WEIGHT PYRAMID AND IRREGULAR IRREPS OF $U_q(sl(3))$

AT ROOTS OF UNITY

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ABSTRACT

The description of the $U_q(sl(3))$ irregular irreps at roots of unity is made
geometrically transparent by an arrangement of the standard $SU(3)$ Gel’fand-
(Weyl)-Zetlin (GWZ) basis in a hexagonal pyramid, which is valid for any
$q$ and seems new even for $q = 1$. The pyramid has as a base the standard
hexagon which gives the weight space of the UIRs of $SU(3)$ in the plane
of third component of isospin $I_z$ and hypercharge $Y$, while the third di-
mension of this pyramid is related to the isospin $I$. The upper part of the
pyramid, which is also an hexagonal pyramid representing another $SU(3)$
irrep of smaller dimension, becomes the submodule to be factored out for
roots of unity.

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1 Introduction

This paper is dedicated to L.C. Biedenharn. The paper stems from a detailed study on polynomial realization of arbitrary lowest weight (holomorphic) representations of $U_q(sl(n))$, most explicitly for $n = 3$, initiated by our joint paper with L.C. Biedenharn [?] and further developed in [?] and [?], the latter for the case in which the deformation parameter $q$ is a root of unity.

The representation theory at roots of unity is drastically different from the case of generic parameter $q$ (the latter being similar to the case $q = 1$). We mention only some early references: [?, ?, ?, ?], the review [?], and for later developments we refer to the book [?].

The representations that we shall call irregular are representations of quantum algebras $U_q(G)$, where $G$ is a simple Lie algebra (or a real form), which are parametrized by the same parameters as the finite-dimensional irreducible representations of $G$ (unitary for the compact form of $G$), yet have dimensions smaller than their classical counterparts. The reason is that such representations which are irreducible for generic $q$ (as well as in the classical case) become reducible and indecomposable, and to obtain an irreducible representation one has to factor out additional submodules [?]. These are the representations which we consider in the present paper. For other types of representations that appear at roots of unity, e.g., periodic, semiperiodic, we refer to the recent paper [?] and references therein.

In principle, the theory of the irregular representations is well developed since the character formulae are known, cf., e.g., [?] (for $U_q(sl(3))$), [?]. (Note that the approach of [?] is different from the one in [?, ?, ?] nevertheless the character formulae coincide for the corresponding classes of representations). However, for the applications in physics it is important to know exactly which states remain in the irreducible representation after the additional factorization at roots of unity. This is what we show in the present paper for $U_q(sl(3))$ by arranging the standard $SU(3)$ GWZ basis in a hexagonal pyramid.

For a fixed finite-dimensional representation of $SU(3)$ there is a range of possible roots of unity for which the corresponding $U_q(sl(3))$ representation is irregular [?]. If we take $q = e^{2\pi i/N}$ the irregular representations are characterized by the following inequalities for the representation parameters [?]:

$$1 < \bar{m}_1, \bar{m}_2 < N < \bar{m}_1 + \bar{m}_2 < 2N \quad (1)$$

In the irregular case one has to factorize also the submodule built on the singular vector [?]

$$v^m_s = \sum_{j=0}^{\bar{m}} (-1)^{\bar{m}-j} q^{j/2} q^{-\bar{m}_1} \left( \frac{\bar{m}}{j} \right)_q \frac{[\bar{m}_1 - 1 - j]_q!}{[\bar{m}_1 - 1 - \bar{m}]_q!} (X^-_2)^j (X^+_3)^{\bar{m}-j} (X^+_1)^j v_0 \quad (2)$$

The resulting finite-dimensional representation in the irregular case has
the following dimension [?]:

\[
\dim L_{\bar{m}_1, \bar{m}_2} = d_{\bar{m}_1, \bar{m}_2} - d_{\bar{m}_1', \bar{m}_2'} = \\
= \frac{1}{2}m_1 \bar{m}_2 (m_1 + \bar{m}_2) - \frac{1}{2}m_1' \bar{m}_2' (m_1' + \bar{m}_2') = \\
= \frac{1}{2}(\bar{m}_1 + \bar{m}_2 - N) (2\bar{m}_1 \bar{m}_2 + N(2N - m_1 - \bar{m}_2))
\]

where \( \bar{m}_1' = N - \bar{m}_2 \), \( \bar{m}_2' = N - \bar{m}_1 \). Actually, we may restrict ourselves to representation parameters \( m_i \leq N \), then \( \bar{m}_i = m_i \), and we shall do so henceforth.

Note that inequalities (??) naturally exclude the flat representations of \( SU(3) \), (obtained when \( \min(\bar{m}_1, \bar{m}_2) = 1 \)), and also show that the minimal possible value of \( N \) is \( N = 3 \). If \( N = 3 \) then there is only one irregular irrep, namely with \( m_1 = m_2 = 2 \) and \( \dim = 7 \), [?, ?, ?], which corresponds to the adjoint irrep of \( SU(3) \) with \( \dim = 8 \). (Note that since there are only two states with the same weight, the existence of a singular vector \( v_s^m \) \( (\bar{m} = m_1 + m_2 - N = 1) \), given already in [?], means that in the irreducible subrepresentation all states have different weights, i.e., the irrep is flat. This was further explicated in [?] by giving the weights of the seven states, cf. (2.16).)

2 Weight pyramid of the \( SU(3) \) UIRs

First let us recall some well known facts about the UIRs of \( SU(3) \) which hold also for the (anti-) holomorphic representations of \( SL(3) \), also for the Lie algebras and quantum groups. Fix such a representation, i.e., the non-negative integers \( r_1, r_2 \), so that we have a representation of dimension \( d_{r_1+1, r_2+1} = (1/2)(r_1 + 1)(r_2 + 1)(r_1 + r_2 + 2) \) cf. (??). (Instead of the notation \( m_i \) of the previous section we use in this section the notation \( r_i = m_i - 1 \) which is usual when considering \( SU(3) \) representations.) It is customary to depict the weight lattice of every such irrep in the \((I_z, Y)\) plane. We recall that the notation comes from the popular application in which \( I_z \) is the third component of isospin, and \( Y \) is the hypercharge. The points of the weight diagram form a hexagon, the sides of the hexagon containing alternatively \( r_1 + 1 \), \( r_2 + 1 \) points. (Thus, the hexagon degenerates into a triangle if \( r_1 r_2 = 0 \).) Each point of the weight diagram represents all states with the same weight and differing only by the values of isospin \( I \), for which the corresponding \( I_z \) is admissible. It is also customary to connect all points with the same multiplicity. Then the resulting figure consists of nested hexagons if \( r_1 r_2 \neq 0 \), the most outward one containing the states with multiplicity one, the next inwards - the states with multiplicity two, etc. When \( r_1 r_2 = 0 \) the resulting figure consists of nested triangles; moreover each weight has multiplicity one and that is why such representations are called flat.

Now for our purposes we shall replace this customary weight diagram with a hexagonal pyramid (when \( r_1 r_2 \neq 0 \)) in the following way. We consider now a three-dimensional picture adding also the direction perpendicular to the \((I_z, Y)\) plane. The points in that plane have coordinates, say, \((i_z, y, 0)\). Next we replace each point of the weight lattice of multiplicity \( m \) and coordinate
(i_z, y, 0) by m equally spaced points in direction perpendicular to the (I_z, Y) plane which points have coordinates: (i_z, y, k), k = 0, 1, ..., m − 1. We consider now each point of the so formed pyramid as one state, i.e., each point has also a fixed value of isospin I and there is no multiplicity. From the algebraic formulae given in next section we shall see that for fixed (I_z, Y) the value of isospin I diminishes as k increases.

Thus, we obtain a pyramid of height \( r_0 \equiv \min(r_1, r_2) \).

Consider now the states with coordinates \((i_z, y, k)\) for a fixed \(k\). We shall say that these states form a layer. We note now that by construction each such layer is actually a weight diagram in the \(I_z\) and \(Y\) axis and has the form of a hexagon. Moreover, this hexagon has exactly the form of a standard \(SU(3)\) weight diagram - the difference is that we put only one GWZ state at each site. Of course, it is important how we distribute the states with the same weight and this is what we explain next.

Let us agree, in order to save space, to omit the first row of the standard \(SU(3)\) Gel'fand pattern

\[
\begin{pmatrix}
  r & r_1 & m_{12} & r_{11} & m_{22} & 0 \\
  m_{12} & m_{11} & r_1 & m_{12} & m_{22} & 0 \\
  m_{11} & m_{11} & r_1 & m_{12} & m_{22} & 0 \\
  m_{11} & m_{11} & r_1 & m_{12} & m_{22} & 0 \\
  m_{11} & m_{11} & r_1 & m_{12} & m_{22} & 0
\end{pmatrix}
\]

since we shall work with fixed representation parameters \(r_1, r_2\). Namely, we set:

\[
\begin{pmatrix}
m_{12} & m_{22} \\
m_{11} & m_{11}
\end{pmatrix} = \begin{pmatrix}
r & r_1 \\
m_{12} & m_{22} \\
m_{11} & m_{11}
\end{pmatrix}
\]

We place the GWZ states on our pyramid in the following manner. On the \(k\)-th layer, \(k \leq r_1, r_2\), we put the following states:

\[
\begin{pmatrix}
r-k & r_1 \\
r_1 & r_1+1 \\
k+1 & k
\end{pmatrix} \ldots \begin{pmatrix}
r-k & r_1 \\
r_1 & r_1+1 \\
k+1 & k
\end{pmatrix} \begin{pmatrix}
r-k & k+1 \\
r-k & k+2 \\
r-k & k
\end{pmatrix} \ldots \begin{pmatrix}
r-k & r_1 \\
r_1 & r_1+1 \\
r-k & r_1+1
\end{pmatrix}
\]

Note that \(k = 0\) represents the bottom layer. It contains the lowest weight state \([r_1 0 0]\) in the bottom left corner and the highest weight state \([r r_1]\) in the top right corner of this hexagon. (Of course, these states and the others on the edges of this initial hexagon are with no multiplicity, so their placement is more or less standard.) Clearly, there are \(r_1 - k + 1\) states on the bottom row of the above hexagon, \(r_1 - k + 2\) states on the next row, etc., and \(r - 2k + 1\) states on the middle (longest) row, then the number of states decreases by one, the top row having \(r - k - r_1 + 1 = r_2 - k + 1\) states. If we
sum these we obtain that the number of states in the $k$-th layer presented in (??) is: $N_{r_1,r_2}^k = \frac{1}{2} (r + 1)(r + 2) + r_1r_2 + 3k^2 - 3k(r + 1)$. From this it is easy to see that the number of states on the first $k$ layers is:

$$\sum_{s=0}^{k-1} N_{r_1,r_2}^s = \frac{k}{2} \left(r^2 + 6r + 2r_1r_2 + 2k^2 - 3k(r + 2)\right)$$ (7)

We make now the observation that the latter number is equal to the difference of two $SU(3)$ dimensions:

$$\sum_{s=0}^{k-1} N_{r_1,r_2}^s = d_{r_1+1,r_2+1} - d_{r_1+1-k,r_2+1-k}$$ (8)

i.e., the dimension of the irrep we are considering minus the dimension of an irrep with each representation parameter $r_i$ decreased by $k$. This seems natural since the latter representation has a weight pyramid with bottom layer the $(k + 1)$-th layer of our pyramid. We note here the similarity of the formula (??) with (??), though they have a different meaning and (??) is valid for any $q$ while (??) is meaningful at roots of unity. The point is that, as we shall show in the next section, it is exactly the structure displayed above that is giving an exact and simple description of the irrep with dimension given by (??).

The algebraic description of the weight pyramid is related to a procedure for obtaining all GWZ states starting from the lowest weight state. First, we describe the states on a fixed layer (hexagon), say, the $k$-th one.

Starting from the state in the lower left corner of the hexagon, i.e., $[r_1 \ k]$, we first obtain the states on the south-west edge of the hexagon:

$$X_2^+ [r_1 \ k] = N_2(s,k) \left[ r_1+s \ k \right], \quad s = 0, 1, \ldots, r_2 - k$$

$$N_2(s,k) = \left( \begin{array}{c} b_{s+1} | r_1+1+s | q^2 | r_2 | q^2 | r_1+1-k-s | q^2 \\ r_1+1 | q^2 | r_2-s | q^2 | r_1+1+k-s | q^2 \end{array} \right)^{1/2}$$ (9)

Now, for $\hat{C} \equiv X_1^+ X_2^+ [H_1]_q + X_2^+ X_1^+ [1 - H_1]_q$, we obtain, [?] , the states on the north-west edge of the hexagon:

$$\hat{C}^t (X_2^+)^{r_2-k} [r_1 \ k] = N_2(r_2 - k, k) \hat{C}^t \left[ r-k \ k \right] = N_2(r_2 - k, k) N_3(t) \left[ r-k \ k+t \right]$$

$$t = 0, 1, \ldots, r_1 - k$$

$$N_3(t) = \left( \begin{array}{c} | r-k+1 | q^2 | r-2k+1 | q^2 | r_1-k-1 | q^2 | k+1 | q^2 \\ r-k+1 | q^2 | r-2k+1-1 | q^2 | r_1-k-1 | q^2 \end{array} \right)^{1/2}$$ (10)

Now all the other states of the $k$-th layer are obtained by the action of
the operator \( X_1^+ \) to the states on the edges (??), (??):

\[
(X_1^+)^u (X_2^+)^s \left[ \begin{array}{c} r_2 \\ k \end{array} \right] = N_2(s, k) (X_1^+)^u \left[ \begin{array}{c} r_1 + s \\ k \end{array} \right] = N_1(u, s, k) N_2(s, k) \left[ \begin{array}{c} r_1 + s \\ k + u \end{array} \right],
\]

\[ s = 0, 1, \ldots, r_2 - k, \quad u = 0, 1, \ldots, r_1 - k + s \]

\[
N_1(u, s, k) = \left( \frac{|r_1 + s - k + u| q_1^{r_1 + s}}{|r_1 + s - k - u| q_1} \right)^{1/2}
\]

\[
(X_1^+)^u \hat{C}^n (X_2^+)^{r_2 - k} \left[ \begin{array}{c} r_1 \\ k \end{array} \right] = N_2(r_2 - k, k) N_3(t) (X_1^+)^u \left[ \begin{array}{c} r_1 - k + t \\ k + t \end{array} \right] = N_1^n(u) N_2(r_2 - k, k) N_3(t) \left[ \begin{array}{c} r_1 - k + t \\ k + t + u \end{array} \right],
\]

\[ t = 0, 1, \ldots, r_1 - k, \quad u = 0, 1, \ldots, r_1 - 2k - t, \]

\[
N_1^n(u) = \left( \frac{|r_2 - 2k - t - u| q_1^{r_2 - 2k}}{|r_2 - 2k - t - u| q_1} \right)^{1/2}
\]

Finally, we explain how to obtain the lower-left corner states \( \left[ \begin{array}{c} r_1 \\ k \end{array} \right] \) starting from the lowest weight state \( \left[ \begin{array}{c} r_1 \\ 0 \end{array} \right] \) using again the operator \( \hat{C} \):

\[
\hat{C}^k \left[ \begin{array}{c} r_1 \\ 0 \end{array} \right] = N_3^k(k, r) \left[ \begin{array}{c} r_1 \\ k \end{array} \right], \quad k = 0, 1, \ldots, r_0 = \min(r_1, r_2)
\]

\[
N_3^k(k, r) = \left( \frac{|r_1 + 1 - k| q_1^{r_1 + 1 - k}}{|r_1 + 1 - k| q_1} \right)^{1/2}
\]

We have shown \[?\] that relation (??) can be rewritten in two alternative ways:

\[
N_3^k(k, r) \left[ \begin{array}{c} r_1 \\ k \end{array} \right] = \Pi_{s=1}^k \hat{C}_s \left[ \begin{array}{c} r_1 \\ 0 \end{array} \right] = \sum_{j=0}^k (-1)^{k-j} q_2^{j(r_1 - 1)} \binom{k-j}{q_1} \times \left( X_2^+ \right)^j \left( X_3^+ \right)^{k-j} \left( X_1^+ \right)^j \left[ \begin{array}{c} r_1 \\ 0 \end{array} \right]
\]

\[
\hat{C}_s \equiv X_1^+ [s - 1 - r_1] q_1 + X_2^+ X_1^+ q_2^{(s-1-r_1)} = X_1^+ X_2^+ [s - 1 - r_1] q_1 + X_2^+ X_1^+ [r_1 - s + 2] q_1
\]
(hexagon):

\[
\left\lfloor r_1^+ s \right\rfloor_{k+t+u}^k \quad \Rightarrow \quad (X_1^+)^u (X_2^+)^s \hat{C}^k \left[ r_1^- 0 \right]^t,
\]

\[
s = 0, 1, \ldots r_2 - k, \quad u = 0, 1, \ldots r_1 - k + s;
\]

\[
\left\lfloor \tau - k \quad k+t+u \right\rfloor_{k+t+u}^t \quad \Rightarrow \quad (X_1^+)^u \hat{C}^u (X_2^+)^{r_2-k} \hat{C}^k \left[ r_1^- 0 \right]^t,
\]

\[
t = 0, 1, \ldots r_1 - k, \quad u = 0, 1, \ldots r - 2k - t,
\]

where \( \left[ r_1^- 0 \right]^t \) represents the lowest weight vector. If we identify now \( \left[ r_1^- 0 \right]^t \) with the lowest weight vector \( v_0 \) of the Verma module \( V^\Lambda \) then the states \( \left[ m_{12}^m m_{12}^m \right] \) would represent states in the Verma module. More than that, since the GWZ basis is the basis of the finite-dimensional irrep \( L_\Lambda \) which is a factor-module of the Verma module we may consider the unnormalized GWZ basis states also as states in the factor-module. For further use we denote the unnormalized GWZ basis states in the factor-module by \( \left[ m_{12}^m m_{12}^m \right]^n \), the lowest weight vector by \( \left[ r_1^- 0 \right]^n \), those quantities being related as their primed counterparts in (??).

Alternatively, if we want to have the GWZ states realized as polynomials (in three (complex or real) variables \( x, y, z \)) we first identify the lowest weight state \( \left[ r_1^- 0 \right] \) with the function 1 and then use the following representation constructed in [?]:

\[
\Gamma_3(X_1^+) = x[ r_1 - N_x ]_q \ q^{\frac{1}{4}(N_x - N_y)} + z D_y q^{\frac{1}{4}(r_1 - 2N_y)},
\]

\[
\Gamma_3(X_1^-) = D_x q^{\frac{1}{4}(N_x - N_y)} + y D_z q^{\frac{1}{4}(r_1 - 2N_z)},
\]

\[
\Gamma_3(H_1) = 2N_x - r_1 + N_x - N_y,
\]

\[
\Gamma_3(H_2) = 2N_y - r_2 + N_x - N_x,
\]

\[
\Gamma_3(X_2^-) = D_y q^{\frac{1}{4}N_x},
\]

\[
\Gamma_3(X_2^+) = y [r_2 + N_x - N_y]_q \ q^{-\frac{1}{4}N_x} - z D_x q^{-\frac{1}{4}(2r_1 - r_2 - 2N_x)}
\]

where \( N_t \) is the number operator, i.e., \( N_t k^t = k \ t^t, \ t = x, y, z, \) and \( D_t = (1/t) \ [N_t]_q \) is a \( q \) - difference operator.

Clearly, \( \Gamma_3(X_1^-) \) annihilate the function 1, and \( \Gamma_3(H_t) \ 1 = -r_t \), while the action of any monomial of \( \Gamma_3(X_1^+) \) will produce a polynomial in the variables \( x, y, z \). The explicit formulae expressing the GWZ states as such polynomials involve \( q \)-hypergeometric functions and these formulae were found in [?].

Finally we note the similarity of the second formula (??) with the formula giving the singular vector in (??). It is this similarity that will be exploited in the next section in order to prove the explicit realization of the irregular irreps in terms of GWZ states.
The irregular irreps in terms of GWZ states

We consider the irregular representations characterized by (??), and we restrict the representation parameters \( r_i = m_i - 1 \) s.t.

\[
1 < r_1 + 1, r_2 + 1 < N < r_1 + r_2 + 2 = r + 2 < 2N
\]

With this the relevant singular vectors are

\[
v_i = (X_i^+)\overset{r_i+1}{v_0}, \quad i = 1, 2
\]

\[
\mathcal{P}^\bar{m}(X_1^+, X_2^+, X_3^+) v_0
\]

\[
\mathcal{P}^\bar{m} = \sum_{j=0}^{\bar{m}} (-1)^{\bar{m}-j} q^{2(j-r_1-1)} \left( \frac{\bar{m}}{j} \right) q^{[r_1-j][j]} \times
\]

\[
\times (X_2^+)^j (X_3^+)^{\bar{m}-j} (X_1^+)^j
\]

\[
\bar{m} = r + 2 - N
\]

In order to obtain an irreducible representation we have to factor out the Verma submodule built on these singular vectors, which means for the corresponding factor-module or in a function space realization of the lowest weight representations, to impose the corresponding null (vanishing) conditions. For the unnormalized states this is straightforward, while for the normalized states some redefinitions of the basis is needed when we consider the root of unity case (see [?] for details).

Using the realization of the factor-module in terms of the unnormalized GWZ states as given above, the null conditions following from (??), (??), are written as:

\[
(X_i^+)\overset{r_i+1}{[r_1 \ 0]}'' = 0, \quad i = 1, 2
\]

\[
\mathcal{P}^\bar{m}(X_1^+, X_2^+, X_3^+) \overset{r_1 \ 0}{[r_1 \ 0]}'' = 0
\]

These are fulfilled by construction, the first being valid also for generic \( q \). We also note (using the analog of (??) for the factor-module states and formula (??)) that:

\[
\mathcal{P}^\bar{m}(X_1^+, X_2^+, X_3^+) \overset{r_1 \ 0}{[r_1 \ 0]}'' = \prod_{s=1}^{\bar{m}} \hat{C}_s \overset{r_1 \ 0}{[r_1 \ 0]}'' = \overset{r_1 \ \bar{m}}{[r_1 \ \bar{m}]}'' = 0
\]

We thus see that the second condition in (??) contains more information. It means also that the lower-left corner state \( \overset{r_1 \ \bar{m}}{[r_1 \ \bar{m}]}'' \) on the \( \bar{m} \)-th layer of our pyramid vanishes (as expected in a factor-module of a Verma module) and decouples from the irrep. The vanishing of these states in turn implies the vanishing of the lower-left corner states on the higher layers, i.e., the states \( \overset{r_1 \ k}{[r_1 \ k]}'' \), with \( k > \bar{m} \), which are descendants of the states with \( k = \bar{m} \). This follows from (??).

Clearly, together with the lower-left corner states decouple also the states on their layers, i.e., all states on layers \( k = \bar{m}, \bar{m}+1, \ldots, r_0 \). Thus, we are
left with the states on the first $m$ layers. Their number is given by (??), (??), with $k = \bar{m}$ which then coincides exactly with (??) if we take into account that:

$$
\begin{align*}
\bar{m}_1' &= N - \bar{m}_2 = N - r_2 - 1 = r_1 + 1 - \bar{m} \\
\bar{m}_2' &= N - \bar{m}_1 = N - r_1 - 1 = r_2 + 1 - \bar{m}
\end{align*}
$$

where we have used: $\bar{m} = r + 2 - N$.

Thus, we have obtained the explicit description of the irregular representations of $\mathcal{U}_q(\mathfrak{sl}(3))$ in terms of the GWZ basis. These are the states displayed in (??) for $k = 0, 1, \ldots, \bar{m} - 1 = r + 1 - N$, or their unnormalized analogs.

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**References**


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