THEORY TO DETERMINE THE CRITICAL CHARGE DENSITY

Floran Vila
Physics Department, Faculty of Natural Sciences
Tirana University, Tirana, Albania
and
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

In this paper we theoretically determine the critical charge density in the system earthed metallic sphere-uniformly charged dielectric plane, in presence of earthed surfaces. This is a situation frequently encountered in industrial condition and has a great importance to evaluate the danger of the electrostatic discharges.

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¹Permanent Address.
1 Introduction

The critical charge density ($\sigma_c$) is the surface charge density that corresponds to the critical intensity of the field that provokes a discharge. Its determination is important because in most cases in industry, when relatively not small and not conductive surfaces are charged and produce electric fields able to create the discharge in gases. If the atmosphere is explosive, the situation may become dangerous [1-11].

Theoretically, in a first approximation, we have examined the problem of determining critical charge density, for clean configuration earthed metallic sphere-uniformly charged dielectric plane [12-14]. But in concrete industrial conditions there always exists an earthed surface. For this reason in this work we will determine $\sigma_c$ taking into consideration earthed surface.

2 Presentation of the Problem

The system we will examine is the earthed sphere-uniformly charged dielectric plane (Fig. 1) and represents a great interest for the estimation of the danger of electrostatics charges [15-22].

The plane uniformly charged, with $\sigma$ density, is rectangular with sides $a$ and $b$, placed respectively in the distance $d$ and $D$ from the earthed surface and earthed sphere with $r$ radius.

The intensity of electrostatic field in the point $s$ (Fig. 1) $E_s = f(D)$ is a decreasing function and the maximal value ($|E_s|_{\text{max}}$), is achieved at $D = r$ (Fig. 2). We are precisely interested at this maximum value for the determination of the critical charge density.

3 Determination of $(E_s)_{x=r}$

The model equivalent scheme of (Fig. 2) for the determination of $(E_s)_{\text{max}}$ is represented in (Fig. 3), where plane $P'$ and sphere $A'$ are respectively the mirror images of plane $P$ and sphere $A$.

The electric field intensity in the $S$ point is

$$E_s = \vec{E}_{P,s_1} + \vec{E}_{P',s_2} + \vec{E}_{s_1} + \vec{E}_{s_2} + \vec{E}_{s_1}^* + \vec{E}_{s_2}^* + \vec{E}_{s_1}^{**} + \vec{E}_{s_2}^{**}$$

(1)

where: $\vec{E}_{P,s_1}$ and $\vec{E}_{P',s_2}$ are the fields created respectively by the uniformly charged $P$ and $P'$, together with their charges $q_{s_1}$ and $q_{s_2}$, which are the mirror images in sphere $A$;
\( \vec{E}_{s_1} \) and \( \vec{E}_{s_2} \) are the fields created by the charges \( q_{s_1} \) and \( q_{s_2} \), which are respectively the mirror images of the charged planes \( P' \) and \( P \) in sphere \( A' \);

\( \vec{E}_{s_1}^* \) and \( \vec{E}_{s_2}^* \) are the fields created by the charges \( q_{s_1}^* \) and \( q_{s_2}^* \), which are respectively the mirror images of \( q_{s_1} \) and \( q_{s_2} \) in the sphere \( A \);

\( \vec{E}_{s_1}^\prime \) and \( \vec{E}_{s_2}^\prime \) are the fields created by the charges \( q_{s_1}^\prime \) and \( q_{s_2}^\prime \), which are respectively the mirror images of \( q_{s_1} \) and \( q_{s_2} \) in the sphere \( A^\prime \).

The sphere \( A \) represents the measuring electrode, and taking into consideration the experimental industrial conditions it can be rightly supposed:

\[ r \ll a, b, d \]  \hspace{1cm} (2)

Then according to [13] we can write:

\[
E_{P,s_1} = \left( \frac{-\partial \varphi_1}{\partial x} \right)_{\varphi = \frac{r}{a}} + \left( \frac{-\partial \varphi_2}{\partial x} \right)_{\varphi = \frac{r}{b}} = \frac{-\sigma}{2\pi \varepsilon_0} \left( \frac{2\pi + \frac{k}{r}}{2\pi} \right) \]  \hspace{1cm} (3)

\[
E_{P',s_2} = \left( \frac{-\partial \varphi_1}{\partial x} \right)_{\varphi = \frac{r}{a}} + \left( \frac{-\partial \varphi_2}{\partial x} \right)_{\varphi = \frac{r}{b}} = \frac{-\sigma}{4\pi \varepsilon_0} \left( \frac{8d}{r} K_2 - \frac{K_3}{r} - \frac{K_4}{d} \right) \]  \hspace{1cm} (4)

\[
E_{s_1} = \left( \frac{-\partial \varphi_2}{\partial x} \right)_{r = \frac{d}{r}} \]  \hspace{1cm} (5)

\[
E - S_2 = \left( \frac{-\partial \varphi_2}{\partial x} \right)_{r = \frac{d}{r}} = \frac{\sigma}{4\pi \varepsilon_0} \left( -\frac{2r}{d} K_2 + \frac{r}{4d^2} K_3 \right) \]  \hspace{1cm} (6)

where

\[
K = a \ln \frac{b + \sqrt{a^2 + b^2}}{a} + b \ln \frac{a + \sqrt{a^2 + b^2}}{b} \]  \hspace{1cm} (7)

\[
K_1 = \sqrt{16d^2 + a^2 + b^2} \]  \hspace{1cm} (8)

\[
K_2 = \arctg \frac{ab}{4dk_1} \]  \hspace{1cm} (9)

\[
K_3 = a \ln \frac{K_1 + b}{K_1 - b} + b \ln \frac{K_1 + a}{K_1 - a} \]  \hspace{1cm} (10)

\[
K_4 = \frac{a^2 b^2 (16d^2 + K_1^2)}{K_1^3 (16d^2 K_1^2 + a^2 b^2)} \]  \hspace{1cm} (11)

Based on relation (2) the charges \( q_{s_1}, q_{s_2}, q_{s_1}', q_{s_2}' \) will be considered as point charges, situated, respectively, at the center of the spheres \( A \) and \( A' \).

According to [8,23]

\[
dq' = -\frac{r}{\ell} dq \]  \hspace{1cm} (12)
where \( dq \) and \( dq' \) are, respectively, the elementary plane charge and its mirror image from
the sphere, while \( \ell \) is the distance between the two elementary charges. The full mirror
image of the plan charge on the sphere is [24]

\[
q' = -r \sigma I_1,
\]

where

\[
I_1 = 2a \ln \frac{b + \sqrt{A^2 + a^2 + b^2}}{\sqrt{A^2 + a^2}} + 2b \ln \frac{a + \sqrt{A^2 + a^2 + b^2}}{\sqrt{A^2 + b^2}}
\] \[ -\frac{4D \arctg ab}{2D \sqrt{A^2 + a^2 + b^2}} \] (14)

In accordance with relations (2), (12-14) and (Fig. 3) we can find out:

\[
q_{s_1} < 0 = -r \sigma (I_1)_{D=-r} = -2r \sigma (K - 2r K_3) \] (15)

\[
q_{s_2} > 0 = r \sigma (I_1)_{D=2d+r} = 2r \sigma (K - 4d K_3) \] (16)

\[
q_{s_1'} > 0 = r \sigma (I_1)_{D=r} = 2r \sigma (K - 2r K_5) \] (17)

\[
q_{s_2'} < 0 = -r \sigma (I_1)_{D=(2d+r)} = -2r \sigma (K - 4d K_2) \] (18)

\[
q_{s_1}^* > 0 = \frac{-r}{2(d + r)} = \frac{r^2 \sigma (K - 2r K_3)}{d} \] (19)

\[
q_{s_2}^* < 0 = \frac{-r}{2(d + r)} q_{s_2} = \frac{-r^2 \sigma (K - 4d K_3)}{d} \] (20)

\[
q_{s_1}^* < 0 = \frac{-r}{2(d + r)} q_{s_1} = \frac{-r^3 \sigma (K - 2r K_5)}{d} \] (21)

\[
q_{s_2}^* > 0 = \frac{-r}{2(d + r)} q_{s_2} = \frac{r^2 \sigma (K - 4d K_2)}{d} \] (22)

\[
L_{s_1}^* = -\frac{\sigma}{4\pi \varepsilon_0} \cdot \frac{r^2 (K - 2r K_3)}{4d^3} \] (23)

\[
F_{s_2}^* = \frac{\sigma}{4\pi \varepsilon_0} \cdot \frac{r^2 (K_6 - 4d K_2)}{4d^3} \] (24)

\[
L_{s_1}^* = -\frac{\sigma}{4\pi \varepsilon_0} \cdot \frac{(K - 2r K_3)}{d} \] (25)

\[
L_{s_2}^* = \frac{\sigma}{4\pi \varepsilon_0} \cdot \frac{(K_6 - 4d K_2)}{d} \] (26)

where

\[
K_5 = \arctg \frac{ab}{2r \sqrt{a^2 + b^2}} \] (27)

\[
K_6 = a \ln \frac{K_1 + b}{K_1 - b} + b \ln \frac{K_1 + a}{K_1 - a} \] (28)
Taking into consideration the relations (2-6), (23-26) and neglecting the term
\[
\frac{(2K - K_2)r}{4d^2} \ll 1
\]  
finally the maximal resultant intensity \((E_s)_{\text{max}}\) can be found from the relation:
\[
(E_s)_{\text{max}} = -\frac{\sigma}{4\pi \varepsilon_0} \left(4\pi + \frac{a}{r} + \frac{B}{d}\right),
\]  
where
\[
A = 2K + 8dK_2 - K_3
\]  
\[
B = K + 2K_2(2d + r) - K_4 - 2rK_5 - K_6
\]  

4 Determination of Critical Charge Density \((\sigma_c)\)

To determine the critical charge density, we must use the condition presented in [14]. Normally the sphere radius is taken to be \(r = 10^{-2} \text{m}\), because different authors [20,25], think that in general when \(r > 10^{-2} \text{m}\) the situation will be dangerous for the beginning of the lamp discharge. While to determine \((E_s)_{\text{max}}\), we shall use Bower’s method [14, 20], according to which for an ionizing tension of arounded air \(V = 12,5 \text{v}\), for average path of electron \(\lambda = 3,7 \times 10^{-7} \text{m}\), and the sphere’s radius \(r = 10^{-2} \text{m}\), discharge at the points of the sphere is reached when \(E_0 = 4,54 \times 10^6 \text{v/m}\). After the determination of \(|E_s|_{\text{max}} = E_0\) and of \(r\), according to (30) at last we can write
\[
\sigma_c = \frac{4\pi \varepsilon_0 E_0}{4\pi + \frac{A}{r} + \frac{B}{d}}
\]  
where \(\sigma_c\) is the critical charge density, while \(A = f_1(a, b, d)\) and \(B = f_2(a, b, r, d)\) are functions of the geometrical parameters of the system.

5 Discussion

The critical charge density \(\sigma_c\) is a function of five parameters
\[
\sigma_c = f(a, b, r, d, E_0)
\]  
among which four of them determine the geometry of the system, while the intensity \(E_0\) represents the Bower’s condition for the discharge creation. Among these five parameters, two of them will definitely be fixed at the parameter’s values \((r = 10^{-2} \text{m}; E_0 = 4.56 \times 10^6 \text{v/m})\). While the parameters \(a, b, d\) are considered variable subjected to the condition (2).
To verify the accuracy of relation (33) it is important to study the function \( \sigma_c = f(d) \).

In Fig. 1) we have graphically presented \( \sigma_c = f(d) \) for parameter's values \( (a = 1.5m; b = 2m) \). As \( d \) increases, \( \sigma_c \) decreases and asymptotically tends to the critical density of the pure system earthed metallic sphere-uniformly charged dielectric plane, as \( d \) goes to infinity, \( \sigma_c^\infty = 8.14 \cdot 10^{-7} \frac{C}{m^2} \).

The critical density of the clear system can be found analytically from the relation (33)

\[
\sigma_c^\infty = \lim_{d \to \infty} \frac{4\pi \varepsilon_0 E_0}{1 + \frac{A}{r} + \frac{B}{d}} = \frac{2\varepsilon_0 E_0}{2 + \frac{k}{r}}
\]

because

\[
\lim_{d \to \infty} \frac{A}{r} = \frac{2k}{r} \quad \text{and} \quad \lim_{d \to \infty} \frac{B}{d} = 0
\]

Relation (35) is in full agreement with our previous results [12-14], a fact which definitely points out the accuracy of relation (33).

In (Fig. 5) and (Fig. 6) \( \sigma_c = f(d) \) is shown graphically, respectively for parameters values \( (a = 2m; b = 2m) \) and \( (a = 2.5m; b = 2m) \). The graphs show the change of \( \sigma_c^\infty \) (respectively \( \sigma_c^\infty = 7.03 \cdot 10^{-7} \frac{C}{m^2} \) and \( \sigma_c^\infty = 6.33 \cdot 10^{-7} \frac{C}{m^2} \)), a fact that is clearly visible from relation (35), because \( k = f(a, b) \). The physical reason of the decrease of \( \sigma_c^\infty \), is the increase on the charge because of the enlargement of the dielectric plane area.

The graphs (4-6) show that \( \sigma_c \) increases as \( \alpha \) decreases. The physical explanation of this phenomena is the following: as \( d \) decreases, the field created by the system at point \( s \) decreases, so that the critical charge density must increase, in order to fulfil the Bowers condition for the start of the discharge.

In conclusion, the advantage of using relation (33) is, because such a relation for a given system allows firstly the computation of \( \sigma_c \), and secondly, is possible, varying the parameters \( a, b, d \) we can give to \( \sigma_c \) the desired value.

6 Conclusion

In this paper we theoretically determine the critical charge density in the system earthed metallic sphere - uniformly charged dielectric plane, in the presence of earthed surfaces, a situation frequently encountered in industrial condition and has a great importance to evaluate the danger of the electrostatic discharges.

Our conclusions easily allows one to apply them in special cases, in order to compute the value of \( \sigma_c \) for given systems, and to vary it as a function of the variable parameters always under the constrain of (2).
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References


Figure Captions

Fig. 1 The system earthed sphere-uniformly charged dielectric plan in the presence of the earthed surface.

Fig. 2 The realistic scheme to determine the critical charge density.

Fig. 3 The model equivalent scheme to determine the critical charge density.

Fig. 4 The graphical representation of $\sigma_c = f(d)$ for parameter’s values: $a = 1.5m$; $b = 2m$.

Fig. 5 The graphical representation of $\sigma_c = f(d)$ for parameter’s values: $a = 1.5m$; $b = 2m$.

Fig. 6 The graphical representation of $\sigma_c = f(d)$ for parameter’s values: $a = 2.5m$; $b = 2m$. 
Fig. 1
Fig. 3
Fig. 4

Fig. 5
Fig. 6