THE RELATION BETWEEN FRACTAL DIMENSION AND ROUGHNESS INDEX FOR FRACTAL SURFACES

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ABSTRACT

This paper discusses the relation between fractal dimension and roughness index for fractal surfaces of solids. It shows that in some cases, it is difficult to distinguish which one is self-similar or self-affine by experiments. An analysis on this problem and a method to distinguish them by experiments are proposed.
1 Introduction

Since Mandelbrot et al. [1] showed that fractured surfaces are fractals in nature and that
the fractal dimension of fractured surfaces correlates well with the toughness of materials,
the present author [2] has analyzed the critical crack extension force with the fractal model
and pointed out that the true areas of the fractured surfaces in materials are actually larger
than that calculated from a flat fractured surface. Since then, many authors have done
experiments on this problem [2-7]. Now, it is generally accepted that the fractured surface
is a fractal with self-affine property [8]. However, the fractal dimension of even the simplest
self-affine fractals is not uniquely defined [9]. The difficulties have been illustrated with
the case of fBm by Voss [10] and Pietronero [11]. In his paper [10], Voss showed that \( D = 2 - H \) in box counting method and \( 1/H \) in the limiting case of the ‘coastline’ method
on smaller scales and 1 on larger scales. Pietronero came to the same conclusion [11] by
taking normal Brownian motion as an example.

In the case of fractured surfaces, ‘coastline’ method seems nearer to real situation and
discussions on limiting cases are not enough to show the detailed relations between \( D \)
and \( H \) in the entire range of scales. In this paper, we use the ‘coastline’ method to show
the relations between \( D \) and \( H \) and show some new interesting findings which previous
authors have not pointed out.

2 Relation of \( D \) to \( H \) for self-affine surfaces

Since most real surfaces scale differently in the plane of fracture and in the vertical
direction, they are self-affine rather than self-similar. What will happen if one measures
the self-affine surfaces with \( D \) artificially, or describes the self-similar surfaces with \( H \)?

If the surface is self-affine, we may determine \( H \) from double logarithmic plots of
\( \Delta V(t) \) vs \( \Delta t \), where \( V \) is the vertical height and \( t \) is the horizontal axis. Then, \( H \) might be
a constant which is independent on the yardstick. On the other hand, if we measure the
fractal dimension of the surface artificially, the \( D \) value might be yardstick dependent.
However, in the earlier works on fracture [1,8], constant \( D \) values have been obtained in
many cases, even though the concept of self-affine property of fractured surfaces has been
accepted generally. Straight lines on double logarithmic plots both on \( \Delta V \) vs \( \Delta x \) [12,13]
and on \( L(\varepsilon) \) vs \( \varepsilon \) [1,8] have been obtained. The measured \( D \) value seems also yardstick
independent. The reason why may be explained in the following:

For a coastline, one may divide the curve into \( N \) segments by walking a ruler of size \( l \)
along the curve. The length along each segment is

\[ \frac{l}{l_0} = \sqrt{\left(\frac{\Delta l}{l_0}\right)^2 + \left(\frac{\Delta V}{l_0}\right)^2} \]  

(1)

Using the relation,

\[ \frac{\Delta V_H}{V_0} = \left(\frac{\Delta l}{l_0}\right)^H = N^{-H} \]  

(2)

and the relation, \( l_0 = \sqrt{2l_0} \),

\[ \frac{\sqrt{2l}}{l_0} = \frac{\Delta l}{l_0} \sqrt{1 + \left(\frac{\Delta l}{l_0}\right)^{2H-2}} \]  

(3)

Then,

\[ N\left(\frac{l}{l_0}\right) = \left(\frac{\Delta l}{l_0}\right)^{-H} = \frac{1}{\sqrt{2}} \left(\frac{l}{l_0}\right)^{-1} \left[1 + \left(\frac{\Delta l}{l_0}\right)^{2H-2}\right]^{\frac{1}{2}} \]  

(4)

\[ N\left(\frac{l}{l_0}\right) = \left(\frac{l}{l_0}\right)^{-H} \]  

(5)

Therefore,

\[ D = 1 - \frac{\ln[1 + \left(\frac{\Delta l}{l_0}\right)^{2H-2}]}{2 \ln\left(\frac{1}{\sqrt{2}}\right)} \]  

(6)

let

\[ \zeta = \frac{1}{2} \left[ 1 + \left(\frac{\Delta l}{l_0}\right)^{2H-2} \right] \]

\[ D(H, \frac{\Delta l}{l_0}) = 1 - \left[1 + \frac{2 \ln(\frac{\Delta l}{l_0})}{\ln \zeta}\right]^{-1} \]

(7)

(i) When \( \frac{\Delta l}{l_0} \ll 1 \), \( N = (\frac{\Delta l}{l_0})^{-1} \approx (\frac{l}{l_0})^{-\frac{1}{H}} \); and then \( D = \frac{1}{H} \).

(ii) When \( \frac{\Delta l}{l_0} = 1 \), \( D \) approaches the limiting value \( \frac{2}{(1+H)} \).

Fig. 1 shows the relationship of \( D(H, \frac{\Delta l}{l_0}) \) with \( \frac{l}{l_0} \). In the double logarithm plots, the dependence of \( D \) with \( \frac{\Delta l}{l_0} \) becomes weaker and weaker as \( H \) rises from 0.5 to 1. One cannot judge whether \( D \) is dependent on the yardstick when \( H \) value rises up to a certain value near unity (say \( 0.8 < H < 1 \)) within the accuracy of measurements. In this case, we cannot distinguish which one, self-similarity or self-affinity, is better to describe the fractal surface. Therefore, in the early works of fracture, nice fractal dimensions have been obtained in previous measurements (for example, see [8]).

Fig. 2 shows the relationship of \( L(H, \frac{\Delta l}{l_0}) \) with \( \frac{l}{l_0} \). In this figure, \( L(H, \frac{l}{l_0}) \) is calculated by \( (\frac{l}{l_0})^{1-D} \) where \( D(l) \) is calculated by eqn. (7). Because of \( D \) a function of \( H \) and \( l \), \( L(H, l) \) and \( l \) are not a linear relation in double logarithmic plots. From Fig. 2, the deviation of
linear relationship is larger when $H = 0.5$ and smaller when $H = 0.8$. $L(i_l)$ is independent of $i_l$ when $H = 1$. Similarly, as $H$ approaches 1 (say $0.8 < H < 1$), we cannot judge whether it is a curve or a straight line within the range of experimental error.

Moreover, the values of $L(l)$ are points on the $L$ vs $l$ double logarithmic plots with various values of $H$. The slope of the curve does not have the meaning of fractal dimension. The real fractal dimension $(1 - D)$ is the slope of the straight line connected the point $(L/Lo, i_l)$ and the zero point $(0,0)$.

### 3 Relation of $H$ to $D$ for self-similar surfaces

On the other hand, if the fracture surface is of self-similar structure, one may determine $D$ from the double logarithmic plots of the $L(i_l)$ and $i_l$ relationship. The value of $D$ might be a constant and is independent of $i_l$. If one measures the roughness exponent of the surface, the value of $H$ might be yardstick dependent.

Now, with similar derivation, we have

$$H(l, D) = 1 + \frac{\ln[2(\frac{l}{l_0})^{2-2D} - 1]}{2D \ln(\frac{l}{l_0})}$$

(i) When $\frac{l}{l_0} << 1$

$$H(l, D) = \frac{1}{D} + \frac{\ln 2}{2D \ln(\frac{l}{l_0})} \approx \frac{1}{D}$$

(ii) When $\frac{l}{l_0} = 1$, the limiting value of $H(l, D) = \frac{2}{D} - 1$

Fig. 3 shows the double logarithmic plots of $H(l, D)$ and $\frac{l}{l_0}$. Similar to the above, the dependence of $H$ on $\frac{l}{l_0}$ becomes weaker and weaker as $D$ approaches unity.

Fig. 4 shows the double logarithmic plots of $\Delta V(D, \frac{l}{l_0})$ and $\frac{l}{l_0}$. As above, the deviation from a straight line is smaller and smaller when $D$ approaches unity.

### 4 Summary

From the above analysis, one may draw the following conclusions:

If one describes a surface of a self-affine structure with fractal dimension, the apparent fractal dimension might be yardstick dependent. However, the dependence cannot be distinguished as the $H$ value is near unity. On the other hand, if one describes a surface of self-similar structure, with roughness exponent, the apparent $H$ value might be yardstick dependent. However, the dependence cannot be correctly appraised when the $D$ value is near unity.
In principle, comparing the linearity of $H$ vs $l$ and $D$ vs $l$ relation in double logarithmic plots, one may make an appraisal as to which structure, either self-affine or self-similar, is the real one. However, if the surface appears to flatten, this experimental method is not sensitive, one should then adopt another experimental method to make the appraisal.

In addition, comparing Figs. 1 and 3, we can see that the dependence of $H(D, \frac{l}{l_0})$ on $\frac{l}{l_0}$ is weaker than that of $D(H, \frac{l}{l_0})$ on $\frac{l}{l_0}$. Then, the measurement of the roughness exponent is a less sensitive way to judge the deviation from self-affinity than the fractal dimension to the deviation from self-similarity. The range of $H$ values from 0.5 to 1 is half the range of $D$ values from 1 to 2. Using the measured values of $H$ parameters to characterize the roughness of materials, the differences among them can easily be ignored or the universal properties can easily be exaggerated.

Moreover, judging a structure to be a fractal or nonfractal, we recommend measuring both the double logarithmic relations of $L(e)$ vs $e$ and of $\Delta V$ vs $\Delta x$. If you find that $D$ is scale dependent, perhaps it is still a fractal of self-affinity. On the other hand, if you find that $H$ is scale dependent, perhaps it is still a fractal of self-similarity. One may remember that the measured values of $H$ parameters is less sensitive, an approximate straight line in Fig.4 is not rigorous enough to judge its self-affine property. Comparing Figs. 2 and 4, it is more conclusive, because the nonlinear behaviour in Fig.2 shows that it is not a fractal of self-similar definitely.

The above discussion on $D$ and $H$ is important. Considering the complicated mechanisms of fracture in materials, it is worthwhile to check if the fractured surface is self-similar or self-affine experimentally. The usual case we discussed is of Mode-I fracture, but how are Mode-II, Mode III and Complex Mode? If we consider the fractured surface on an atomic level, say the dislocation mechanism of micro-crack nucleation; the percolation model for brittle fracture, or the fragmentation, friction, wear, corrosion and other processes, what will happen?

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References


Figure captions

Fig.1. Relationship of $D(H, \Delta t/t_0)$ with $l/l_0$.

Fig.2. Double logarithmic relationship of $L(l/l_0)$ with $l/l_0$.

Fig.3. Relationship of $H(D, l/l_0)$ with $l/l_0$.

Fig.4. Double logarithmic relationship of $\Delta V(D, l/l_0)$ with $\Delta t/l_0$. 
Fig. 3

Fig. 4