ITERATIVE SOLUTION OF NONLINEAR EQUATIONS
IN ARBITRARY BANACH SPACES

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ABSTRACT

Let \( E \) be an arbitrary real Banach space and let \( K \) be a proximinal subset of \( E \) with a nonempty interior \( K^\circ \). Let \( T : K \rightarrow E \) be a uniformly continuous strongly accretive operator such that the operator equation \( Tx = f \) has a solution \( x^* \in K^\circ \). It is proved that modified iteration processes of the Mann and Ishikawa types converge strongly to \( x^* \). If \( E \) is reflexive and \( T \) is locally uniformly continuous with an open domain \( D(T) \), similar results are obtained. Related results deal with the solution of the operator equation \( x + Tx = f \) and the approximation of fixed points of strong pseudocontractions. Explicit convergence rates are also given. The technique of the proof is of independent interest.
1 Introduction

Let $E$ be an arbitrary real Banach space. An operator $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be accretive if $\forall x, y \in D(T)$ and $\forall r > 0$ the following inequality holds:

$$\|x - y\| \leq \|x - y + r(Tx - Ty)\|$$

(1)

If the inequality (1) is strict whenever $x \neq y$ then $T$ is said to be strictly accretive and if $\exists k > 0$ such that $T - kI$ is accretive, where $I$ denotes the identity operator on $E$, then $T$ is said to be strongly accretive. Let $N(T) := \{x^* \in D(T) : f = Tx^*\}$ denote the solution set of the equation $Tx = f$. Then $T$ is said to be quasi-accretive if the inequality (1) holds $\forall x \in D(T)$ and $\forall x^* \in N(T)$. If $\forall r > 0 \ (I + rT)(E) = E$ then $T$ is said to be $m$-accretive.

Closely related to the class of accretive operators is the class of dissipative operators where an operator $T$ is said to be dissipative if and only if $(-T)$ is accretive. The concepts of strict, strong and $m$-dissipativity are similarly defined.

Another class of operators intimately tied to the accretive operators is the class of pseudocontractions. A mapping $T$ is said to be strongly pseudocontractive if $\exists t > 1$ such that $\forall x, y \in D(T)$ and $\forall r > 0$ the following inequality holds:

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

(2)

and is said to be pseudocontractive if (2) holds with $t = 1$. A close study of inequalities (1) and (2) shows that $T$ is pseudocontractive if and only if $A = I - T$ is accretive. Thus the mapping theory of accretive operators is intimately tied to the fixed point theory of pseudocontractive mappings.

These classes of mappings have attracted a lot of interest because of their involvement in evolution systems modeling several physically significant problems. Consequently, several authors have studied the existence, uniqueness and iterative construction of solutions to nonlinear equations involving such mappings (see e.g., [1-29] and the references cited therein). Recently, Chidume and Osilike [12] proved that if $K$ is a bounded closed convex and nonempty subset of a real Banach space $E$, $T : K \mapsto K$ is a uniformly continuous strong pseudocontraction with a fixed point $x^* \in K$ and $(I - T)$ has a bounded range then the Mann and Ishikawa iteration processes converge strongly to $x^*$. Subsequently, Chidume and Moore [8] extended this to the case where $T : K \mapsto E$ with $K$ bounded and proximal. Two questions of interest naturally arising from the foregoing discussion are:

(i) Can the boundedness condition on $K$ and/or the operator $(I - T)$ be removed?

(ii) Can the uniform continuity condition on $T$ be weakened?

If $E$ is a uniformly smooth Banach space, then both questions are answered in the affirmative (see e.g., [7, 11, 21]). No answer has been provided for spaces more general than uniformly smooth.

It is our purpose in this paper to provide an affirmative answer to question (i) and a partially affirmative answer to question (ii). Our theorems, therefore, extend, generalize and unify the most important known results in this connection. Furthermore, we discuss the explicit convergence rates and the stability of our iteration processes (under the introduction of “small” perturbation terms). Our method of proof is also of independent interest.
2 Preliminaries

Let $K$ be a nonempty subset of $E$. An element $x^* \in K$ is called a nearest point or point of best approximation in $K$ to $x \in E$ if

$$\|x - x^*\| = \inf\{\|x - y\| : y \in K\} =: d(x, K)$$ (3)

An operator $P_K : E \mapsto K$ defined by

$$P_K x : = \{x^* \in K : \|x - x^*\| = d(x, K)\}; \ \forall x \in E$$ (4)

is called a best approximation (or variously, proximal, metric projection) operator on $K$. If $P_K x \neq \emptyset$, $\forall x \in E$ then $K$ is said to be proximal in $E$ and if $P_K x = \{x^*\}$, a singleton, $\forall x \in E$ then $K$ is called a Chebyshev subset of $E$. It is known that proximal subsets of Banach spaces are closed and every closed convex nonempty subset of a reflexive Banach space is at least proximal.

Let $E^*$ denote the dual space of $E$. For $1 < p < \infty$ we denote by $J_p : E \mapsto 2^{E^*}$ the duality mapping defined by

$$J_p x : = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^p; \ \|f^*\| = \|x\|^{p-1}\}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing between elements of $E$ and of $E^*$. If $p = 2$, then $J = J_2$ is called the normalized duality mapping. Observe that $J_p 0 = 0$ hence, $J_p x = \|x\|^{p-2} J x, \ \forall 0 \neq x \in E$. It is also known that $J_p$ is the subdifferential of the convex functional $f : E \mapsto [0, \infty)$ defined by

$$f(x) := \frac{1}{p} \|x\|^p$$

where the subdifferential of $f$ is a map

$$\partial f : E \mapsto 2^{E^*}$$

defined by

$$\partial f(x) : = \{x^* \in E^* : f(y) \geq f(x) + \langle y - x, x^* \rangle; \ \forall y \in E\}$$

Recall that two elements $x, y \in E$ are said to be orthogonal if $\forall \alpha > 0$ the following inequality holds:

$$\|x\| \leq \|x + \alpha y\|$$

which is equivalent to: $\exists J_p(x) \in J_p(x)$ such that the following inequality holds

$$\langle y, J_p(x) \rangle \geq 0$$

Recall also that if $K \subset E$ is proximal then $\forall x^* \in K, x \in E$ and $z \in P_K x$ we have that $x^* - z$ and $z - x$ are orthogonal.

The following lemmas shall be used in the sequel.

Lemma 1 Let $E$ be a real Banach space and let $K$ be a proximal subset of $E$. Suppose that $P_K : E \mapsto K$ is the proximal projection operator on $K$. Then, $\forall x \in E$ and $\forall x^* \in K$ the following inequality holds:

$$\|z - x^*\| \leq \|x - x^*\|; \ z \in P_K x$$ (5)
Proof Since \( J_p = \partial f \), where \( f \) is as defined above, then \( \forall u, w \in E \) we have that
\[
f(w) - f(u) \geq \langle w - u, j_p(u) \rangle, \quad j_p(u) \in J_p(u)
\]
so that
\[
\|u\|^p \leq \|w\|^p + p\langle u - w, j_p(u) \rangle
\]
If we now set \( u := x + y \) and \( w := x \) we obtain that \( \forall x, y \in E \) the following inequality holds:
\[
\|x + y\|^p \leq \|x\|^p + p\langle x + y, j_p(x + y) \rangle, \quad j_p(x + y) \in J_p(x + y)
\]
Now, let \( x^* \in K, x \in E \) and \( z \in P_Kx \) be arbitrarily chosen. Since \( x^* - z \) and \( z - x \) are orthogonal then the following inequality holds:
\[
\langle z - x, j_p(x^* - z) \rangle \geq 0
\]
We, therefore, have the following estimates.
\[
\|x^* - z\|^p = \|x^* - x + x - z\|^p \leq \|x^* - x\|^p + p\langle x - z, j_p(x^* - z) \rangle \leq \|x^* - x\|^p
\]
which completes the proof.

Lemma 2 Let \( \{\delta_n\}_{n \geq 0} \subseteq [0, 1] \) and \( \{\sigma_n\}_{n \geq 0} \) be real sequences such that \( \sum_{n \geq 0} \delta_n = \infty \) and \( \sigma_n = o(\delta_n) \). Suppose that \( \{\Psi_n\}_{n \geq 0} \) is a sequence of nonnegative real numbers satisfying the following condition:
\[
\Psi_{n+1} \leq (1 - \delta_n)\Psi_n + \sigma_n
\]
Then \( \lim_{n \to \infty} \Psi_n = 0 \).

Proof Since \( \sigma_n = o(\delta_n) \) then \( \exists \lambda_n \) such that \( \sigma_n = \delta_n \lambda_n, \forall n \geq 0 \) and \( \lambda_n \to 0 \) as \( n \to \infty \). Given any \( \varepsilon > 0 \) there exists an integer \( N > 0 \) such that \( \lambda_n \leq \varepsilon, \forall n \geq N \). Moreover, for any \( k > N \) we have that
\[
\prod_{j=k}^n (1 - \delta_j) \leq \exp\{-\sum_{j=k}^n \delta_j\} \to 0 \quad \text{as} \quad n \to \infty
\]
and
\[
\sum_{j=k}^n \delta_j \prod_{i=j+1}^n (1 - \delta_i) \leq 1 \quad \forall n
\]
now inducting down from \( n \) to \( k \) in (??) yields
\[
0 \leq \Psi_{n+1} \leq \prod_{j=k}^n (1 - \delta_j) \Psi_k + \sum_{j=k}^n \left[ \delta_j \prod_{i=j+1}^n (1 - \delta_i) \right] \lambda_j
\]
from which we easily deduce the following relation
\[
0 \leq \liminf_{n \to \infty} \Psi_n \leq \limsup_{n \to \infty} \Psi_n \leq \varepsilon
\]
Since \( \varepsilon \) is arbitrary, the result follows and the proof is complete.

Lemma 1 was proved in [8] and Lemma 2 was proved in [27]. The details are included here for completeness.
3 Main Theorems

3.1 Iteration of Fixed Points of Strong Pseudocontractions

**Theorem 1** Let $E$ be an arbitrary real Banach space and let $K$ be a proximal subset of $E$ with a nonempty interior $K^o$. Let $T : K \rightarrow E$ be a uniformly continuous strongly pseudocontractive mapping with a fixed point $x^* \in K^o$. Then there exist a real number $\gamma_0 > 0$ and a neighbourhood $B$ of $x^*$ such that the sequence $\{x_n\}_{n \geq 0}$ generated from an arbitrary $x_0 \in B$ by

$$y_n \in P_K w_n; \quad w_n := (1 - \beta_n)x_n + \beta_nTx_n; \quad n \geq 0$$  

(7) $$x_{n+1} \in P_K z_n; \quad z_n := (1 - \alpha_n)x_n + \alpha_nTy_n; \quad n \geq 0$$  

(8)

remains in $B$ and converges strongly to $x^*$ provided that the real sequences $\{\alpha_n\}_{n \geq 0}, \{\beta_n\}_{n \geq 0} \subseteq [0, \gamma_0]$ satisfy the following conditions

(i) $\lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \beta_n$ and (ii) $\sum_{n \geq 0} \alpha_n = \infty$  

(9)

**Proof** Observe that $(I - T)$, being accretive, is locally bounded at each point of its effective domain. Hence, there exists a real number $r > 0$ such that the ball $B = B_r(x^*) := \{x \in E : \|x - x^*\| \leq r\} \subset K$ and $(I - T)(B)$ is bounded. Let $d = diam.(I - T)(B)$. Let $k = (t - 1)/t$ where $t > 1$ is the constant appearing in (??). Since $T$ is uniformly continuous then given $\varepsilon = \frac{kr}{6} > 0 \ \exists \delta > 0$ such that

$$\|x - y\| \leq \delta \Rightarrow \|Tx - Ty\| \leq \frac{kr}{6} = \varepsilon$$

Choose any $0 < \delta \leq \delta_\varepsilon$ and define

$$\gamma_0 := \min \left\{ \frac{1}{2\delta}, \frac{kr}{6(3 - k)d} \right\}$$

With this $\gamma_0$ define the sequence $\{x_n\}$ as given in (??) and (??). Observe that, in view of Lemma 1, the following estimates hold:

$$\|x_{n+1} - y_n\| \leq \|z_n - w_n\| \leq \alpha_n \|x_n - Ty_n\| + \beta_n \|x_n - Tx_n\|$$

$$\|y_n - x_n\| \leq \|w_n - x_n\| = \beta_n \|x_n - Tx_n\|$$

$$\|x_{n+1} - x^*\| \leq \|z_n - x^*\|$$

$$\|y_n - x^*\| \leq \|w_n - x^*\|$$

To keep notations simple, let $S := (I - T - kI)$ and observe that $S$ is accretive. We now prove that $x_n \in B \ \forall n \geq 0$. To do this, we first establish that $x_n \in B \implies y_n \in B$. Suppose that $x_n \in B$. Then,

$$\|(1 + \beta_n)(y_n - x^*) + \beta_n(Sy_n - Sx^*)\| = \|1 + (2 - k)\beta_n\| \|y_n - x^* - \beta_n(Ty_n - x^*)\|$$

$$\leq \|x_n - x^*\| + (1 - k)\beta_n \|x_n - x^*\|$$

$$+ (3 - k)\beta_n^2 \|x_n - Tx_n\| + \beta_n \|Ty_n - Tx_n\|$$

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so that
\[ \|x_n - x^*\| \geq (1 + \beta_n)\|y_n - x^*\| + \frac{\beta_n}{1 + \beta_n}(Sy_n - Sx^*) - (1 - k)\beta_n\|x_n - x^*\| \\
- (3 - k)\beta_n^2\|x_n - Tx_n\| - \beta_n\|Ty_n - Tx_n\| \]
\[ \geq (1 + \beta_n)\|y_n - x^*\| - (1 - k)\beta_n\|x_n - x^*\| - (3 - k)\beta_n^2\|x_n - Tx_n\| \\
- \beta_n\|Tx_n - Ty_n\| \]
But
\[ \|x_n - y_n\| \leq d\beta_n \leq d\gamma_0 \leq \frac{\delta_\varepsilon}{2} < \delta_\varepsilon \]
so that
\[ \|Tx_n - Ty_n\| \leq \frac{kr}{6} \]
Hence
\[ \|y_n - x^*\| \leq \left[ 1 + (1 - k)\beta_n \right]\|x_n - x^*\| + (3 - k)\beta_n^2\|x_n - Tx_n\| + \beta_n\|Tx_n - Ty_n\| \]
\[ \leq (1 - k\beta_n + k\beta_n^2)\|x_n - x^*\| + (3 - k)\beta_n^2\|x_n - Tx_n\| + \beta_n\|Tx_n - Ty_n\| \]
\[ \leq (1 - k\beta_n + k\beta_n^2)r + (3 - k)d\beta_n^2 + \frac{kr}{6}\beta_n \]
\[ = r - \beta_n \left\{ kr - kr\beta_n - (3 - k)d\beta_n - \frac{kr}{6} \right\} \]
\[ \leq \left( 1 - \frac{k}{2}\beta_n \right)r < r \]
Thus, the assertion holds. Now, by the choice of the initial guess, \(x_0 \in B\). Suppose that \(x_n \in B\) (and hence, \(y_n \in B\)). We now prove that \(x_{n+1} \in B\).
\[ \| (1 + \alpha_n)(x_{n+1} - x^*) \| + \alpha_n(Sx_{n+1} - Sx^*) \| = \| [1 + (2 - k)\alpha_n](x_{n+1} - x^*) - \alpha_n(Tx_{n+1} - x^*) \| \\
\leq \|x_n - x^*\| + (1 - k)\alpha_n\|x_n - x^*\| \\
+ (3 - k)\alpha_n^2\|x_n - Ty_n\| + \alpha_n\|Tx_{n+1} - Ty_n\| \]
so that
\[ \|x_n - x^*\| \geq (1 + \alpha_n)\|x_{n+1} - x^* + \frac{\alpha_n}{1 + \alpha_n}(Sx_{n+1} - Sx^*)\| - (1 - k)\alpha_n\|x_n - x^*\| \\
\geq (1 + \alpha_n)\|x_{n+1} - x^*\| - (1 - k)\alpha_n\|x_n - x^*\| - (3 - k)\alpha_n^2\|x_n - Ty_n\| \\
- \alpha_n\|Tx_{n+1} - Ty_n\| \]
But
\[ \|x_{n+1} - y_n\| \leq d(\alpha_n + \beta_n) \leq \delta_\varepsilon \]
so that, by the uniform continuity of \(T\), we have that
\[ \|Tx_{n+1} - Ty_n\| \leq \frac{kr}{6} \]
Thus,
\[
\|x_{n+1} - x^*\| \leq \left[1 + \frac{(1-k)\alpha_n}{1+\alpha_n}\right]\|x_n - x^*\| + (3-k)\alpha_n^2\|x_n - Ty_n\|
+ \alpha_n\|Tx_{n+1} - Ty_n\|
\leq (1 - k\alpha_n + k\alpha_n^2)\|x_n - x^*\| + (3-k)\alpha_n^2\|x_n - Ty_n\|
+ \alpha_n\|Tx_{n+1} - Ty_n\|
\leq (1 - k\alpha_n + k\alpha_n^2)r + (3-k)d\alpha_n^2 + \frac{kr}{6}\alpha_n
\leq r - \alpha_n\left\{kr - k\alpha_n - (3-k)d\alpha_n - \frac{kr}{6}\right\}
\leq \left(1 - \frac{k}{2}\alpha_n\right)r < r
\]

By the inductive hypothesis, the sequence \{x_n\} remains in B. Now, let
\[
\Psi_n := \|x_n - x^*\|
\delta_n := k\alpha_n
\sigma_n := \alpha_n[kr\alpha_n + (3-k)d\alpha_n + \|Tx_{n+1} - Ty_n\|]
\]
so that we now have the following inequality
\[
\Psi_{n+1} \leq (1 - \delta_n)\Psi_n + \sigma_n
\]
which is precisely the inequality (??) so that a routine application of Lemma 2 yields that \(x_n \to x^*\) as \(n \to \infty\). This completes the proof.

**Theorem 2** Let \(E\) be a real reflexive Banach space and let \(T : D(T) \subset E \mapsto E\) be a strongly pseudocontractive mapping with open domain \(D(T)\). Suppose, further, that \(T\) is locally uniformly continuous and has a fixed point \(x^* \in D(T)\). Then there exists a closed neighbourhood \(K\) of \(x^*\) such that the sequence \(\{x_n\}\) generated from an arbitrary \(x_0 \in K\) by (??) and (??) converges strongly to \(x^*\) provided the real sequences \(\{\alpha_n\}, \{\beta_n\} \subset [0,1)\) satisfy the conditions (??).

**Proof** Since \((I - T)\) is accretive, then there exists a closed neighbourhood \(B_1\) of \(x^*\) such that \(B_1 \subset D(T)\) and \((I - T)(B_1)\) is bounded. Also, since \(T\) is locally uniformly continuous, then there exists a closed neighbourhood \(B_2\) of \(x^*\) such that \(B_2 \subset D(T)\) and \(T\) is uniformly continuous on \(B_2\). Let \(K = B_1 \cap B_2\) and observe that \(K\) is a proximal subset of \(E\) since it is closed, convex and nonempty. Thus, the sequence \(\{x_n\}\) generated by (??) and (??) is well-defined and remains in \(B\). Mimicking the relevant computations in the proof of Theorem 1 easily yields an inequality of the form (??) from which the strong convergence follows routinely by applying Lemma 2. This completes the proof.

### 3.2 Iterative Solution of the Equation \(Tx = f\)

**Theorem 3** Let \(E\) be an arbitrary real Banach space and let \(K\) be a proximal subset of \(E\) with \(K^o \neq \emptyset\). Let \(T : K \mapsto E\) be a uniformly continuous strongly accretive mapping
such that the equation $Tx = f$ has a solution $x^* \in K^o$ for each $f \in E$. Then, there exists a real number $\gamma_0 > 0$ and a neighbourhood $B$ of $x^*$ such that the sequence $\{x_n\}_{n \geq 0}$ iteratively generated from an arbitrary $x_0 \in B$ by

$$y_n \in P_K w_n; \quad w_n := (1 - \beta_n)x_n + \beta_n(f + x_n - Tx_n), \quad n \geq 0$$

$$x_{n+1} \in P_K z_n; \quad z_n := (1 - \alpha_n)x_n + \alpha_n(f + y_n - Ty_n), \quad n \geq 0$$

remains in $B$ and converges strongly to $x^*$ provided that the sequences $(\alpha_n)_{n \geq 0}, (\beta_n)_{n \geq 0} \subset [0, \gamma_0]$ satisfy the conditions (??).

**Proof** Observe that the mapping $A = (I - T)$ is a uniformly continuous strongly pseudocontractive mapping. Theorem 1, therefore, applies and completes the proof.

**Theorem 4** Let $E$ be a real reflexive Banach space and let $T : D(T) \subset E \mapsto E$ be a strongly accretive mapping with an open domain $D(T)$. Suppose further that $T$ is locally uniformly continuous and the equation $Tx = f$ has a solution $x^* \in D(T)$ for each $f \in E$. Then, there exists a closed neighbourhood $K$ of $x^*$ such that the sequence $\{x_n\}_{n \geq 0}$ iteratively generated from an arbitrary $x_0 \in K$ by (??) and (??) converges strongly to $x^*$ provided that the real sequences $(\alpha_n), (\beta_n) \subset [0, 1)$ satisfy the conditions (??).

**Proof** This follows from Theorem 2 once it is observed that the mapping $A = (I - T)$ is strongly pseudocontractive and locally uniformly continuous.

### 3.3 Iterative Solution of the Equation $x + \lambda Tx = f$

**Theorem 5** Let $E, K$ be as in Theorem 1, let $T : K \mapsto E$ be a uniformly continuous $m$-accretive mapping and let $x^* \in K$ denote the solution to the equation $x + \lambda Tx = f$ for an arbitrary but fixed $f \in E$. Then, there exist a real number $\gamma_0 > 0$ and a neighbourhood $B$ of $x^*$ such that the sequence $\{x_n\}_{n \geq 0}$ iteratively generated from an arbitrary $x_0 \in B$ by

$$y_n \in P_K w_n; \quad w_n := (1 - \beta_n)x_n + \beta_n(f + \lambda Tx_n), \quad n \geq 0$$

$$x_{n+1} \in P_K z_n; \quad z_n := (1 - \alpha_n)x_n + \alpha_n(f + \lambda Ty_n), \quad n \geq 0$$

remains in $B$ and converges strongly to $x^*$ provided that the sequences $(\alpha_n), (\beta_n) \subset [0, \gamma_0]$ satisfy the conditions (??).

**Proof** Observe that $(I + T)$ is a uniformly continuous strongly accretive mapping and that the equation $x + \lambda Tx = f$ has a unique solution $x^* \in K$ since $T$ is $m$-accretive. Theorem 3 now applies and completes the proof.

**Corollary 1** Let $E, K$ be as in Theorem 1, let $T : K \mapsto E$ be a uniformly continuous $m$-dissipative mapping and let $x^* \in K$ denote the solution to the equation $x - \lambda Tx = f$ for an arbitrary but fixed $f \in E$. Then, there exist a real number $\gamma_0 > 0$ and a neighbourhood $B$ of $x^*$ such that starting with an arbitrary $x_0 \in B$ the iterative sequence $\{x_n\}_{n \geq 0}$ generated by

$$y_n \in P_K w_n; \quad w_n := (1 - \beta_n)x_n + \beta_n(f + \lambda Tx_n), \quad n \geq 0$$

$$x_{n+1} \in P_K z_n; \quad z_n := (1 - \alpha_n)x_n + \alpha_n(f + \lambda Ty_n), \quad n \geq 0$$

converges strongly to $x^*$ provided that the sequences $(\alpha_n), (\beta_n) \subset [0, \gamma_0]$ satisfy the conditions (??).
Theorem 6 Let $E$ be a real reflexive Banach space and let $T : D(T) \subset E \mapsto E$ be an $m$-accretive mapping with an open domain $D(T)$. Suppose further that $T$ is locally uniformly continuous and let $x^*$ denote the solution to the equation $x + Tx = f$ for an arbitrary but fixed $f \in E$. Then, there exists a closed neighbourhood $K$ of $x^*$ such that starting with an arbitrary $x_0 \in K$ the iterative sequence $\{x_n\}_{n \geq 0}$ generated by (??) and (??) converges strongly to $x^*$ provided that the real sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1)$ satisfy the conditions (??).

Corollary 2 Let $E$ be a real reflexive Banach space and let $T : D(T) \subset E \mapsto E$ be an $m$-dissipative mapping with an open domain $D(T)$. Suppose further that $T$ is locally uniformly continuous and let $x^*$ denote the solution to the equation $x - \lambda Tx = f$ for an arbitrary but fixed $f \in E$. Then, there exists a closed neighbourhood $K$ of $x^*$ such that the iterative sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by (??) and (??) converges strongly to $x^*$ provided that the real sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1)$ satisfy the conditions (??).

4 Remarks and Conclusion

1. It is easily observed that by setting $\beta_n \equiv 0$ in the theorems above, it readily follows that iteration processes of the Mann-type converge strongly under the hypotheses of the theorems.

2. It is possible to obtain the additional information of explicit convergence rates for specific choices of the real sequences in the iteration processes above. For instance, choosing $\alpha_n = (n + \gamma_0^{-1})^{-1}$ yields that $\Psi_n = 0(n^{-1})$.

3. The conclusions of the theorems above still hold in the cases where the mapping $T$ is strongly hemicontractive, quasi-accretive or quasi-dissipative.

4. The techniques of our proof - particularly for Theorems 2,4,6 and Corollaries 2 and 4 - easily extend to the case where the mapping $T$ may be locally strongly accretive or locally strongly pseudocontractive, and for the rest of our theorems to the case where the mapping $T$ may be locally uniformly continuous.

5. If $E$ is locally compact then it suffices for the mapping $T$ to be merely locally continuous for the theorems in this paper to hold.

6. The iteration processes in the theorems above are stable in the sense that the introduction of “small” perturbation terms, the so-called error terms, does not affect their convergence properties and asymptotic behaviours. We summarize this in the following corollaries.

Corollary 3 Let $E$ be an arbitrary real Banach space and let $K$ be a proximal subset of $E$ with $K^0 \neq \emptyset$. Let $T : K \mapsto E$ be a uniformly continuous strongly pseudocontractive map with a fixed point $x^* \in K^0$. Then, there exist a real number $\gamma_0 > 0$ and a neighbourhood $B$ of $x^*$ such that the sequence $\{x_n\}$ iteratively generated from an arbitrary $x_0 \in B$ and some $u_0, v_0 \in E$ by

$$y_n \in P_K w_n; \quad w_n := (1 - \beta_n)x_n + \beta_n(Tx_n + u_n), \quad n \geq 0 \quad (16)$$

$$x_{n+1} \in P_K z_n; \quad z_n := (1 - \alpha_n)x_n + \alpha_n(Ty_n + v_n), \quad n \geq 0 \quad (17)$$
remains in $B$ and converges strongly to $x^*$ provided that the real sequences $\{\alpha_n\}, \{\beta_n\} \subseteq \{0, \gamma_0\}$ satisfy the conditions (??) and the sequences $\{u_n\}, \{v_n\}$ satisfy the following conditions

$$(i) \|u_n\|, \|v_n\| \leq \gamma_0 \quad \text{and} \quad (ii) \lim_{n \to \infty} \|v_n\| = 0 = \lim_{n \to \infty} \|u_n\|$$

**Proof** Proceeding as in the proof of Theorem 1, we choose $B, r, d$. Given $\varepsilon = \frac{kr}{8} > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\|x - y\| \leq \delta_\varepsilon \Rightarrow \|Tx - Ty\| \leq \frac{kr}{8} = \varepsilon$$

Choose any $0 < \delta \leq \delta_\varepsilon$ and define

$$\gamma_0 := \min \left\{ \frac{kr}{8}, \frac{\delta}{4d}, \frac{kr}{8(3 - k)d}, \frac{1}{8} \right\}$$

The rest now follows as in the proof of Theorem 1.

**Corollary 4** Let $E$ be a real reflexive Banach space and let $T : D(T) \subset E \rightrightarrows E$ be a strongly pseudocontractive map with open domain $D(T)$. Suppose further that $T$ is locally uniformly continuous and has a fixed point $x^*$. Then there exists a closed neighbourhood $K$ of $x^*$ such that starting from an arbitrary $x_0 \in K$ and some $u_0, v_0 \in E$ the sequence $\{x_n\}$ iteratively generated by (??) and (??) converges strongly to $x^*$ provided that the real sequences $\{\alpha_n\}, \{\beta_n\} \subseteq \{0, 1\}$ satisfy the conditions (??) and the sequences $\{u_n\}, \{v_n\}$ satisfy the following condition

$$\lim_{n \to \infty} \|v_n\| = 0 = \lim_{n \to \infty} \|u_n\|$$

Similar corollaries can easily be obtained for the rest of our theorems.

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References


20. Chika Moore; The solution by iteration of nonlinear equations involving psi-strongly accretive operators, to appear Nonlinear Anal. TMA.


25. S. Reich; Constructing zeros of accretive sets, Applicable Anal. 8 (1979), 349-352.

26. S. Reich; Constructing zeros of accretive sets II, Applicable Anal. 9 (1979), 150-163.

