ABSTRACT

The duality in the Chalker-Coddington network model is examined. We are able to write down a duality relation for the edge state transmission coefficient, but only for a specific symmetric Hall geometry. Looking for a broader implication of the duality, we calculate the transmission coefficient $T$ in terms of the conductivities $\sigma_{xx}$ and $\sigma_{xy}$ in the diffusive limit. The edge state scattering problem is reduced to solving the diffusion equation with two boundary conditions $(\partial_y - \frac{\sigma_{xy}}{\sigma_{xx}} \partial_x)\phi = 0$ and $[\partial_x + \frac{\sigma_{xy} - \sigma_{xx}}{\sigma_{xx}} \partial_y]\phi = 0$. We find that the resistances in the geometry considered are not necessarily measures of the resistivity and $\rho_{xx} = \frac{W}{I} \frac{h}{R} \frac{h}{T}$ ($R = 1 - T$) holds only when $\rho_{xy}$ is quantized. We conclude that duality alone is not sufficient to explain the experimental findings of Shahar et al., and that the Landauer-Buttiker argument does not render the additional condition, contrary to previous expectation.
I. INTRODUCTION

We consider the transport properties of a two-dimensional (2D) disordered strip connected to two disorder-free leads, as shown in figure 1. In the transverse direction, the system is restricted to a finite width of $W$ by confinement potentials. The entire system is subject to a perpendicular magnetic field. In the leads the edge states are the only current carrying states. Therefore the problem can also be viewed as that of edge states scattering through a disordered region. From the viewpoint of the Landauer-Buttiker formula [1,2], the DC transport properties are determined by the scattering matrix at the Fermi energy. It has been shown that the total transmission or the total back scattering of edge states leads to the quantization of Hall conductance [3–6]; from the bulk point of view the quantization requires the localization of the bulk states. The equivalence between the edge state description and the bulk point of view has generated interesting discussions [3–8]. However, there has been no quantitative analysis as to how the edge state transmission coefficient relates to the bulk conductivity or resistivity. It has been shown [4,5] that the longitudinal resistance $R_{xx}$ of the above system is

$$R_{xx} = \frac{R \ h}{T \ e^2},$$

where $T$ and $R$ are transmission and reflection coefficient of a single edge state. The following relation

$$\rho_{xx} = \frac{L \ R \ h}{W \ T \ e^2}$$

has also been used in a number of articles [5,8,9], where $\rho_{xx}$ is the longitudinal resistivity, $W$ and $L$ are the width and length of the sample respectively. The latter relation is based on the assumption that the resistance is a measure of the resistivity. This assumption is not justified.

Our work is also motivated by the recent experimental and subsequent theoretical work by Shahar et al. [10,11] concerning the duality between phases on either sides of the quantum Hall transitions. Using the Chalker-Coddington network model [12], we are able to obtain a previously speculated duality relation [11] for the transmission coefficient and the longitudinal resistance in the geometry specified above. To decide whether or not the duality relation explains the experiment, which was done in a different geometry, one again needs to resolve the relation between resistance and resistivity.

This paper addresses two issues. First we are concerned with the duality in the Chalker-Coddington model and how this duality manifests itself in the resistance measurement. Secondly we render a microscopic calculation for the edge state transmission coefficient in the diffusive limit in terms of the bulk parameters, the longitudinal and Hall conductivity $\sigma_{xx}$ and $\sigma_{xy}$. We find a non-trivial relation between resistance and resistivity. Although the calculation helps to put a restriction to the implication of the duality, it resolves an independent issue of its own.

Section II serves as a brief review of the quantum linear response theory. There we set up the starting point for the microscopic calculations. We emphasize that for finite-sized systems with phase coherence, all measured quantities are conductances or resistances and are in principle sensitive to the way the measurement is set up. We make a contrast between
the bulk current density and the Landauer-Buttiker scattering point of view. In following both approaches in later calculations, we demonstrate their equivalence.

In section III we express the Hall and longitudinal resistance in terms of the transmission coefficient using the Landauer-Buttiker formula. Our derivation makes explicit the involvement of probes and leads in the measurement. In section IV, we proceed to discuss the duality in the discrete Chalker-Coddington model, which leads to an inverse relation for the longitudinal resistance, but only for samples with reflection symmetry. We also look for the implication of the duality in the continuous limit. There the duality is between the phase with its bare Hall conductivity at $\sigma_{xy}^0$ and the phase at $n - \sigma_{xy}^0$, where $n$ is an integer. (Note: from now on we use $e^2/h$ as the unit for conductivity and $h/e^2$ for resistivity.) The duality relation in the continuous limit translates to a relation between the renormalized conductivity: $\sigma_{xy}(\sigma_{xy}^0) + \sigma_{xy}(n - \sigma_{xy}^0) = n$ (here $\sigma_{xy}(\sigma_{xy}^0)$ denotes the renormalized $\sigma_{xy}$ as a function its bare value $\sigma_{xy}^0$). We show that this duality relation alone is not enough to explain the experimental claim of [10]. To do so an additional constraint between $\sigma_{xx}$ and $\sigma_{xy}$ is required.

In section V we calculate the transmission coefficient and the resistances in the perturbative limit ($\sigma_{xx} > 1$). We show that the ideal leads affect the outcome of the measurement by imposing a boundary condition on the electro-chemical potential and the Hall resistance is thus artificially fixed to a quantized value. We find that only under the condition that the Hall resistivity is quantized the resistance is proportional to resistivity. The edge state scattering problem in the diffusive limit is reduced to a special boundary problem. Its analytical solution, by conformal mapping, is discussed in section VI. We conclude with a discussion on the missing connections between the duality found in experiment and that in existing models for the quantum Hall effects.

II. TWO FORMS OF THE LINEAR RESPONSE THEORY

The quantum mechanical linear response theory for non-interacting electron gas can be put in two forms. In one form one writes the local current density as a functional of the external field:

$$j_\mu(r) = \int d\mathbf{r}' \sigma_{\mu\nu}(r, r') E_\nu(r'),$$

where $\sigma_{\mu\nu}(r, r')$ is the bilocal conductivity and can be expressed in terms of the single particle Green’s function $G^\pm(E) = 1/(E - H \pm i\eta)$ where $H$ is the single particle Hamiltonian. (For a detailed form of $\sigma(r, r')$ see [13].) The relation between the current and the external field is generally non-local and in the presence of magnetic field $\sigma_{\mu\nu}(r, r')$ contains not only Fermi surface contribution, but also contributions from all energies below the Fermi energy.

Due to a set of current conservation constraints [13], the total current in a lead (say, the $i$th lead) to linear order depends only on the voltages in the leads, the $V_j$s:

$$I_i = g_{ij} V_j,$$

where the conductance coefficients, the $g_{ij}s$, are surface integrals of $\sigma(r, r')$ in the $i$th and $j$th lead:
Although the off-Fermi-surface terms in $\sigma(\mathbf{r}, \mathbf{r}')$ contribute to the local current response, they give zero net contribution upon surface integral [13]. As a result the $g_{ij}$s can be written in terms of the scattering matrix at the Fermi energy. One arrives at the Landauer-Buttiker formula [1,2]

$$g_{ij} = -\int \int d\mathbf{S}_i \cdot \sigma(\mathbf{r}, \mathbf{r}') \cdot d\mathbf{S}_j.$$  

(2.3)

where $T_{ij}$ is the total transmission coefficient from the $i$th to the $j$th lead,

$$T_{ij} = Tr \left\{ t_{ij} t_{ij}^\dagger \right\};$$

$t_{ij}$ is the scattering matrix between the states of the $i$th and the $j$th probe and $N$ is the total number of scattering states (or total number of edge states in the present case) at the Fermi-level. The above formula best illustrates the non-local aspect of quantum transport and has been instrumental to our understanding of Anderson localization and to the formulation of the random matrix theory for quasi one-dimensional disorder systems. The DC conductance $g_{ij}$ is well defined for a finite-sized disordered region (embedded in an infinite open system), therefore are good candidates for scaling analysis; moreover, they are directly measurable in experiments. However, the formula is rarely used for microscopic calculations, because it introduces confinements and leads which can be cumbersome to address theoretically, particularly in the presence of magnetic field. Only recently it was understood that the presence of a confinement potential at the edges changes the boundary condition for diffusion from $(\partial_n \phi) = 0$ to $(\partial_n + \frac{\sigma_{xx}}{\sigma_{xx}} \partial_t) \rho = 0$ where $t$ is the tangential and $n$ is the normal direction of the edge [14]. The boundary condition imposed by the perfect 2D leads will be discussed in this paper.

In practice, the conductivity, a concept from classical physics, is still widely used, although for quantum mechanical systems it no longer bears the local interpretation. It can be defined as the ratio between the average current density and the average field. Caution has to be exercised for finite-sized closed systems, where the DC dissipative conductivity $\sigma_{xx}$ is zero due to the discreteness of energy levels. The DC $\sigma_{xx}$ is usually asymptotically defined from AC conductivity $\sigma_{xx}(\omega)$ by taking the system size to infinite before taking the frequency ($\omega$) of the external field to zero. Experimentally, most of the measurements of $\sigma_{xx}$ and $\sigma_{xy}$ are deduced from longitudinal and Hall conductance(resistance) by extending the following classical relation to quantum mechanical systems:

$$g_{xx} = \sigma_{xx} \frac{W}{L}, \quad g_{xy} = \sigma_{xy},$$

(2.5)

where $g_{xx}$ and $g_{xy}$ are combinations of the transmission coefficients. The “conductivities” thus defined are in fact conductances. The conductance or the transmission coefficients depend on a number of factors: the random scattering potential in the disordered region, the confinement potential, the property of the leads and even that of the thermal reservoirs. The result of a measurement is not independent of the way in which the measurement is
done. Without checking that the measurement results are robust against size and geometry variations, one cannot be sure that the conductivity (or resistivity) is obtained. As we will show in section V, even in the perturbative regimes where quantum corrections are small, $R_{xx}$ and $R_{xy}$ exhibit the above Ohmic type of relations only under certain conditions. For the critical regimes between the quantum Hall plateaus, experimental and numerical studies for different geometries have so far come up with a range of values for the critical value of $\sigma_{xx}$ [15], indicating the need for a more careful study of finite-size scaling under different boundary conditions. We would like to address this point in a future study.

III. LANDAUER-BUTTIKER FORMULA FOR RESISTANCE

In this section, we express the longitudinal and Hall resistance of our system in terms of the total transmission coefficient. These Landauer-Buttiker type of expressions have been derived before [4,5] and recently have been evoked by Shahar et al. as a possible vehicle for the understanding of certain duality relations in the quantum Hall effect. Our derivation stresses a peculiarity of the measurement involved.

The system, as it is shown in figure 1, cannot be used to extract both the Hall and longitudinal resistance. To do so, while preserving the simplicity of the S-matrix, one can attach additional voltage probes outside the scattering region. Suppose we attach 4 voltage probes and make a 6-probe Hall bar, as shown in figure 2a. The edge states outside the scattering region go from one probe to the next with probability 1 and all measurement results are related to one parameter, the transmission coefficient $T = Tr\{tt^\dagger\}$, where $t$ is the $N \times N$ transmission matrix of the right-going edge states over the disordered region. The transmission coefficient of the left-going edge states is also $T$ by the unitary requirement of the S-matrix. The only non-zero elements among the $T_{ij}$s are: $T_{21} = T_{34} = T_{43} = T_{54} = T_{16} = N$, $T_{32} = T_{65} = T$, $T_{62} = T_{35} = R$, where $R = N - T$. Passing a total current $I$ in the $x$-direction, between probe 1 and 4, the resulting voltages can be solved (up to a constant shift) using the Landauer-Buttiker formula (2.4), bearing in mind that the total current in the voltage probes is fixed to be zero. The Hall resistance $R_{xy} = (V_3 - V_5)/I = (V_2 - V_6)/I$ in this particular setup is totally dictated by the property of the ideal leads since the voltage probes are across the leads. In the leads, the Fermi energy is pinned to its gap of bulk density of states. In such case, it is easy to show that the integrated current is equal to $N$ multiplied by the voltage difference of the two edges. Therefore,

$$R_{xy} = \frac{1}{N}. \quad (3.1)$$

The longitudinal voltage drop can be shown to be [4,5]

$$R_{xx} = \frac{V_3 - V_2}{I} = \frac{V_5 - V_6}{I} = \frac{1}{NT} \frac{R}{N}. \quad (3.2)$$

We emphasize that while the above expressions hold for every realization of the disorder, they are not expected to hold for other geometries. As we have mentioned, the quantization of $R_{xy}$ is a direct consequence of the measurement arrangement; it does not imply that $\sigma_{xy}$ is also quantized. Note that the quantization of $\rho_{xy}$ implies that $\sigma_{xx}^2 + \sigma_{xy}^2 = N\sigma_{xy}$. Were it
to be true, this would give, for critical $\sigma_{xy} = \sigma_{xy}^c = N - 1/2$, the critical $\sigma_{xx} = \sigma_{xx}^c = \sqrt{2N-1}/2$, which contradicts the current consensus that all integer quantum Hall transitions are the same and that $\sigma_{xx}^c$ should be independent of $N$. This is one more reason why $R_{xx}$ should not be taken as $\sigma_{xx}$ for $N > 1$. (This simple measurement setup is not able to detect whether $N$ Landau levels are coupled by disorder or decoupled from one another in the scattering region, since it is only sensitive to the trace of the transmission matrix. Equilibrium among edges states are re-enforced by the thermal reservoirs attached to the probes.)

IV. DUALITY RELATION FOR THE EDGE TRANSMISSION COEFFICIENT

Recently, studying the critical transitions between the quantum Hall plateaus, Shahar et al. [10,11] found that the longitudinal I-V curves near the critical magnetic field $B_c$ are non-linear and demonstrate a certain reflection symmetry with respect to the linear line at $B_c$; more precisely, the longitudinal voltage and current appear to reverse roles at two filling fractions, $\nu$ and $\nu^d$, on either sides of the quantum Hall transition:

$$\{V_x(\nu^d), I_x(\nu^d)\} = \{I_x(\nu), V_x(\nu)\}, \quad (4.1)$$

with $\nu$ and $\nu^d$ satisfying a definite relation suggestive of charge-flux symmetry in the bosonic view of the quantum Hall effect [10,11]. The authors pointed out that duality between charge and flux in the effective bosonic action can be the explanation of the observed relation (4.1), however there is one catch to this scenario-- it requires the extra condition that the bosonic Hall resistivity remains zero across the phase transition, to which there has been no satisfying explanation. As an alternative the same authors also suggested a fermionic scenario, appealing to the Landauer-Buttiker expression similar to (3.1) and (3.2) (their version is amendable to linear response). Noticing that for $N = 1$, $R_{xx}$ goes to its inverse as $T \rightarrow 1 - T = R$, Shahar et al. propose that (4.1) can be explained within the Landauer-Buttiker framework if the following is true:

$$T(E_c + \Delta E) = 1 - T(E_c - \Delta E). \quad (4.2)$$

We show that the above relation can be the consequence of certain duality embedded in the Chalker-Coddington model for the quantum Hall effect. However, one can only write down the duality relation for the transmission coefficient of a symmetric sample with $L = W$.

The Chalker-Coddington model [12] consists of a lattice of directed links and scattering nodes (see figure 3), representing the semi-classical orbits along the equipotential contours of the random potential and the tunneling among these orbits at the nodes. Each node is described by a $2 \times 2$ scattering matrix with random phases and a fixed probability $T_0$ to scatter to the right and $1 - T_0$ to the left. $T_0$ is a function of the Fermi-energy. In figure 3, we show one such network coupled to two edge states. At fixed Fermi energy, there are two equivalent ways to view the network: as one built out of clockwise guiding center orbits (the white squares) with probability $1 - T_0$ to tunnel to the neighboring orbits, or equivalently, of counter-clockwise orbits (black squares) with tunneling probability $T_0$. (At $T_0 = 1$, the network breaks down to decoupled clockwise orbits and two edge states at the top and bottom; at $T_0 = 0$, it breaks down to decoupled counter-clockwise orbits and two
edge states at the left and right entrance of the sample.) The two states at energies related by \( T(E_f') = 1 - T_0(E_f) \) are dual in the sense that one ensemble can be mapped onto the other, if the system is infinite, by reflection with respect to any of the discrete ridges in the direction of \( \hat{x} \pm \hat{y} \). If the system is finite and of arbitrary shape, the above symmetry is broken by the boundary. However, if the system is a square, two of the reflection axes survive. In this case the reflection along the diagonal brings the white squares onto the back square, and at the boundary, transmission channel becomes the reflection channel. Among the many relations that one can write down between the two states (or two phases), one is the following

\[
\langle T(1 - T_0) \rangle = \langle R(T_0) \rangle. \tag{4.3}
\]

(In fact the distribution of \( T \) at \( 1 - T_0 \) is identical to that of \( R \) at \( T_0 \).) The self-dual point is apparently \( T_0 = 1 - T_0 = T_c = 1/2 \). Chalker and Coddington have made use of this fact in locating the critical point of the model. Linearizing the function \( T_0 = T_0(E_f) \) near \( T_c(E_c) \), \( E - E_c \approx (T_0 - T_c)/T_0'(E_c) \), the above gives equation (4.2) and subsequently

\[
R_{xx}(E_c + \Delta E) = \frac{1}{R_{xx}(E_c - \Delta E)}, \tag{4.4}
\]

for \( (T_0 - T_c)/T_c \ll 1 \). (The above is rigorously true if the random potential distribution is symmetric with respect to positive and negative potentials. In that case it is appropriate to assume \( T_0(E_c + \Delta E) = 1 - T_0(E_c - \Delta E) \). In this case there is no more need for the expansion.) If experiment is done in the geometry we consider, relation (4.4) will result as a simple consequence of the duality between the phase at \( E_f = E_f(T_0) \) and the phase at \( E_f' = E_f(1 - T_0) \).

The experiment by Shahar et al. was done in a Hall bar geometry with the voltage probes placed in the interior of the sample as shown in figure 2b. The S-matrix for the realistic Hall bar is significantly more complicated. Moreover, the argument leading up to relation (4.4) requires the reflection symmetry with respect to the two diagonal axis. There is no apparent reason why it should apply to a realistic Hall bar geometry with no such reflection symmetry. To find the possible connection between the experiment and the duality in the microscopic non-interacting models for the quantum Hall effect, one has to find out: 1) whether or not a more general duality relation can be written which is independent of geometry and boundary conditions; 2) whether or not under certain conditions resistance measured in complicated geometries exhibit tractable scaling laws with some universal coefficients and exponents.

The relation we wrote down for the discrete lattice model is somewhat artificial, since the network is an idealization of the mutually tunneling guiding orbits of arbitrary size and shape. It is then sensible to look for the implication of duality in continuous theories. The network model has been mapped onto the non-linear sigma model with a topological term [17]. The latter was shown by Pruisken et al. to exhibit the appropriate asymptotic scaling property required of the quantum Hall transitions [7]. If the network model is coarse-grained at length scale much larger than the lattice spacing, the clockwise and counter-clockwise orbits are lost, i.e. one cannot tell the difference between the \( T_0 \) and \( 1 - T_0 \) state in the bulk, however differences do show up at the boundaries. Consequently, the bulk characteristic of the two phases, i.e. the diffusion constant or \( a_{xx}^0 \) are the same, while their bare Hall
conductivity $\sigma_{xy}^0$ differ by one quanta. (It has been shown that the bare conductivities of the network model are $\sigma_{xx}^0 = \frac{T_0(1-T_0)}{T_0^2(1-T_0)^2}$ and $\sigma_{xy}^0 = \frac{T_0^2}{T_0^2+(1-T_0)^2}$ [8,19]). Since reflection is the operation that maps the $T_0$ phase to the $1-T_0$ phase, we consider how the non-linear sigma model transforms under the one reflection operation ($x \rightarrow -x$ or $y \rightarrow -y$). The topological term, with $\sigma_{xy}^0$ as its coefficient, changes sign while the $\sigma_{xx}^0$ term remains the same. Since the theory is periodic in $\sigma_{xy}^0$, the corresponding dual phases are those parameterized by $\sigma_{xy}^0$ and $n - \sigma_{xy}^0$, with $n = \ldots, -2, -1, 0, 1, 2, \ldots$ (Note: $\sigma_{xy}^0$ and $n - \sigma_{xy}^0$ correspond to two filling factors or two Fermi-energies). Using the renormalization equations given by Pruisken et al. [7], one can verify the following relation between the renormalized $\sigma_{xx}$, $\sigma_{xy}$

$$
\sigma_{xx}(\sigma_{xy}^0) = \sigma_{xx}(n - \sigma_{xy}^0)
$$

$$
\sigma_{xy}(\sigma_{xy}^0) = n - \sigma_{xy}(n - \sigma_{xy}^0),
$$

if $\sigma_{xx}(\sigma_{xy}^0) = \sigma_{xx}(n - \sigma_{xy}^0)$. The above does not necessarily lead to the desired relation

$$
\rho_{xx}(\sigma_{xy}^0) = \rho_{xx}^{-1}(n - \sigma_{xy}^0).
$$

To obtain the above, one more constraint is required between $\sigma_{xx}$ and $\sigma_{xy}$ at the same $\sigma_{xy}^0$ (filling factor, Fermi energy). For example, the above relation is satisfied if $\sigma_{xy}$ and $\sigma_{xx}$ obey (4.5), as well as the semi-circle relation

$$
\sigma_{xx}^2 + (\sigma_{xy} - 1/2)^2 = 1/4,
$$

which is equivalent to

$$
\rho_{xy} = 1.
$$

(Up to this point, another form of constraint is also possible. See later discussions on this issue.) Therefore, in both the fermionic and the bosonic picture for the quantum Hall effect duality alone is not sufficient to explain the experimental result (4.1). Both approaches require one more constraint: constant or vanishing Hall resistivity. As we explained before and will make more clear in the calculation to follow, the quantization of Hall resistance in our particular setup does not necessarily imply the quantization of Hall resistivity. It has been shown that equation (4.6) or (4.7) holds for a classical version of the network model without phase interference [19,23]. There is also numerical evidence that it holds for the original quantum version of the network model [18]. Theoretically, the reason of the constraint is not clear.

V. RESISTANCE IN TERMS OF THE BARE CONDUCTIVITY, PERTURBATIVE CONSIDERATIONS

We next calculate the transmission coefficients, $R_{xx}$ and $R_{xy}$ in our geometry in terms of $\sigma_{xx}$ and $\sigma_{xy}$ in the perturbative limit. (The Chalker-Coddington model for one Landau level has very small bare conductivity $\sigma_{xx}^0 < 1$, therefore it cannot be treated perturbatively.) We perform our calculation for high Landau levels ($N > 1$) and for the short-ranged random
potential model, of which the bare conductivity $\sigma_{xx}^0$ can be large. The calculation serves to strengthen our view regarding duality in quantum Hall systems, it also has an interest of its own, since the problem of edge state transmission in the multi-scattering, diffusive regime has not been analyzed before. Our treatment is at most phenomenological. Its rational is as follows. In our previous work [19], we have calculated the bilocal conductivity $\sigma_{\mu\nu}(r, r')$ to leading order in $1/\sigma_{xx}^0$. We find that to leading order the non-local relation between current $j_\mu(r)$ and external field $E(r)$ can be decoupled and reduced to the familiar Drude equation (with some modification to be specified later)

$$j_\mu(r) = \sigma_{\mu\nu}^0 E_\nu(r), \quad (5.1)$$

where $E = E + \nabla \mu / e$ is the electro motive field ($\mu$ is the local chemical potential). In the same work we also give the generating function from which one obtains the full quantum mechanical conductance. The effect of the edges can be treated within the non-linear sigma model, with $\sigma_{xx}^0$ and $\sigma_{xy}^0$ as the coupling constants and the presence of the topological term alters the boundary condition for diffusion. In treating the quantum mechanical system as though it was classical, we are assuming that the field theory is renormalizable, i.e., all the quantum corrections to conductance can be accounted for by replacing the bare coupling constants with the renormalized reversion. The assumption so far has met no contradiction at perturbative level [19]. The new ingredient is the discussion on the boundary condition imposed by the 2D perfect leads. We show that the computation of transmission coefficient and of current distribution reduce to solving the Laplace equation with identical boundary conditions.

In reference [19] it has been shown using the self-consistent Born approximation (SCBA) [20], which gives the leading order in $1/\sigma_{xx}^0$, that the bilocal conductivity tensor is of the following form:

$$\langle \sigma_{\mu\nu}(r, r') \rangle = \left[ \sigma_{xx}^0 \delta_{\mu\nu} + (\sigma_{xy}^0 + \sigma_{xy}^{II,0}) \epsilon_{\mu\nu} \right] \delta(r - r')$$

$$- \frac{1}{\sigma_{xx}^0} \left[ \sigma_{xx}^0 \partial_\mu + \sigma_{xy}^0 \epsilon_{\mu\nu} \partial_\nu + \sigma_{xy}^{II,0} \delta_{\nu\sigma} (\delta(y' - W) - \delta(y')) \right]$$

$$\times \left[ \sigma_{xx}^0 \partial_\nu' - \sigma_{xy}^0 \epsilon_{\nu\sigma'} \partial_\sigma' - \sigma_{xy}^{II,0} \delta_{\nu\sigma} (\delta(y' - W) - \delta(y')) \right] d(r, r')$$

$$+ O(1/\sigma_{xx}^0, l/L), \quad (5.2)$$

where $d(r, r')$ is the diffusion propagator satisfying $-\nabla^2 d(r, r') = \delta(r - r')$, $\sigma_{xx}^0$, $\sigma_{xy}^0$, and $\sigma_{xy}^{II,0}$ are the SCBA version of the Streda conductivities [21, 7, 19]. The higher gradient terms are of higher order in $l/L$, where $l$ is the mean free path and $L$ is the system length. The physics implied by the leading order expression is simple. The first term, the contact term, arises, in terms of Drude’s picture for conduction, from electrons accelerating in the combined external field $E - e\nu \times B$. The second term is the diffusion term and it arises from the charge density fluctuation $\delta n_ e$ in response to the external potential, which is long-ranged. A local relation between the current and the electromotive field can be obtained if one introduces an effective local chemical potential $\mu(r)$ to account for the density fluctuation [19]. However, equation (5.2) does not quite recover equation (5.1). One notices that in the diffusive term, $\sigma_{xy}^0$ splits into two parts, $\sigma_{xy}^{II,0}$ which appears in the bulk diffusion current and $\sigma_{xy}^{II,0}$, which is proportional to the edge current density and shows up only at the reflecting
edges \((\sigma_{xy} = \sigma_{xy}^I + \sigma_{xy}^{II})\). It was shown that \(\sigma_{xy}^{II}\) is proportional to the rate at which the total density changes with the magnetic field at fixed Fermi energy \([7]\). In the low field limit \(\omega_c \tau_0 \ll 1\) \((\omega_c\) is the cyclotron radius and \(\tau_0\) is the zero field elastic scattering rate), the Landau levels merge into a continuum and the density of states is only weakly dependent on the magnetic field, therefore the \(\sigma_{xy}^{II}\) term can be ignored. This is not the case in the high field limit with \(\omega_c \tau_0 > 1\), when separate Landau bands are formed, and the density of states oscillates with the field. In general, equation (5.1) should be replaced by \([19]\)

\[
j_\mu = \sigma_{\mu \nu}^0 E_\nu + [\sigma_{\mu \nu}^0 \delta_{\mu \nu} + \sigma_{\mu \nu}^{I0} \epsilon_{\mu \nu}] \partial_\mu \mu / \epsilon + \sigma_{xy}^{II0} [\delta(y - W) - \delta(y)] \delta_{\mu, x} \mu / \epsilon. \tag{5.3}
\]

in which we have included an edge current \(I_e^0 = \sigma_{xy}^{II0} \mu / \epsilon\), which is nothing but the extra edge current induced by an increase in chemical potential.

In the disorder-free leads, \(\sigma_{xy}^I = 0\) and \(\sigma_{xy}^{II} = \sigma_{xy}^{lead} = N\) is quantized, but \(\sigma_{xy}^{II}\) is not always quantized in the disordered region \([7]\). It is apparent that when the edge current in the lead and the sample are different there has to be an edge current along the border with the leads amounting to \(I_e^0 = [\sigma_{xy}^{lead} - \sigma_{xy}^{II0}] \mu / \epsilon\), in order to satisfy current conservation at the corners. Therefore, for the geometry under consideration the local equation is further modified to be

\[
j_\mu = \sigma_{\mu \nu}^0 E_\nu + [\sigma_{\mu \nu}^0 \delta_{\mu \nu} + \sigma_{\mu \nu}^{I0} \epsilon_{\mu \nu}] \partial_\mu \mu / \epsilon + \sigma_{xy}^{II0} [\delta(y - W) - \delta(y)] \delta_{\mu, x} \mu / \epsilon
+(\sigma_{xy}^{lead} - \sigma_{xy}^{II0}) \delta_{\mu, y} [\delta(x - L) - \delta(x)] \mu. \tag{5.4}
\]

To calculate the resistance, there are two approaches. One approach is to find the static electro-chemical potential and subsequently the current distribution, by requiring \(\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t = 0\), in which case, one can write \(\mathbf{E}(\mathbf{r}) = -\nabla \varphi(\mathbf{r})\). The current conservation condition for the static case is \(\nabla \cdot \mathbf{j} = 0\), which gives in the bulk

\[
\nabla^2 (-\varphi + \mu / \epsilon) = 0. \tag{5.5}
\]

At the top and bottom edge, we have shown \([19]\) that current conservation leads to the following boundary condition

\[
\sigma_{xx}^0 \partial_y (-\varphi + \mu / \epsilon) - \sigma_{xy}^0 \partial_x (-\varphi + \mu / \epsilon) = 0. \tag{5.6}
\]

In other words, the electromotive field \(\mathbf{E}\) has to make an angle \(\theta = \tan^{-1} \sigma_{xy}^0 / \sigma_{xx}^0\) with the edge along \(\hat{x}\)-direction:

\[
\frac{\mathbf{E}_y}{\mathbf{E}_x} = \frac{\sigma_{xy}^0}{\sigma_{xx}^0}.
\]

At the left and right border with the leads, one can derive the following boundary condition by constructing a surface that encloses the edge current \(I_e^0\) and imposing current conservation:

\[
j_x - \partial_y I_e^0(y) = j_x^{lead}. \tag{5.7}
\]

In the leads where \(\sigma_{xx} = 0\), one can write down the following equation for the local current density \([22]\):

\[
j_x^{lead} = \sigma_{xy}^{lead} \epsilon_{\mu \nu} E_\nu + \sigma_{xy}^{lead} \delta_{\mu, x} [\delta(y - W) - \delta(y)] \mu / \epsilon. \tag{5.8}
\]
Combining (5.4), (5.7), (5.8), we get

$$\sigma_{xx}^0 \partial_x (-\varphi + \mu/e) + [\sigma_{xy}^0 - \sigma_{xy}^{\text{lead}}] \partial_y (-\varphi + \mu/e) = 0,$$

(5.9)
i.e., \( E \) has to make an angle \( \theta' = \tan^{-1}(\sigma_{xy}^0 - \sigma_{xy}^{\text{lead}})/\sigma_{xx}^0 \) with the border with the leads (along \(-\hat{y}\)-direction):

$$\frac{E_x}{E_y} = -\frac{\sigma_{xy}^0 - \sigma_{xy}^{\text{lead}}}{\sigma_{xx}^0}.$$ 

Thus the problem of finding the resistance to leading order of \( 1/\sigma_{xx}^0 \) is that of solving the Laplacian equation for the electrical chemical potential with two tilted boundary conditions. Notice that as far as the conductance or resistance is concerned, one can arrive at the correct equations by using the simpler but incorrect local relation (5.1).

Alternatively, one can follow the Landauer-Buttiker approach, i.e., one finds the transmission coefficient. This also requires the computation of the bilocal-conductivity tensor. For our simple geometry,

$$T = \int_{S_1} dy_1 \int_{S_2} dy_2 \sigma_{xx}(r_1, r_2)$$

(5.10)

where \( S_1 \) and \( S_2 \) are two the cross-sections at the left and right end of the scattering region. Making use of the SCBA expression (5.2) for \( \sigma_{xx}(r, r') \), we get

$$T = -\frac{1}{\sigma_{xx}^0} \int_0^W dy \int_0^W dy' (\sigma_{xx}^0 \partial_x + \sigma_{xy}^0 \partial_y)(\sigma_{xx}^0 \partial_x' - \sigma_{xy}^0 \partial_y')d(r, r') \big|_{x=0, x'=L}$$

$$= \frac{(\sigma_{xy}^0)^2}{\sigma_{xx}^0} [d(L, W; 0, W) + d(0, 0; L, 0) - d(L, W; 0, 0) - d(L, 0; 0, W)].$$

(5.11)

In reference [19], we have shown that the diffusion propagator satisfies boundary condition (5.6) at the reflecting edges, with \( d \) replacing \(-\varphi + \mu/e\). One can easily check that it also satisfies boundary condition of the form (5.9) at borders with the leads. The above expression can be simplified if one takes into account the additional detailed boundary condition that the returning probability via the incoming links is zero, the coarse-grained diffusive version of which is \( d(L, 0; r) = d(0, W; r) = 0 \), and the fact that transmission probability of left-going and right-going edge states are the same, i.e., \( d(L, W; 0, W) = d(0, 0; L, 0) \). We get

$$T = 2 \frac{(\sigma_{xy}^0)^2}{\sigma_{xx}^0} d(L, W; 0, W)$$

(5.12)

Defining \( \phi(r) = 2 \frac{(\sigma_{xy}^{\text{lead}})^2}{\sigma_{xx}^0} d(r; 0, W) \), \( \phi \) satisfies (5.5), (5.6), (5.9) and, in addition, the following initial condition \( \phi(L, 0) = 0 \), and \( \phi(0, W) = N \), since in the limit \( L \to 0, T \to N \). Thus within the diffusive limit the Landauer-Buttiker approach and the current distribution approach boil down to the same mathematical problem. Since the problem contains non-self-adjoint boundary conditions, its solution in the general case requires a special method, which we will discuss in the next section. However, some conclusions can be made before solving the equation.
Again the fact that the Hall resistance is quantized can be demonstrated using boundary condition (5.9) alone. The current across the border with the leads is $I_x = \int_0^W dy (\sigma_{xx}^0 \mathcal{E}_x + \sigma_{xy}^0 \mathcal{E}_y)$ and the transverse drop in electro-chemical potential at the borders is $\Delta \phi_y = \int_0^W dy \mathcal{E}_y (x, y)_{x=0,L}$. Making use of the relation between $\mathcal{E}_x$ and $\mathcal{E}_y$ from (5.9), the result $R_{xy} = \frac{1}{\sigma_{xx}^0}$ is recovered.

The next conclusion follows from the two boundary conditions. Since the field $\mathcal{E}$ at the reflecting edges and at the borders with the leads is required to point in two specific directions, the field and current distribution is uniform only when the two directions coincide, i.e. when the bare conductivities satisfy the following constraint:

$$\sigma_{xx}^0 \sigma_{xy}^0 - \sigma_{yx}^0 \sigma_{yy}^0 = 0,$$

which is equivalent to

$$\rho_{xy} = \frac{\sigma_{xy}^0}{(\sigma_{xx}^0)^2 + (\sigma_{yx}^0)^2} = \frac{1}{\sigma_{xx}^0 \rho_{xx}}.$$

For $\sigma_{yx}^0 = 1$, the above gives the same semi-circle constraint (4.6) of Ruzin et al. Under condition (5.13), the field distributions takes the simple form:

$$\phi_0(x, y) = -E_0 (x + \sigma_{xy}^0 \sigma_{yx}^0 y), \quad E_0(x, y) = E_0 (\hat{x} + \sigma_{xy}^0 \hat{y}),$$

where $E_0$ is a constant. The total current, the transverse and longitudinal electro-chemical potential differences can be easily calculated. Indeed $R_{xx, SCBA} = \frac{L}{W} \rho_{xx}$.

The $\sigma_{xx}^0$ and $\sigma_{xy}^0$ of the Chalker-Coddington model satisfies the above. However such bare values were computed by forcefully leaving out the quantum interference of the original model [19]. Since $\sigma_{xx}^0 < 1$, the model has no perturbative regime, i.e., there does not exist a length scale within which these bare values are the leading contributions. For the Gaussian white noise potential model $\sigma_{xx}^0 = (2N - 1) \pi^{-1} \sin^2 \alpha$ and $\sigma_{xy}^0 = N - 1 + \alpha / \pi$, where $\alpha$ is a function of the Fermi energy (we do not need its expression here, see [7]), they do not satisfy the constraint. In this case the field and current distribution is not uniform, $R_{xx}$ is not proportional to $\rho_{xx}$ and $R_{xy}$ is not equal to $\rho_{xy}$.

As we mentioned before, the geometry we consider (figure 2a) does not correspond to those used in the experiments, we would like to know how $R_{xx}$ and $R_{xy}$ change if we move the voltage probes into the scattering region as shown in figure 2b. Assuming that the voltage probes are very small in dimensions (much smaller than the sample length $L$ and width $W$), we can treat the probes as perturbations in boundary conditions and get a rough picture by looking at the field distribution in the main strip in the absence of these probes. If condition (5.13) is met and the field and current distribution is uniform, the electro-chemical potential drop transverse to the current is the same everywhere along the strip and the longitudinal potential difference is in proportion with the distance between the probes. Such systems have a quantized Hall resistivity, as is the case with the Chalker-Coddington network with no phase coherence or a random version of it as discussed by [23]. In the general case when the field is not uniform, the Hall resistance departs from the quantized value as we move the voltage probes to the interior of the sample. However if the Hall bar is long and narrow ($L \gg W$), the field distribution in the interior far from the ends takes the form of equation
(5.14), forced by boundary condition (5.6). If the probes are placed in the region where the
field is nearly uniform, \( R_{xx} \) and \( R_{xy} \) are roughly proportional to the intrinsic \( \rho_{xx}^0 \) and \( \rho_{xy}^0 \).
In this way the effect of the perfect leads is removed.

VI. ANALYTICAL SOLUTION OF THE BOUNDARY PROBLEM

To solve the boundary problem we use the method of conformal mapping, which was first
used by Girvin and Rendell [24] to find the two-probe conductance in the case of absorbing
leads. The idea is as follows. Let us consider a parallelogram in the \( z' \)-plane \( (z' = x' + iy') \),
with its top and bottom edge parallel to \( x' \)-axis and its left and right side tilted from the
vertical direction by \( \delta \theta = \theta - \theta' + 90 \). In this geometry, the uniform field distribution of (5.14)
satisfies both boundary conditions. The solution for the rectangle geometry in the \( z \)-plane
\( (z = x + iy) \) can be found if one finds a conformal mapping \( z' \to z \) which transforms the
parallelogram to the rectangle. We start by writing down the complex potential \( \phi(z') \) which
can give the correct physical field \( \mathcal{E}_0(x',y') \). The field is related to the complex potential through
\[
-\frac{d\phi}{dz} = \mathcal{E}_y + i\mathcal{E}_x.
\]
It is easy to see that the following complex potential
\[
\phi(z') = -iE_0e^{-i\theta}z'. \tag{6.1}
\]
gives \( \mathcal{E}_0(x',y') \) in the \( z' \)-plane. The conformal transformation from \( z \)-plane to the \( z' \)-plane
rotates the angle at \( x = 0, L \) by \( \delta \theta \), i.e., the rotation defined by
\[
\frac{dz'}{dz} = e^{f(z)} \tag{6.2}
\]
has the boundary conditions
\[
Im[f(z)] = \delta \theta, \text{ for } x = 0, L; \quad Im[f(z)] = 0 \quad \text{for } y = 0, W. \tag{6.3}
\]
The transformation has been worked out by Girvin and Rendell to be
\[
f(z) = \sum_{n=odd} \frac{4\delta \theta}{n\pi} \{ \sinh[n\pi(z - L/2)/W]/\cosh(n\pi L/2W) \}. \tag{6.4}
\]
The potential drops are:
\[
\Delta \phi_x = -E_0\cos(\theta)\int_0^L dx \exp \left[ \sum_{n=odd} \frac{4\delta \theta}{n\pi} \sinh[n\pi(x - L/2)/W]/\cosh(n\pi L/2W) \right], \tag{6.5}
\]
\[
\Delta \phi_y = -E_0\sin(\theta - \delta \theta)\int_0^W dy \exp \left[ \sum_{n=odd} \frac{4\delta \theta}{n\pi} \tanh(n\pi L/2W) \cos(n\pi y/W) \right], \tag{6.6}
\]
The transmission coefficient is
\[ T = \phi(L, W) = N \frac{\Delta \phi_y}{\Delta \phi_x + \Delta \phi_y} \quad (6.7) \]

In the limit \( W = L \), one can show that the two integrands in (6.5) and (6.6) are equal, the expression simplifies to be

\[ T_{L=W} = N \frac{\sin(\theta - \delta \theta)}{\cos \theta + \sin(\theta - \delta \theta)} = N \frac{-\cos(\theta')}{\cos \theta - \cos(\theta')}. \quad (6.8) \]

The resistance \( R_{xx} \) can be obtained by either plugging the above into the Landauer-Buttiker expression (3.2) or by calculating the total current from the field distribution, then dividing \( \Delta \phi_x \) by it.

**VII. CONCLUSIONS AND DISCUSSIONS**

Prompted by the experimental evidence of duality near quantum Hall transitions, we probe the implication of duality in the Chalker-Coddington model within the fermionic non-interacting picture. For finite systems with boundaries, a duality relation holds for the edge state transmission coefficient, however, only for a class of systems with reflection symmetry. For such systems duality leads to measurable inverse relations of the longitudinal resistance of two states at opposite sides of a quantum Hall transition. In a general situation (for arbitrary geometry) the duality translates to a diluted version

\[ \sigma_{xy}(a^0) + \sigma_{xy}(n - a^0) = n. \]

Our key difference with previous authors on the subject [11] is that we think that the Landauer Buttiker argument constructed for resistance in symmetric geometry cannot be expected to hold in other geometries, such as a Hall bar, since our calculations show that resistance is not necessarily resistivity. If the experiment of Shahar et al. is correct and the reversal of the role of longitudinal voltage and current is indeed observed in the quantum scaling regime, it points to the possibility that the renormalized \( \sigma_{xx} \) and \( \sigma_{xy} \) satisfy the constraint (4.6) first suggested by the numerical work of Ruzin et al. This has so far not been understood within any theoretical framework.

Our work unravels the mystery surrounding the Landauer-Buttiker argument. We performed a perturbative calculation for the edge state transmission coefficient in terms of the conductivities and analyze the current distribution. The presence of the 2D perfect leads gives rise to a tilted boundary condition for diffusion, which is similar and dual to the one at the reflecting edges, (dual in the sense that one boundary condition becomes the other as \( \sigma_{xy}^0 \rightarrow n - \sigma_{xy}^0 \)). Only when \( \sigma_{xx} \) and \( \sigma_{xy} \) satisfy the semi-circle constraint (equivalent to the quantization of \( \rho_{xy} \)), the current distribution is uniform and the resistance \( R_{xx,xy} \) is in proportion to the resistivity \( \rho_{xx,xy} \). Our phenomenological treatment is based on our knowledge of the bilocal conductivity \( \sigma_{\mu\nu}(r, r') \) in the perturbative regime. To access the finite-size scaling of the resistance in the non-perturbative critical regime, one needs to study the form of \( \sigma_{\mu\nu}(r, r') \) afresh.

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REFERENCES


FIGURES

FIG. 1. Edge states scattering through a disordered region.

FIG. 2. The Hall bar geometry. a: fictitious version with the voltage probes set outside the disordered region. b: more realistic version with the voltage probes set in the interior of the sample.

FIG. 3. Duality in the Chalker-Coddington network model for the quantum Hall effect. At fixed Fermi energy $E_f$, the system can be viewed as a network built of counter-clockwise orbits enclosing the shaded areas with tunneling probability $1 - T_0(E_f)$, or equivalently, a network built of clockwise orbits enclosing the white regions with tunneling probability $T_0(E_f)$. The phase at $T_0(E_f)$ can be mapped onto the phase at $T_0(E'_f) = 1 - T_0(E_f)$ upon reflection with respect to one of the diagonal axis.
Figure 1
Figure 2
Figure 3
Figure 3
Confinement

lead

Figure 1