STRONG CONVERGENCE AND STABILITY OF FIXED POINT ITERATION PROCESSES

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ABSTRACT

Let $K$ be a bounded closed convex nonempty subset of a real uniformly smooth Banach space $E$. Let $T : K \mapsto K$ be a strongly pseudocontractive mapping. It is proved that fixed point iteration processes of the Mann and Ishikawa types converge strongly to the fixed point of $T$ and are $T$-stable. Related results deal with strong convergence and stability of the iteration processes for certain nonlinear operator equations.

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1 Introduction

Let $E$ be a real Banach space and let $T$ be a mapping defined on $E$. Let $x_0 \in E$ be arbitrarily chosen and let

$$x_{n+1} = f(T, x_n) \quad \forall n \geq 0$$

be an iteration process generating the sequence $\{x_n\}_{n \geq 0}$ in $E$. Suppose $T$ has at least one fixed point $x^* \in E$ and $x_n \to x^*$ as $n \to \infty$. Let $\{z_n\}_{n \geq 0} \subseteq E$ be any sequence and set

$$\epsilon_n = \|z_{n+1} - f(T, z_n)\|, \quad n \geq 0$$

The iteration process $\{x_n\}_{n \geq 0}$ is said to be $T$-stable or stable with respect to $T$ if

$$\lim_{n \to \infty} \epsilon_n = 0 \implies \lim_{n \to \infty} \|z_n - x^*\|$$

Several results on the strong convergence and stability of fixed point iteration processes for various classes of mappings have been established (see e.g., [1-12], [14-21] and the references cited therein). In [15], Osilike gave a brief but concise account of the current trend with regard to stability results for fixed point iterations and proved, in $q$-uniformly smooth Banach spaces, results that generalized and extended most known results.

It is our purpose in this paper to prove theorems more general than the ones established in [15]; whereas most of the earlier results have required that the map be Lipschitz continuous, our theorems do not require any continuity condition on the map. The strong convergence results in our theorems are new and interesting in their own right. Moreover, our theorems are established in the more general uniformly smooth Banach spaces.

2 Preliminaries

Let $E$ be a real Banach space with $\dim E \geq 2$. The modulus of smoothness of $E$ is the function $\rho_E : [0, \infty) \mapsto [0, \infty)$ defined by

$$\rho_E(\tau) = \sup \left\{ \frac{1}{2} \left( \|x + \tau y\| + \|x - \tau y\| \right) - 1 : x, y \in E, \|x\| \leq 1, \|y\| \leq 1 \right\}$$

The Banach space $E$ is said to be uniformly smooth if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$, and for $q > 1$, is said to be $q$-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c \tau^q$. Typical of the $q$-uniformly smooth spaces are the sequence spaces $\ell^p$, the Lebesgue spaces $L^p$ and the Sobolev spaces $W^{m,p}$ for $1 < p < \infty$. Clearly, every $q$-uniformly smooth space is uniformly smooth.

Let $E^*$ denote the dual space of $E$ and let $\langle \cdot, \cdot \rangle$ denote the generalized duality pairing. Then $J : E \mapsto 2^{E^*}$ denotes the normalized duality mapping defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 ; \|f^*\| = \|x\| \} \quad \forall x \in E$$

It is well known that $J$ is single-valued if $E$ is smooth and is uniformly continuous on bounded subsets of $E$ if $E$ is uniformly smooth. In the sequel, we shall denote the single-valued normalized duality map by $j$. 

2
A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be strongly pseudocontractive if there exists a $t > 1$ such that for all $x, y \in D(T)$ and $r > 0$ the following inequality holds:

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

(1)

and is called pseudocontractive if $t = 1$ in the inequality (1). This class of maps has been studied extensively by various authors (see e.g., [1-20], [22] and the references cited therein) and is of interest mainly because of its firm connection with the important class of accretive maps where a map $A$ is said to be accretive if for all $x, y \in D(A)$ and $s > 0$ the following inequality holds:

$$\|x - y\| < \|y - s(Ax - Ay)\|$$

(2)

The map $A$ is said to be strongly accretive if there exists a constant $k \in (0, 1)$ such that $A - kI$ is accretive where $I$ denotes the identity mapping on $E$. A close study of inequalities (1) and (2) easily shows that $T$ is (strongly) pseudocontractive if and only if $A = (I - T)$ is (strongly) accretive. Thus, the mapping theory of accretive operators is closely related to the fixed point theory of pseudocontractive maps. Inequality (1) implies that $\forall A > 0$ the operator $(I + \lambda A)$ is invertible. In general, the range $R(I + \lambda A)$ of $(I + \lambda A)$ is not the whole of $E$ so that the operator $(I + \lambda A)^{-1}$ is not defined on the whole of $E$. If, however, $(I + \lambda A)^{-1}$ is defined on the whole of $E$, that is, $R(I + \lambda A) = E$, then $A$ is called $m$-accretive.

Using the duality mapping and a result of Kato [13], the accretive condition (2) is equivalently given by: for all $x, y \in D(A)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0$$

(3)

and the condition that $T$ be strongly pseudocontractive (1) is equivalently given by: for all $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ and a constant $k \in (0, 1)$ such that

$$\langle (I - T)x - (I - T)y, j(x - y) \rangle \geq k \|x - y\|^2$$

(4)

The following results shall be needed in the sequel.

**Lemma XR [23]:** Let $E$ be a real Banach space. Then $E$ is uniformly smooth if and only if there exist positive constants $c, D$ such that for all $x, y \in E$ the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x) \rangle + D \max\left\{ \|x\|, \frac{\|y\|}{2} \right\} \rho_E(\|y\|)$$

(5)

**Lemma XLW [21]:** Let $\{\Psi_n\}_{n \geq 0}$ be a nonnegative real sequence such that $\Psi_{n+1} \leq (1 - \delta_n)\Psi_n + \sigma_n$ where $\delta_n \in [0, 1], \sum_{n=0}^{\infty} \delta_n = \infty$ and $\sigma_n = o(\delta_n)$. Then, $\Psi_n \to 0$ as $n \to \infty$.

### 3 Main Theorems

**Lemma 1** Let $E$ be a real smooth Banach space. Then for any $x, y \in E$ the following inequalities hold:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle$$

(6)

$$\|x + y\|^2 \geq \|x\|^2 + 2\langle y, j(x) \rangle$$

(7)
**PROOF:** $j$ is the subdifferential $\partial \varphi$ of the convex functional \( \varphi : E \to \mathbb{R}^+ \cup \{0\} \) defined by

\[ \varphi(u) = \frac{1}{2} \|u\|^2; \quad u \in E \]  

(8)

Then, for any \( u, w \in E \) we have that

\[ \varphi(w) - \varphi(u) \geq \langle w - u, j(u) \rangle \]  

(9)

so that on applying (8) to (9) we obtain

\[ \|u\|^2 \leq \|w\|^2 + 2\langle u - w, j(u) \rangle \]  

(10)

If we now set \( u := x + y \) and \( w := y \) in (9) then we obtain the inequality

\[ \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \]

which is precisely the inequality (9). Furthermore, if we set \( u := x \) and \( w := x + y \) in (9) then we would have

\[ \|x + y\|^2 \geq \|x\|^2 + 2\langle y, j(x) \rangle \]

which is exactly the inequality (9). This completes the proof.

In the sequel, \( E \) denotes a real uniformly smooth Banach space and \( K \) denotes a bounded, closed, convex, and nonempty subset of \( E \) unless defined otherwise.

**THEOREM 1** Let \( T : K \to K \) be a strongly pseudocontractive mapping and let \( x^* \in K \) denote its unique fixed point. Let \( \{\alpha_n\}_{n \geq 0}, \{\beta_n\}_{n \geq 0} \) be real sequences satisfying the following conditions

(i) \( 0 \leq \alpha_n; \beta_n < 1; \quad n \geq 0 \)

(ii) \( \lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \beta_n \)

(iii) \( \sum_{n \geq 0} \alpha_n = \infty \)

Starting from an arbitrary \( x_0 \in K \) define the sequence \( \{x_n\}_{n \geq 0} \) by

\[ y_n = x_n - \beta_n(x_n - Tx_n); \quad n \geq 0 \]  

(11)

\[ x_{n+1} = x_n - \alpha_n(x_n - Ty_n); \quad n \geq 0 \]  

(12)

Let \( \{z_n\}_{n \geq 0} \) be an arbitrary sequence in \( K \) and let

\[ w_n := z_n - \beta_n(z_n - Tz_n); \quad n \geq 0 \]

\[ f(T, z_n) := (1 - \alpha_n)z_n + \alpha_nTw_n; \quad n \geq 0 \]

\[ \epsilon_n := \|z_{n+1} - f(T, z_n)\|; \quad n \geq 0 \]

Then,

(i) \( \{x_n\}_{n \geq 0} \) converges strongly to \( x^* \)

(ii) \( \|z_{n+1} - x^*\| \leq \sqrt{\{(1 - 2k\alpha_n)\|z_n - x^*\|^2 + \alpha_n\lambda_n\}} + \epsilon_n \)

where \( \lambda_n \to 0 \) as \( n \to \infty \)

(iii) \( \lim_{n \to \infty} \|z_n - x^*\| = 0 \) if and only if \( \lim_{n \to \infty} \epsilon_n = 0 \)
PROOF: Let \( d_1 = \text{diam}.K, \ d = 2d_1, \ M = 2d^2 \) and \( A = D \max\{2d, \frac{c}{2} \} \). Then, we have the following estimates:

\[
\langle x_n - Ty_n, j(x_n - x^*) \rangle = \langle x_n - y_n, j(x_n - x^*) \rangle + \langle y_n - Ty_n, j(x_n - x^*) \rangle \\
\geq k\beta_n \|x_n - x^*\|^2 + k\|y_n - x^*\|^2 \\
- \|y_n - Ty_n\|.\|j(y_n - x^*) - j(x_n - x^*)\|
\]

\[
\geq k(1 + \beta_n)\|x_n - x^*\|^2 + k\beta_n^2\|x_n - Ty_n\|.\|x_n - x^*\| \\
- \|y_n - Ty_n\|.\|j(y_n - x^*) - j(x_n - x^*)\|
\]

\[
\geq k\|x_n - x^*\|^2 - 2kd_1\beta_n - d\|j(y_n - x^*) - j(x_n - x^*)\|
\]

Now,

\[
\|x_{n+1} - x^*\|^2 = \|x_n - x^* - \alpha_n(x_n - Ty_n)\|^2 \\
\leq \|x_n - x^*\|^2 - 2\alpha_n\langle x_n - Ty_n, j(x_n - x^*) \rangle \\
+ D \max\left\{\|x_n - x^*\| + \alpha_n\|x_n - Ty_n\|, \frac{c}{2}\right\} \rho_E(\alpha_n\|x_n - Ty_n\|) \\
\leq (1 - 2k\alpha_n)\|x_n - x^*\|^2 + M\alpha_n\beta_n + 2d\alpha_n\|j(y_n - x^*) - j(x_n - x^*)\| \\
+ A\alpha_n \left[ \frac{\rho_E(d\alpha_n)}{\alpha_n} \right]
\]

Let

\[
\mu_n = M\beta_n + 2d\|j(y_n - x^*) - j(x_n - x^*)\| + A \left[ \frac{\rho_E(d\alpha_n)}{\alpha_n} \right]
\]

and observe that \( \mu_n \to 0 \) as \( n \to \infty \) so that we now have

\[
\|x_{n+1} - x^*\|^2 \leq (1 - 2k\alpha_n)\|x_n - x^*\|^2 + \alpha_n\mu_n
\]

A straightforward application of Lemma XLW establishes (i).

Now, using similar arguments as adduced above we have

\[
\langle z_n - Tw_n, j(z_n - x^*) \rangle = \langle z_n - w_n, j(z_n - x^*) \rangle + \langle w_n - Tw_n, j(z_n - x^*) \rangle \\
\geq k\beta_n \|z_n - x^*\|^2 + k\|w_n - x^*\|^2 \\
- \|w_n - Tw_n\|.\|j(w_n - x^*) - j(z_n - x^*)\|
\]

\[
\geq k\|z_n - x^*\|^2 - 2kd_1\beta_n - d\|j(w_n - x^*) - j(z_n - x^*)\|
\]

and

\[
\|f(T, z_n) - x^*\|^2 \leq \|z_n - x^*\|^2 - 2\alpha_n\langle z_n - Tw_n, j(z_n - x^*) \rangle \\
+ D \max\left\{\|z_n - x^*\| + \alpha_n\|z_n - Tw_n\|, \frac{c}{2}\right\} \rho_E(\alpha_n\|z_n - Tw_n\|) \\
\leq (1 - 2k\alpha_n)\|z_n - x^*\|^2 + \alpha_n\lambda_n
\]

where

\[
\lambda_n = M\beta_n + 2d\|j(w_n - x^*) - j(z_n - x^*)\| + A \left[ \frac{\rho_E(d\alpha_n)}{\alpha_n} \right]
\]
Clearly, $\lambda_n \to 0$ as $n \to \infty$. Hence,

$$
\|f(T, z_n) - x^*\| \leq \sqrt{\{(1 - 2k\alpha_n)\|z_n - x^*\|^2 + \alpha_n\lambda_n\}}
$$

Now,

$$
\|z_{n+1} - x^*\| \leq \|z_{n+1} - f(T, z_n)\| + \|f(T, z_n) - x^*\| \\
\leq \epsilon_n + \sqrt{\{(1 - 2k\alpha_n)\|z_n - x^*\|^2 + \alpha_n\lambda_n\}}
$$

(16)

This proves (ii). Furthermore,

$$
\epsilon_n = \|z_{n+1} - f(T, z_n)\| \\
\leq \|z_{n+1} - x^*\| + \|f(T, z_n) - x^*\| \\
\leq \|z_{n+1} - x^*\| + \sqrt{\{(1 - 2k\alpha_n)\|z_n - x^*\|^2 + \alpha_n\lambda_n\}}
$$

Clearly, if $z_n \to x^*$ then $\epsilon_n \to 0$ as $n \to \infty$. Conversely, suppose $\epsilon_n \to 0$ as $n \to \infty$. Then, from (??) we have that

$$
\|z_{n+1} - x^*\|^2 \leq (1 - 2k\alpha_n)\|z_n - x^*\|^2 + \lambda_n\alpha_n + \mu(\epsilon_n)
$$

where $\mu(\epsilon_n) = \epsilon_n^2 + 2\epsilon_n\sqrt{\{(1 - 2k\alpha_n)\|z_n - x^*\|^2 + \alpha_n\lambda_n\}}$ and hence the sequence $\{\|z_{n+1} - x^*\|\}$ converges. Moreover, using the inequality (??) we have that

$$
\epsilon_n = \|z_{n+1} - f(T, z_n)\|^2 \\
= ||(1 - \alpha_n)(z_{n+1} - z_n - \alpha_n(Tw_n - z_{n+1}))||^2 \\
\geq (1 - \alpha_n)^2\|z_{n+1} - z_n\|^2 + 2\alpha_n(1 - \alpha_n)(z_{n+1} - Tw_n, j(z_{n+1} - z_n)) \\
\geq (1 - \alpha_n)^2\{\|z_n - x^*\|^2 + 2(z_{n+1} - x^*, j(z_n - x^*))\} \\
\geq 2\alpha_n(1 - \alpha_n)(z_{n+1} - Tw_n, j(z_{n+1} - z_n)) \\
\geq (1 - \alpha_n)^2\{3\|z_n - x^*\|^2 - 2\|z_{n+1} - z_n\|\|z_n - x^*\|\}
$$

Suppose $\|z_n - x^*\| \to \delta > 0$ as $n \to \infty$. Then, there exists an integer $N > 0$, sufficiently large, such that

$$
\|z_n - x^*\| \geq \frac{\delta}{2} > 0, \ \forall n \geq N
$$

Thus, $\forall n \geq N$ we have

$$
\epsilon_n^2 \geq \frac{3}{4}(1 - \alpha_n)^2\delta^2 - 2(1 - \alpha_n)^2\|z_{n+1} - z_n\|\|z_n - x^*\| + 2\alpha_n(1 - \alpha_n)(z_{n+1} - Tw_n, j(z_{n+1} - z_n))
$$

so that taking limits on both sides yields

$$
0 \geq \frac{3}{4}\delta^2
$$

which is a contradiction. Hence, we must have that $\|z_n - x^*\| \to 0$ as $n \to \infty$. This completes the proof.
COROLLARY 1 Let $T : K \mapsto K$ be a strongly pseudocontractive map and let $x^* \in K$ denote its unique fixed point. Let $\{\gamma_n\}_{n \geq 0}$ be a real sequence such that

(i) $0 \leq \gamma_n < 1; \quad \forall n \geq 0$
(ii) $\lim_{n \to \infty} \gamma_n = 0$
(iii) $\sum_{n \geq 0} \gamma_n = \infty$

Starting with an arbitrary initial guess $x_0 \in K$ define the sequence $\{x_n\}_{n \geq 0}$ by

$$x_{n+1} = x_n - \gamma_n(x_n - Tx_n); \quad n \geq 0$$

(17)

Let $\{w_n\}_{n \geq 0}$ be an arbitrary sequence in $K$ and define

$$f(T, w_n) = w_n - \gamma_n(w_n - Tw_n)$$
$$\epsilon_n = \|w_{n+1} - f(T, w_n)\|$$

Then,

(i) $\{x_n\}_{n \geq 0}$ converges strongly to $x^*$
(ii) $\|w_{n+1} - x^*\| \leq \sqrt{\left\{(1 - 2k\gamma_n)\|w_n - x^*\|^2 + \gamma_n\varphi_n\right\} + \epsilon_n}$
where $\varphi_n \to 0$ as $n \to \infty$
(iii) $\lim_{n \to \infty} \|w_n - x^*\| = 0$ if and only if $\lim_{n \to \infty} \epsilon_n = 0$

For the next theorem, we need the following definition.

Definition: Let $C$ be a subset of a Banach space $E$ and let $y \in E$ be an arbitrary but fixed element. The set $C$ is said to be $y$-symmetric if it is invariant under the operation $x \mapsto y - x$.

THEOREM 2 Let $f \in E$ be arbitrary but fixed and let $K$ be $f$-symmetric. Let $T : K \mapsto K$ be an accretive operator such that the equation $x + Tx = f$ has a solution $x^* \in K$. Let $\{\alpha_n\}_{n \geq 0}, \{\beta_n\}_{n \geq 0}$ be the real sequences in Theorem 1 and define $S : K \mapsto K$ by $Sx := f - Tx$. Starting from any initial guess $x_0 \in K$ define the sequence $\{x_n\}_{n \geq 0}$ by

$$y_n = x_n - \beta_n(x_n - Sx_n), \quad n \geq 0$$
$$x_{n+1} = x_n - \alpha_n(x_n - Sy_n), \quad n \geq 0$$

(18)

(19)

Let $\{z_n\}_{n \geq 0}$ be any sequence in $K$ and set

$$w_n = z_n - \beta_n(z_n - Sz_n); \quad n \geq 0$$
$$f(S, z_n) = z_n - \alpha_n(z_n - Sw_n); \quad n \geq 0$$
$$\epsilon_n = \|z_{n+1} - f(S, z_n)\|$$

Then

(i) $\{x_n\}_{n \geq 0}$ converges strongly to $x^*$
(ii) $\|z_{n+1} - x^*\| \leq \sqrt{\left\{(1 - 2k\alpha_n)\|z_n - x^*\|^2 + \alpha_n\lambda_n\right\} + \epsilon_n}$
where $\lambda_n \to 0$ as $n \to \infty$
(iii) $\lim_{n \to \infty} \|z_n - x^*\| = 0$ if and only if $\lim_{n \to \infty} \epsilon_n = 0$
PROOF: Using the constants as defined in the proof of Theorem 1, we have

\[ \langle x_n - Sy_n, j(x_n - x^*) \rangle = \langle x_n - y_n, j(x_n - x^*) \rangle + \langle y_n - Sy_n, j(x_n - x^*) \rangle + \langle y_n - y_n, j(y_n - x^*) \rangle \]

\[ = \beta_n \langle x_n - Sy_n, j(x_n - x^*) \rangle + \langle y_n - Sy_n, j(y_n - x^*) \rangle \]

\[ \geq \beta_n \| x_n - x^* \|^2 + \| y_n - x^* \|^2 - \| y_n - Sy_n \| \| j(y_n - x^*) - j(x_n - x^*) \| \]

so that similar arguments as adduced for Theorem 1 above yield

\[ \| x_{n+1} - x^* \|^2 \leq (1 - 2\alpha_n) \| x_n - x^* \|^2 + \alpha_n \mu_n \]

The rest now follow as in the proof of Theorem 1. This completes the proof.

COROLLARY 2 For an arbitrary but fixed \( f \in E \) let \( K \) be \( f \)-symmetric. Let \( T : K \rightarrow K \) be an accretive mapping such that the equation \( x + Tx = f \) has the solution \( x^* \in K \). Let \( S \) be as defined in Theorem 2. Starting from an arbitrary initial guess \( x_0 \in K \) define the sequence \( \{x_n\}_{n \geq 0} \) by

\[ x_{n+1} = x_n - \gamma_n (x_n - Sx_n); \quad n \geq 0 \] (20)

where \( \{\gamma_n\}_{n \geq 0} \) is the real sequence in Corollary 1. Let \( \{w_n\}_{n \geq 0} \) be any sequence in \( K \) and define

\[ f(S, w_n) = w_n - \gamma_n (w_n - Sw_n), \quad n \geq 0 \]

\[ \epsilon_n = \| w_{n+1} - f(S, w_n) \| \]

Then,

(i) \( \{x_n\}_{n \geq 0} \) converges strongly to \( x^* \)

(ii) \( \| w_{n+1} - x^* \| \leq \sqrt{(1 - 2k\gamma_n) \| w_n - x^* \|^2 + \gamma_n \phi_n} + \epsilon_n \)

where \( \phi_n \rightarrow 0 \) as \( n \rightarrow \infty \)

(iii) \( \lim_{n \rightarrow \infty} \| w_n - x^* \| = 0 \) if and only if \( \lim_{n \rightarrow \infty} \epsilon_n = 0 \)

Remarks:

- It has just come to our knowledge that very recently Chidume [6] proved results similar to part (i), the strong convergence part, of Theorem 1. However, the method of proof and the geometric inequality he used are different from ours. Thus, both results are complementary and may provide a basis for the comparison of the two geometric inequalities, due to Reich [16] and Xu and Roach [23] respectively, valid in uniformly smooth Banach spaces.

- The stability results in our theorems generalize and extend most important known results on the stability of fixed point iterations. For instance, our theorems extend the corresponding results of [16] to the more general class of uniformly smooth Banach spaces and from Lipschitz continuous maps to maps which need not be continuous.
• Our real sequences are simple and can easily be chosen at the beginning of the iteration process. Prototypes of our real sequences are
\[ \alpha_n = (1 + n)^{-a}; \quad \beta_n = (1 + n)^{-b}; \quad a, b > 0; \quad a \leq 1 \]

• Suppose that \( E = \mathbb{R}^n \) for \( n \geq 1 \) and \( f \) is an arbitrary but fixed element of \( E \). Then there exist \( a, b \in E \) with \( a_i \leq b_i \) for each \( i = 1, 2, \ldots, n \) such that \( f = a + b \). This representation is not unique. It is easy to see that the \( E \)-interval
\[ K = [a, b] = \{ x = (x_1, x_2, \ldots, x_n) : a_i \leq x_i \leq b_i; \ i = 1, 2, \ldots, n \} \]
is \( f \)-symmetric.

• We can also establish the stability of the iteration processes by showing that their asymptotic behaviour is not affected by the introduction of “small” perturbation terms. We summarize this in the following corollaries (given below) to our theorems.

• The methods of our proof easily carry over to the case of set-valued maps since we can make suitable single-valued selections under the hypotheses of our theorems.

**COROLLARY 3** Let \( T, x^*, \{\alpha_n\}_{n \geq 0}, \{\beta_n\}_{n \geq 0} \) be as in Theorem 1. Let \( \{u_n\}_{n \geq 0} \) and \( \{v_n\}_{n \geq 0} \) be sequences in \( E \) satisfying the following conditions

(i) \( \|u_n\|, \|v_n\| \leq \gamma_0 \) (a constant),  
(ii) \( \lim_{n \to \infty} \|v_n\| = 0. \)

Then the sequence \( \{x_n\}_{n \geq 0} \) iteratively generated from an arbitrary \( x_0 \in K \) and some \( u_0, v_0 \in E \) by
\begin{align*}
y_n &= x_n - \beta_n(x_n - Tx_n - u_n); \quad \forall n \geq 0 \\
x_{n+1} &= x_n\alpha_n(x_n - Ty_n - v_n); \quad \forall n \geq 0 \tag{21, 22}
\end{align*}

converges strongly to \( x^* \) provided that \( \{x_n\}, \{y_n\} \subset K. \)

**PROOF:** Let \( d_1 = \text{diam}(K), d = d_1 + \gamma_0, M = 4d_1^2 + 3\gamma_0d_1 \) and \( A = D \max\{2d, \frac{M}{2}\}. \) Then we have the following estimates:
\begin{align*}
\langle x_n - Ty_n - v_n, j(x_n - x^*) \rangle &= \langle x_n - y_n, j(x_n - x^*) \rangle + \langle y_n - Ty_n, j(x_n - x^*) \rangle \\
&\quad - \langle v_n, j(x_n - x^*) \rangle \\
&= \beta_n \langle x_n - Tx_n, j(x_n - x^*) \rangle - \langle v_n, j(x_n - x^*) \rangle \\
&\quad + \langle y_n - Ty_n, j(y_n - x^*) \rangle - \langle y_n - Ty_n, j(y_n - x^*) - j(x_n - x^*) \rangle \\
&\geq k(1 + \beta_n)\|x_n - x^*\|^2 - 2k\beta_n\|x_n - Tx_n\|\|x_n - x^*\| \\
&\quad - (1 + 2k)\beta_n\|u_n\|\|x_n - x^*\| - \|v_n\|\|x_n - x^*\| \\
&\quad - \|y_n - Ty_n\|\|j(y_n - x^*) - j(x_n - x^*)\| \\
&\geq k\|x_n - x^*\|^2 - d_1[2kd + (1 + 2k)\gamma_0]\beta_n - d_1\|v_n\| \\
&\quad - d\|j(y_n - x^*) - j(x_n - x^*)\| \\
&\geq k\|x_n - x^*\|^2 - d_1[2kd + (1 + 2k)\gamma_0]\beta_n - d_1\|v_n\| \\
&\quad - d\|j(y_n - x^*) - j(x_n - x^*)\|
\end{align*}
Hence,
\[
\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - 2\alpha_n \langle x_n - Ty_n - v_n, j(x_n - x^*) \rangle \\
+ D \max \{\|x_n - x^*\| + \alpha_n \|x_n - Ty_n - v_n\|, \frac{c}{2}\} \rho_E(\alpha_n \|x_n - Ty_n - v_n\|)
\]
\[
\leq (1 - 2k\alpha_n)\|x_n - x^*\|^2 + M\alpha_n \beta_n + 2d_\alpha \|j(y_n - x^*) - j(x_n - x^*)\| \\
+ 2d_1 \alpha_n \|v_n\| + A\alpha_n \left[\frac{\rho_E(d\alpha_n)}{\alpha_n}\right]
\]
\[(23)\]

Let
\[
\mu_n = M\beta_n + 2d_1 \|v_n\| + 2d \|j(y_n - x^*) - j(x_n - x^*)\| + A \left[\frac{\rho_E(d\alpha_n)}{\alpha_n}\right]
\]
and observe that \(\mu_n \to 0\) as \(n \to \infty\). Then (22) yields
\[
\|x_{n+1} - x^*\|^2 \leq (1 - 2k\alpha_n)\|x_n - x^*\|^2 + \alpha_n \mu_n
\]
which is the inequality (23). This completes the proof.

**COROLLARY 4** Let \(T, x^*\) and \(\{\gamma_n\}_{n \geq 0}\) be as in Corollary 1 and let \(\{u_n\}_{n \geq 0}\) be a sequence in \(E\) such that \(\|u_n\| \to 0\) as \(n \to \infty\). Then, the sequence \(\{x_n\}_{n \geq 0}\) iteratively generated from an arbitrary \(x_0 \in K\) and some \(u_0 \in E\) by
\[
x_{n+1} = x_n - \gamma_n(x_n - Tx_n - u_n); \quad \forall n \geq 0
\]
converges strongly to \(x^*\) provided that \(\{x_n\} \subset K\).

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References


