton state, (b) anti–soliton state, (c) multi-soliton state.

Figure 2 . (a) Perturbation of the SG potential (full curve) by another periodic interaction (dotted curve); (b) when $|\delta| > \frac{\pi}{n}$, the vacuum moves away from the origin.

Figure 3 . Examples of degeneracy of the vacuum at the origin with one of the other $(n-1)$ minima for $\delta = \frac{\pi}{n}$: (a) $\omega = \frac{m}{n}$ $(m \neq 1)$; (b) $\omega = \frac{1}{n}$.

Figure 4 . Straight line (??) on the torus $\mathcal{T}$ in the rational (a) and irrational case (b). The black point identifies the location of the minimum of the potential on the torus.

Figure 5 . Form of the potential near the phase transition point: (a) $\eta < \eta_c$; (b) $\eta = \eta_c$; (c) $\eta > \eta_c$.

Figure 6 . RG flows from $C = 1$ to $C = \frac{1}{2}$.

Figure 7 . The two–layer Ising model.

Figure 8 . Potential term in the massive Schwinger model: (a) $\Theta \neq \pi$; (b) $\Theta = \pi$.

Figure 9 . (a) Two–particle kink–antikink and (b) antikink–kink states.

Figure 10 . Bound states of the linear potential confining the soliton-antisoliton pairs. The levels above E_T correspond to unstable particles.

Figure 1

(a) (b)

Figure 2

(a) (b)

Figure 3

(a)

(b)

Figure 4

(a) (b)

(c)

Figure 5

Figure 6

Figure 7

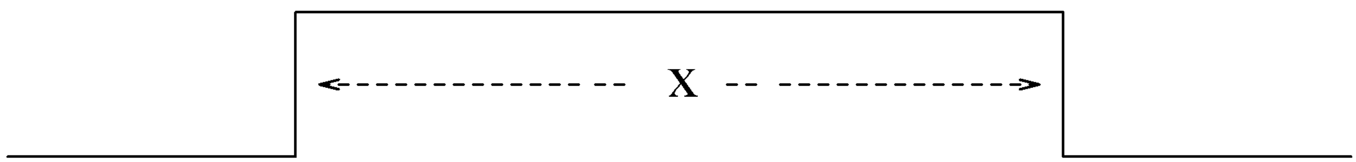
(a)

(b)

Figure 8

Figure 9

Figure 10



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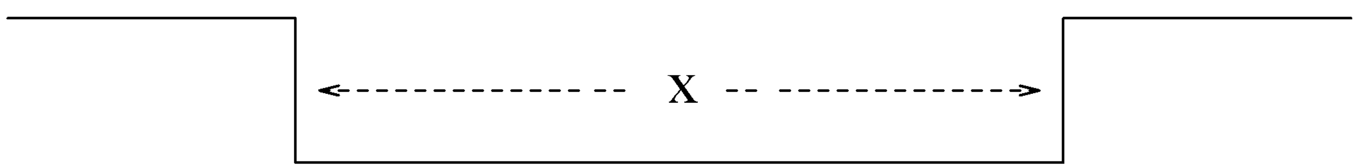

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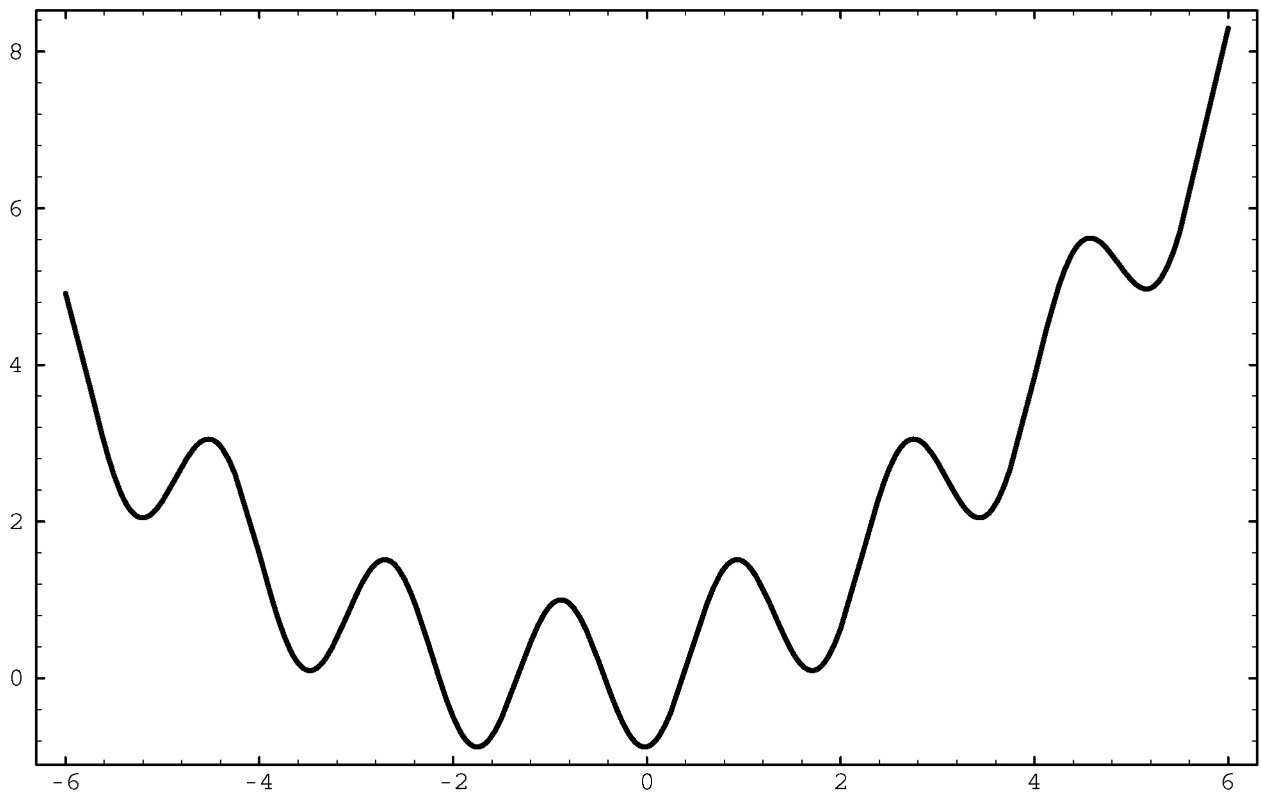
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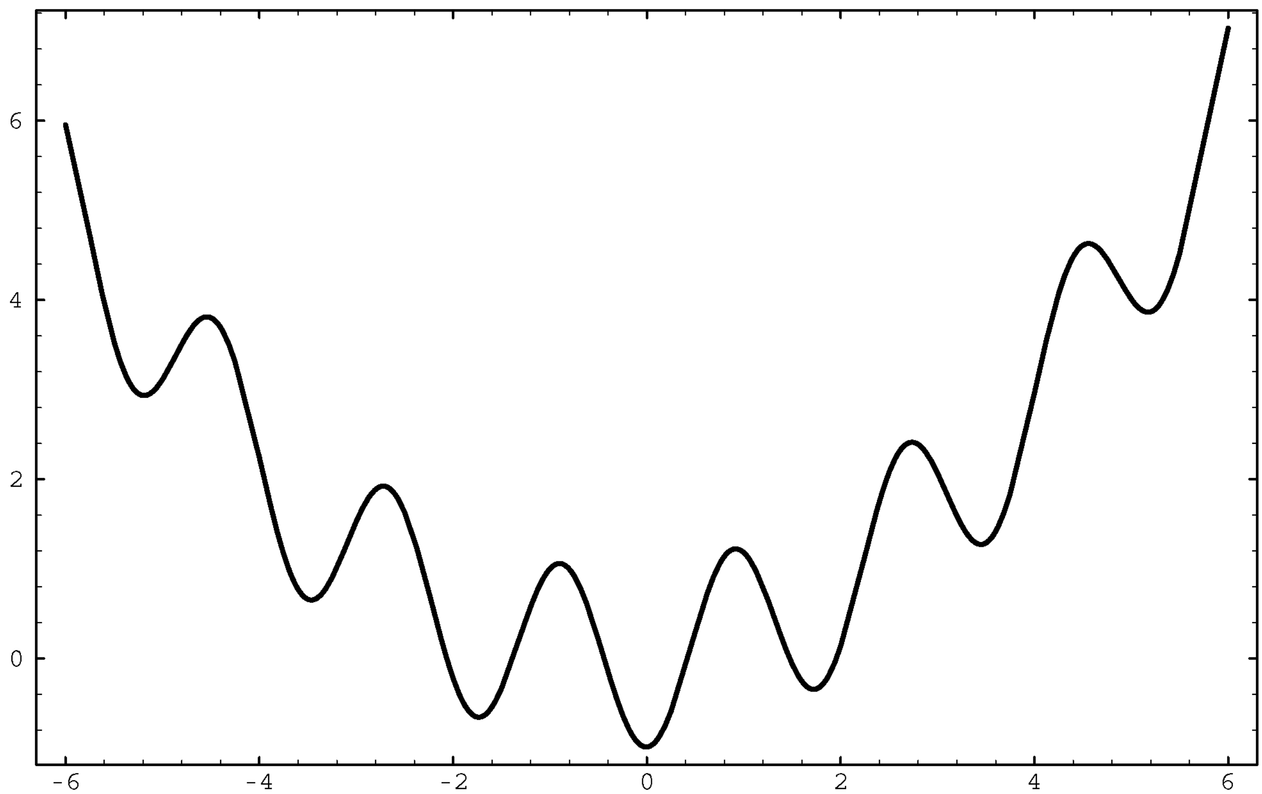
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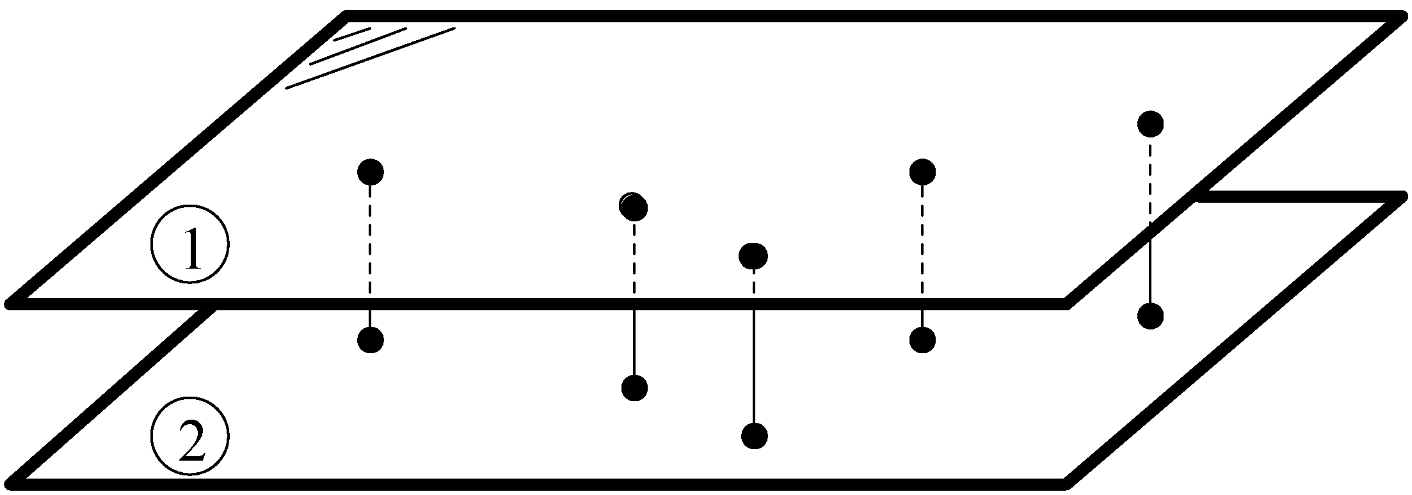
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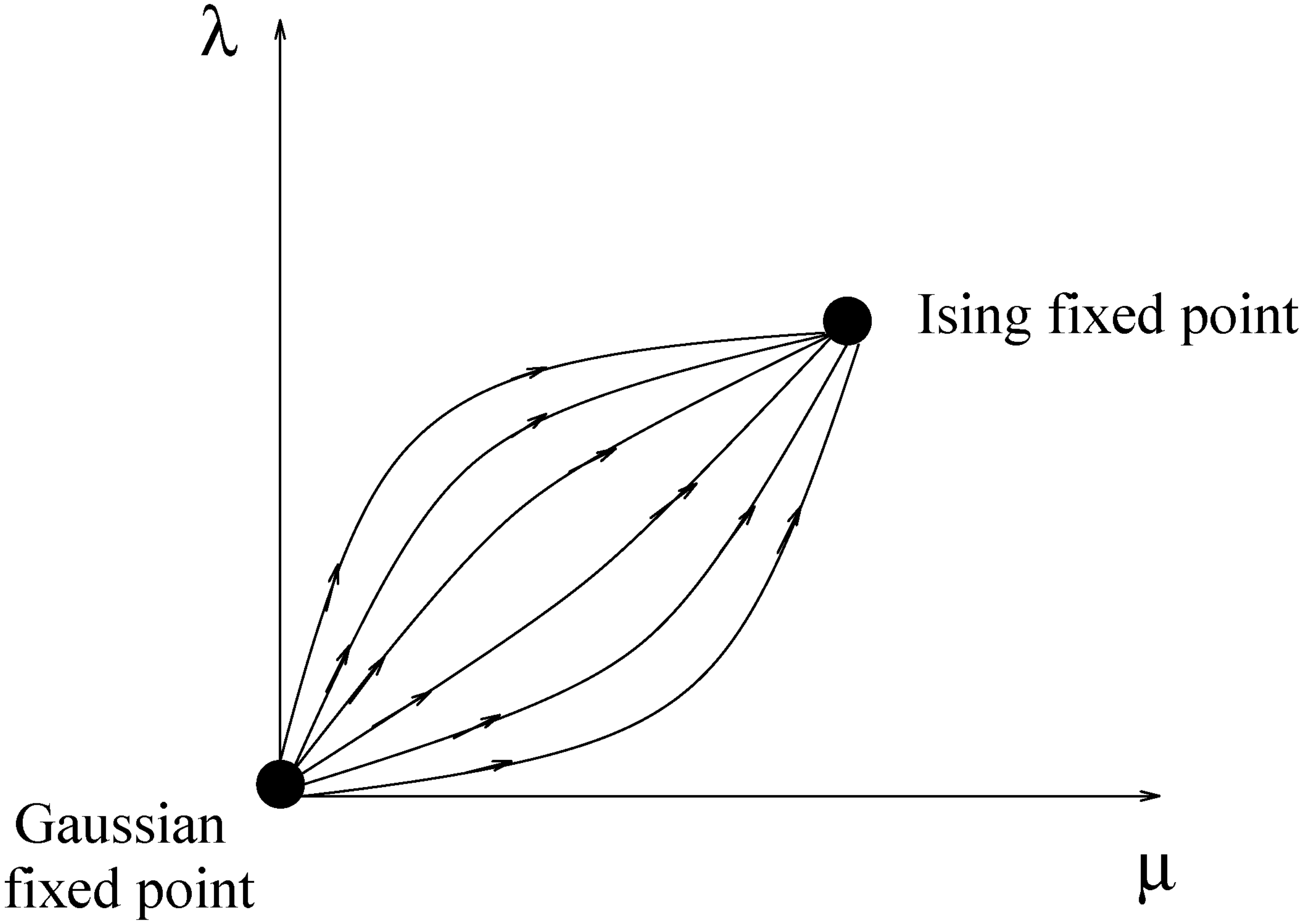
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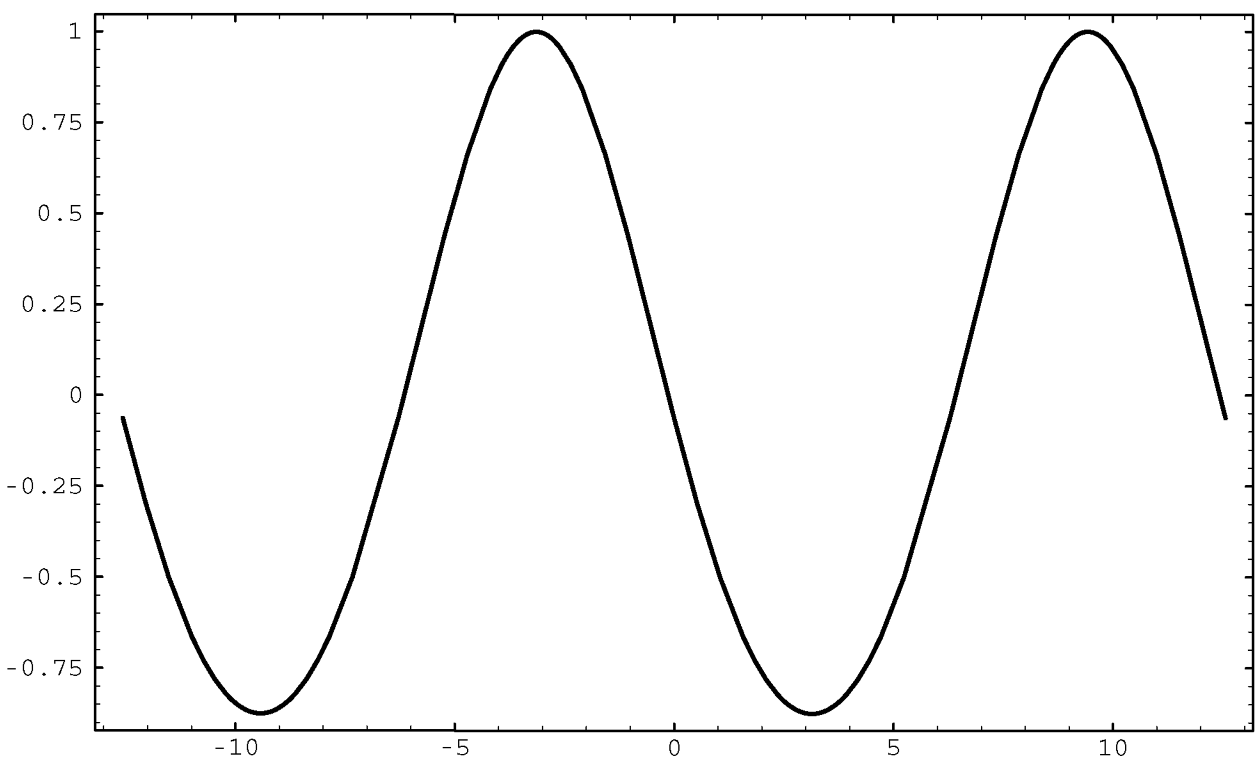
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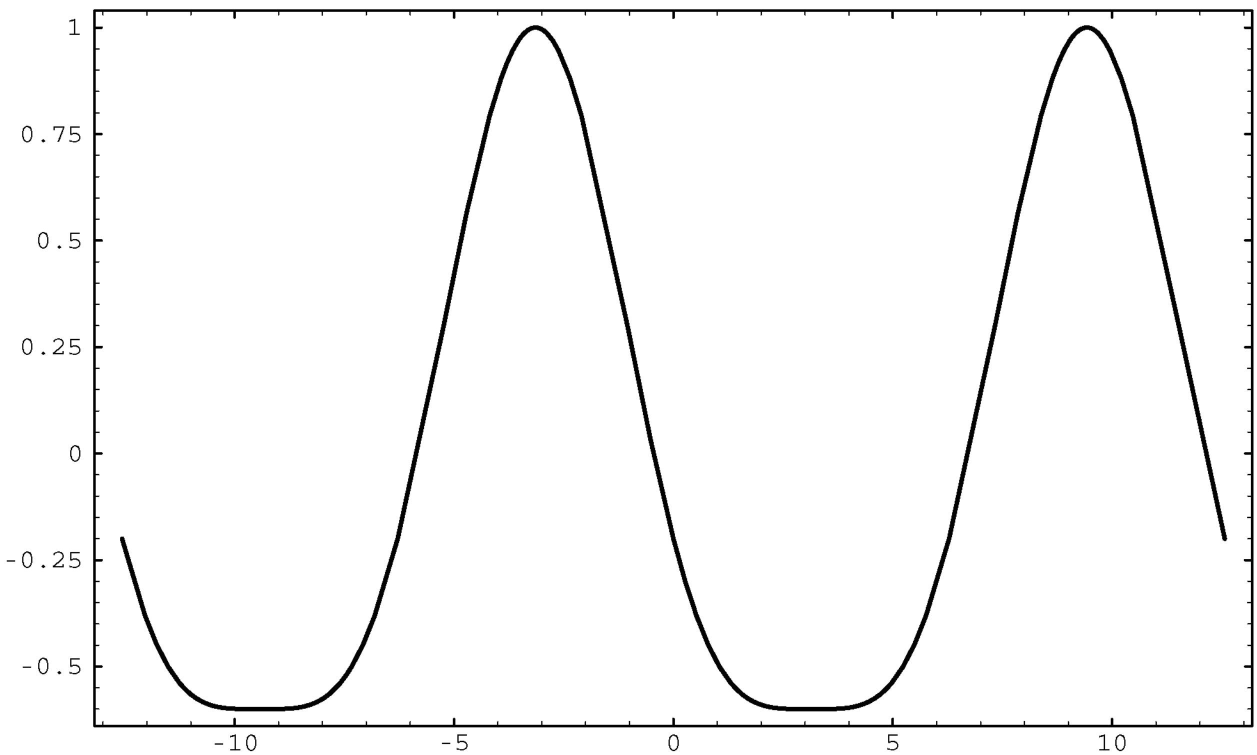

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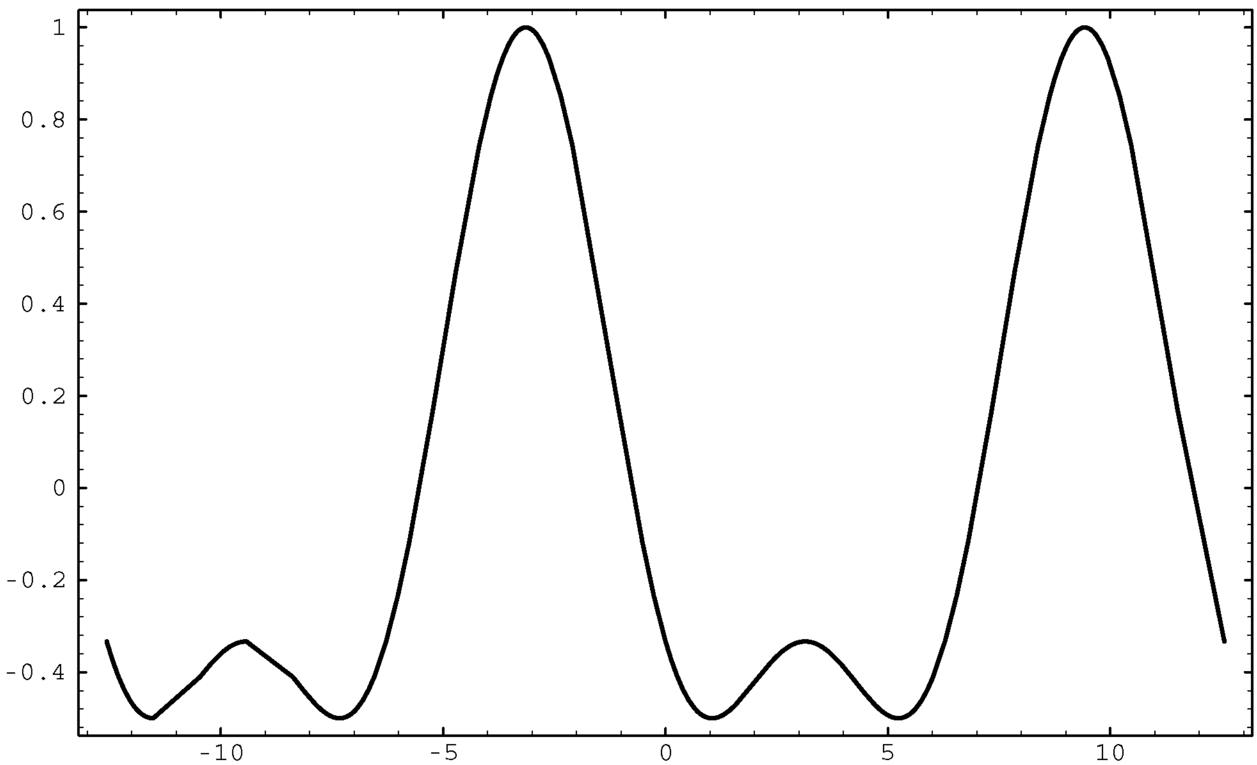
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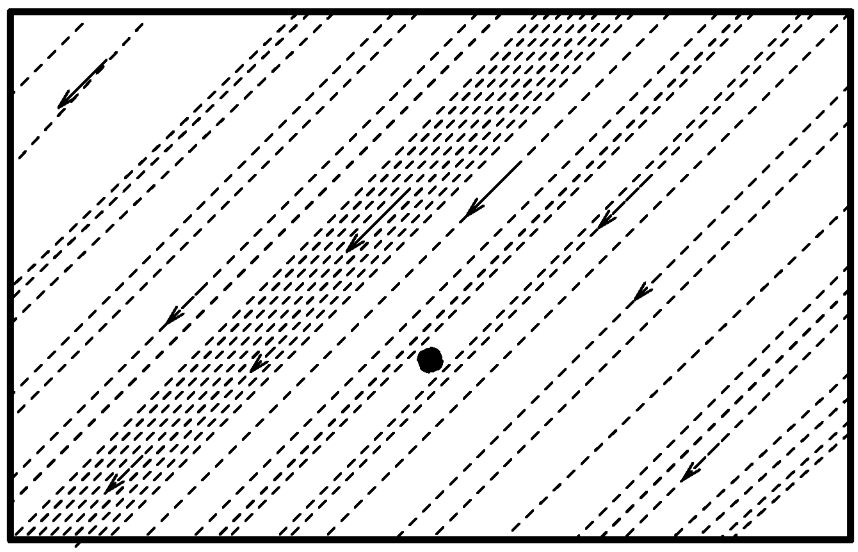

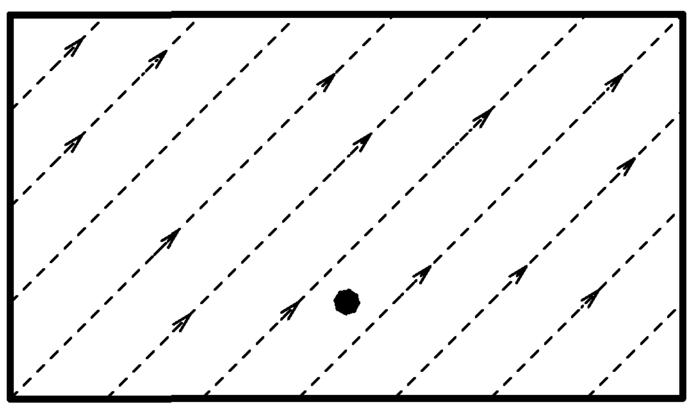

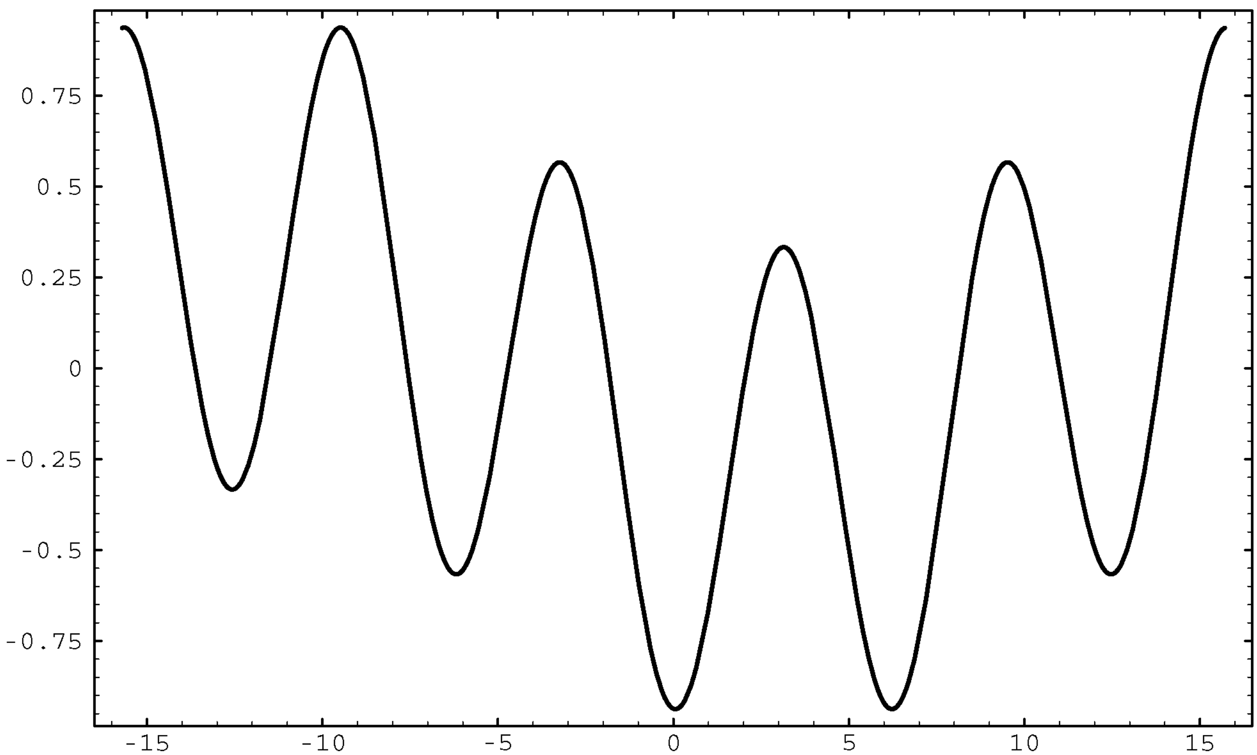

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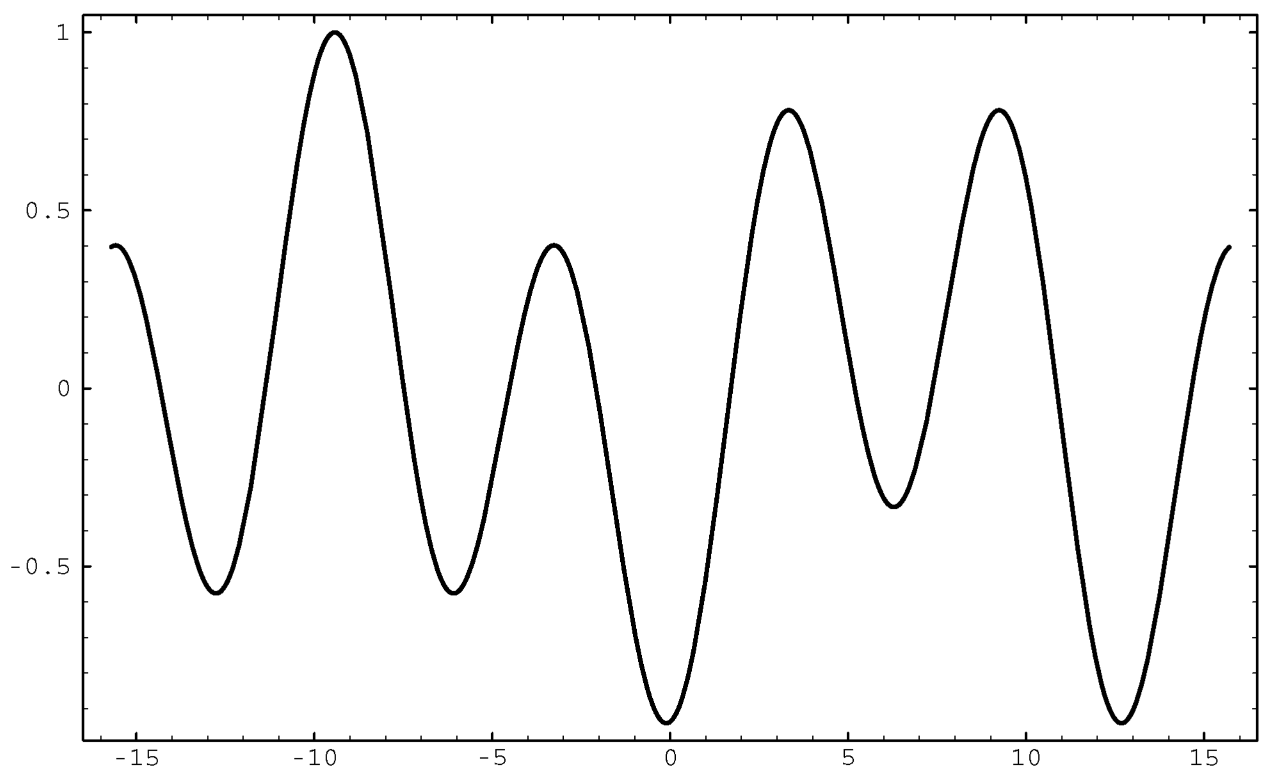

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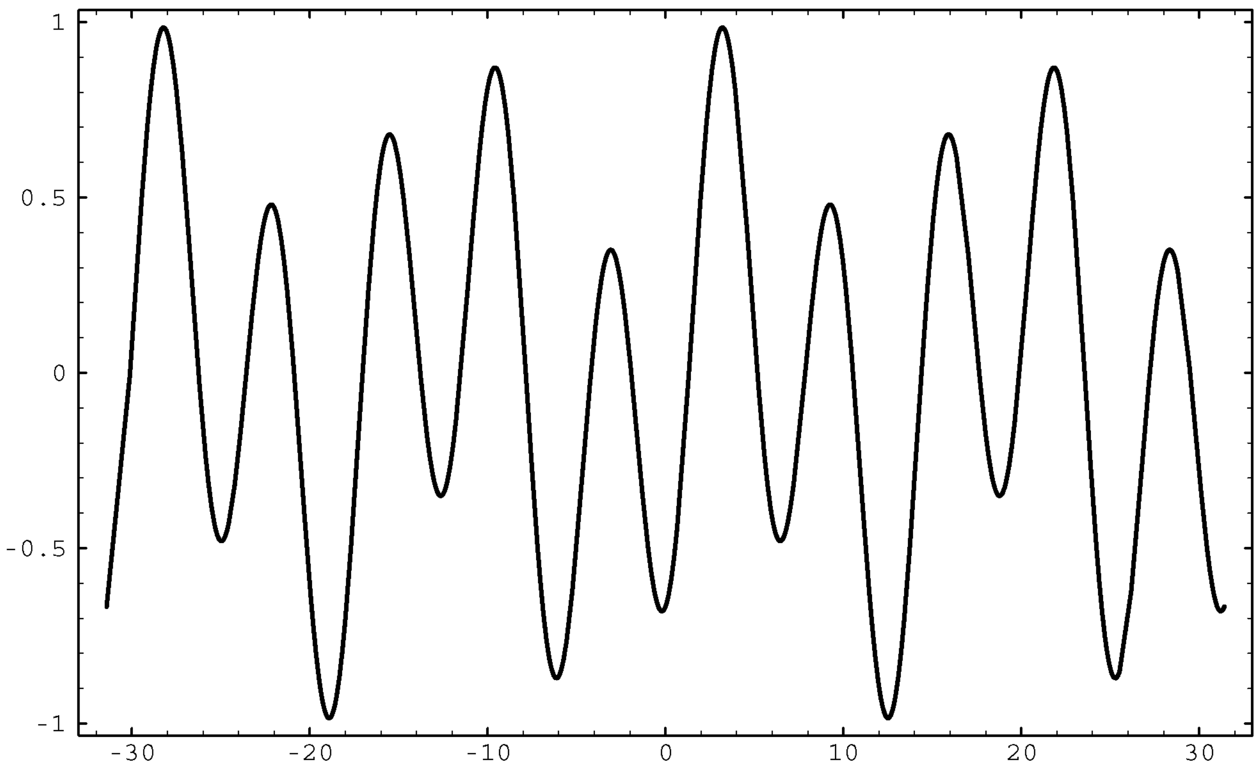

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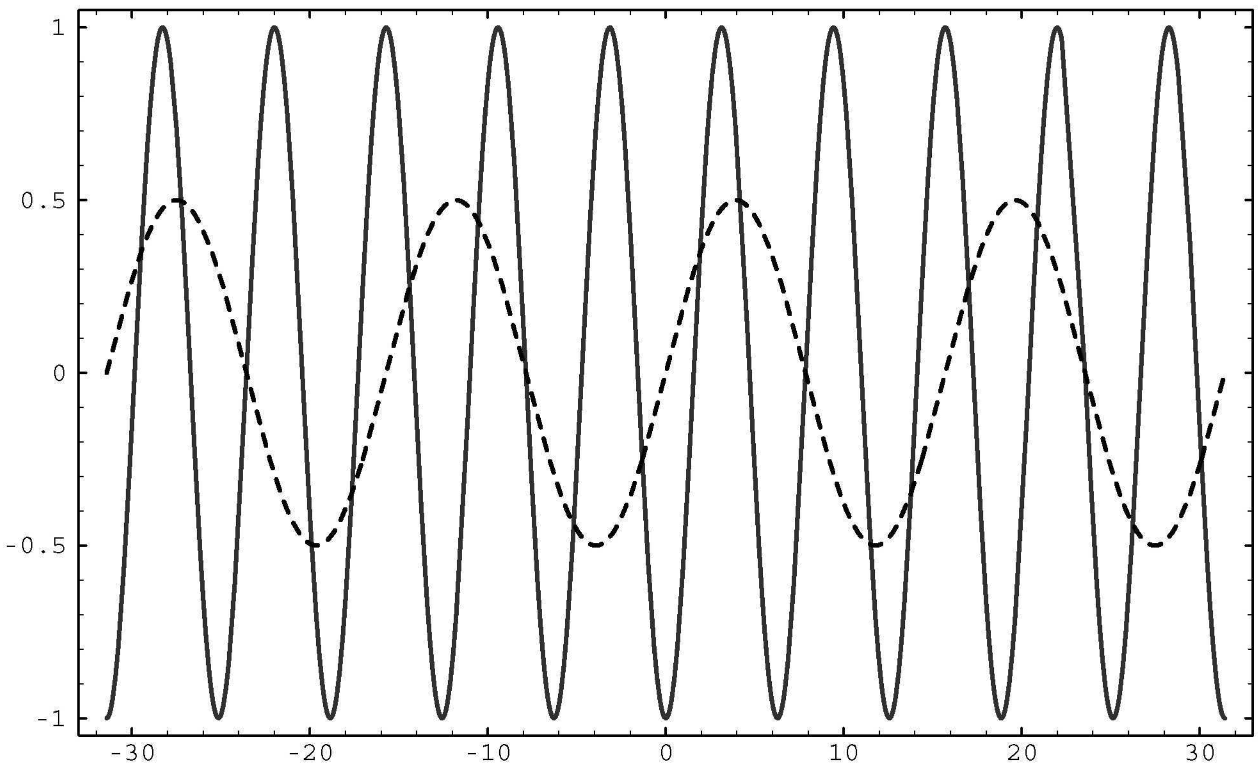

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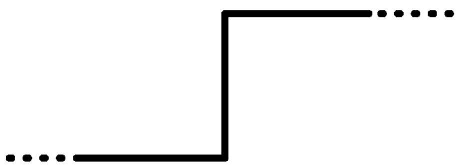

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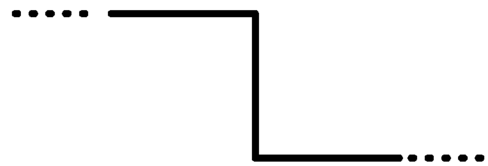

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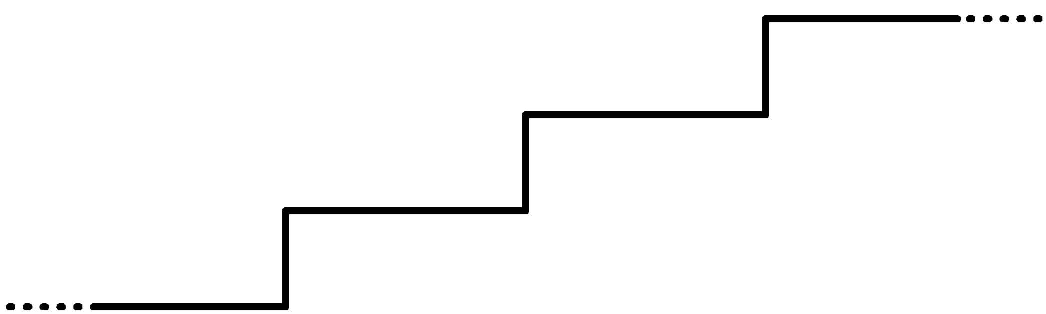

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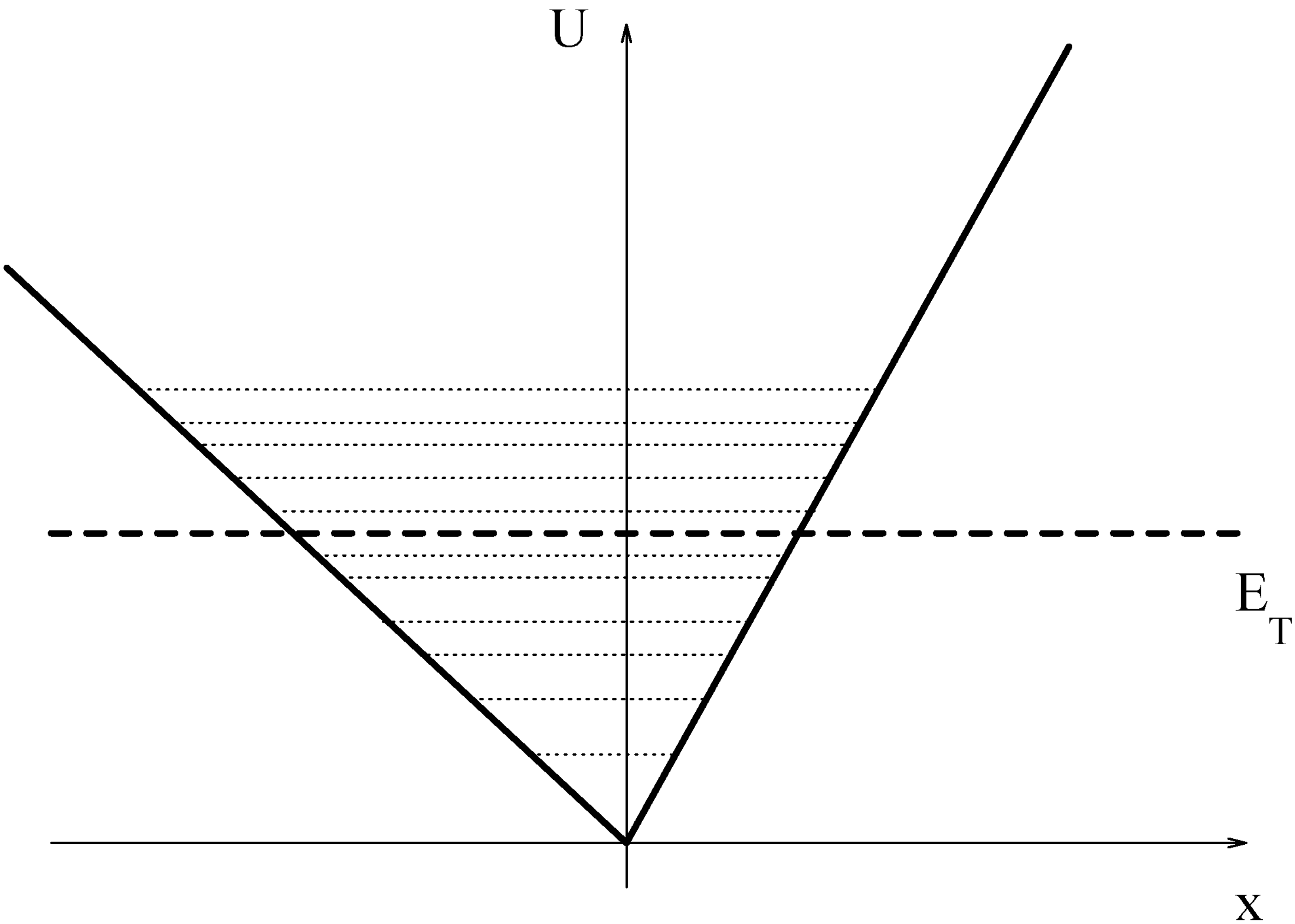
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