EUCLIDEAN D-BRANES
AND HIGHER-DIMENSIONAL GAUGE THEORY

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ABSTRACT.

We consider euclidean D-branes wrapping around manifolds of exceptional holonomy in dimensions seven and eight. The resulting theory on the D-brane—that is, the dimensional reduction of 10-dimensional supersymmetric Yang–Mills theory—is a cohomological field theory which describes the topology of the moduli space of instantons. The 7-dimensional theory is an $N=2$ (or balanced) cohomological theory given by an action potential of Chern-Simons type. As a by-product of this method, we construct a related cohomological field theory which describes the monopole moduli space on a 7-manifold of $G_2$ holonomy.

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1 Introduction

D-branes (see for instance [20]) lead naturally to the study of geometries in dimension greater than four, and in particular to the calibrated geometries introduced in [17]. Part of the BPS spectrum of M-theory or superstring theory compactified down to realistic dimensions consists of branes wrapping around supersymmetric cycles [6,19,5], which are precisely the calibrated submanifolds [7]. Calibrated geometries are particularly rich in dimensions six, seven and eight, and precisely on riemannian manifolds admitting parallel spinors [18,16]; in other words, on manifolds of reduced holonomy.

These very manifolds also play a crucial role in the “Oxford programme” [15] to generalise Donaldson–Floer–Witten theory to higher dimensions. The Yang–Mills equations on these manifolds admit instanton-like solutions, obtained by imposing linear constraints on the Yang–Mills curvature [13,21] which simply project it onto a particular irreducible representation of the holonomy group. Like in four dimensions, the Yang–Mills action on these manifolds satisfies an $L^2$ bound which the instantons saturate [1]. In contrast with the familiar 4-dimensional case, very little is known about the higher dimensional instantons and in particular about their moduli space. One particularly useful approach to the study of 4-dimensional instantons is via topological field theory [22] and it is hoped that a similar approach might prove fruitful also in higher dimensions.

Indeed higher-dimensional generalisations of Witten’s original cohomological field theory were written down independently in [4] and [2]. Unlike the 4-dimensional case, these cohomological field theories are not topological, since they depend upon the reduction of the holonomy group; but as shown in [2] the observables are locally constant in the space of metrics of reduced holonomy. It was remarked in both of these papers that these theories had precisely the same spectrum as 10-dimensional supersymmetric Yang–Mills reduced to the corresponding dimension. Thus it was conjectured that these theories should be obtained in this way. The point of this paper is to prove this conjecture.

The dimensional reduction of 10-dimensional supersymmetric Yang–Mills is of course an old game. This is used, for example, to explain the existence of extended supersymmetric Yang–Mills theories in four dimensions and gives the simplest known method to construct the actions. Much less studied are the reductions to euclidean spaces or more generally to riemannian manifolds. Results in this direction have been obtained in [7,12], who considered euclidean D-branes wrapping around calibrated submanifolds. The resulting theories on the D-brane were seen to be topologically twisted Yang–Mills theory, with the components of the 10-dimensional gauge field in directions normal to the D-brane being no longer simply scalars, but rather sections of the normal bundle to the calibrated submanifold.

The theories we will describe in what follows can be understood as those aris-

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8 It is not known whether this space is connected or, if not, whether these invariants detect the connected component. We are grateful to IM Singer for some correspondence on this point.
ing out of euclidean D-branes wrapping around the full manifold, it being trivially calibrated by the volume form. Our approach is the following. We start with 10-dimensional supersymmetric Yang–Mills theory and reduce it to $d$-dimensional euclidean space, where for the purposes of this paper $d=7,8$. The resulting lagrangian can be promoted to any spin manifold $M$ of dimension $d$ by simply covariantising the derivatives with respect to the spin connection; but the supersymmetry transformations will fail to be a symmetry unless the parameters are covariantly constant. This requires that $M$ admit parallel spinors, and this singles out those manifolds admitting a reduction of the holonomy group to a subgroup of Spin(7) in eight dimensions and to a subgroup of $G_2$ in seven. Covariance of the supersymmetry algebra under the holonomy group implies that the commutator of two supersymmetry transformations with parallel spinors as parameters will result (at least on shell and up to gauge transformations) in a translation by a parallel vector. Since for the irreducible manifolds we consider there are no such vectors, the supersymmetry transformation is a BRST symmetry. Therefore the resulting theory is cohomological. As we will see, the theories constructed in this paper in this fashion agree morally with the ones considered in [4] and [2], whose observables are topological invariants of the moduli space of instantons.

As a by-product of our construction we also arrive at a related cohomological field theory, briefly discussed in [4], which describes what could be termed the monopole moduli space in a 7-manifold of $G_2$ holonomy. Just like monopoles in three dimensions—by which we mean any solution of the Bogomol'nyi equation—can be understood as 4-dimensional instantons with a certain symmetry, every 7-dimensional monopole yields a particular solution of the 8-dimensional instanton equations. Their cohomological field theory is therefore obtained from the one describing the 8-dimensional instantons. In this way, we can understand this theory as a natural higher-dimensional generalisation of the cohomological field theory written down in [8,3].

The plan of this paper is the following. After a cursory look at our notation, we briefly review 10-dimensional supersymmetric Yang–Mills theory in Section 2. In Section 3 we discuss the reduction to 8-dimensional euclidean space, and then to manifolds of Spin(7) holonomy. In Section 4 we consider the reductions to 7-dimensional euclidean space and then to manifolds of $G_2$ holonomy. Finally in Section 5 we will offer some conclusions.

After completion of this work there appeared the paper [9] which also deals with the present topic, but (thankfully!) in a largely complementary manner.
1.1 A word on notation

Throughout this paper we will use the notation $\mathbb{M}^{s+t}$ to refer to $(s+t)$-dimensional Minkowski spacetime. In addition $\mathbb{E}^d = \mathbb{R}^{d+0}$ will denote $d$-dimensional euclidean space. Spinor notation follows the conventions in [18]. In particular, $\text{Cl}(s,t)$ denotes the Clifford algebra (notice the sign!)

$$\Gamma_{\mu}\Gamma_{\nu} + \Gamma_{\nu}\Gamma_{\mu} = -2\eta_{\mu\nu}1,$$

where $\eta_{\mu\nu}$ is diagonal with signature $+s-t$. Our notation for representations of the spin groups is the following. The trivial, vector and adjoint representations are denoted $\Lambda^0$, $\Lambda^1$, and $\Lambda^2$ respectively. For example, for $\text{Spin}(7)$ these are the $1, 7$ and $21$; and for $\text{Spin}(8)$ the $1, 8_8$ and $28$. The half-spin representations are denoted $\Delta$ for odd-dimensional spin groups and $\Delta_\pm$ for the even-dimensional spin groups. For example, for $\text{Spin}(7)$, $\Delta$ is the $8$, whereas for $\text{Spin}(8)$, $\Delta_+$ and $\Delta_-$ are the $8_S$ and $8_C$ respectively. Other group theory notation will be introduced as needed.

2 Ten dimensions

We start by briefly reviewing supersymmetric Yang–Mills in 10-dimensional Minkowski spacetime $\mathbb{M}^{10+1}$.

Let $\hat{\Gamma}_{\mu}$ denote the generators of the Clifford algebra $\text{Cl}(9,1)$. As real associative algebras $\text{Cl}(9,1) \cong \text{Mat}_{32}(\mathbb{R})$, whence the $\hat{\Gamma}_{\mu}$ can be chosen to be $32 \times 32$ real matrices. This means that $\hat{\Gamma}_0$ can be chosen in addition to be symmetric and $\hat{\Gamma}_i$ for $i = 1, \ldots, 9$ antisymmetric. The charge conjugation matrix $\hat{C}$ satisfies $\hat{C}^\dagger = -\hat{C}$, and also $\hat{C}^\dagger = -\hat{C}\hat{\Gamma}_\mu\hat{C}^{-1}$. This means that it anticommutes with $\hat{\Gamma}_0$ and commutes with all the other $\hat{\Gamma}_i$. Therefore we can choose it to be

$$\hat{C} = \hat{\Gamma}_1 \cdots \hat{\Gamma}_9. \quad (1)$$

One can check that indeed $\hat{C}^\dagger = -\hat{C}$. The chirality operator $\hat{\Gamma}_{11}$ is defined by $\hat{\Gamma}_0\hat{\Gamma}_1 \cdots \hat{\Gamma}_9$. One can check that $\hat{\Gamma}_{11}^2 = 1$, whence in this realisation it is both real and symmetric. In other words, Majorana–Weyl spinors exist in $\mathbb{M}^{10+1}$.

The action for supersymmetric Yang–Mills theory in $9+1$ dimensions can be formulated in terms of a gauge field $A_\mu$ and a negative chirality Majorana–Weyl spinor $\Psi$, taking values in a Lie algebra $\mathfrak{g}$ assumed to possess an invariant metric. The lagrangian density is given by

$$\mathcal{L} = \frac{1}{4}(F_{\mu\nu}, F^{\mu\nu}) + \frac{i}{2}(\bar{\Psi}, \hat{\Gamma}_\mu D_\mu \Psi), \quad (2)$$
where

- \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]; \)
- \( D_\mu \Psi = \partial_\mu \Psi - [A_\mu, \Psi]; \) and
- \((-, -)\) is a fixed invariant metric on the Lie algebra \( \mathfrak{g}. \)

This action is invariant under gauge transformations and under the super Poincaré group in 9+1 dimensions. Infinitesimally, the gauge transformations take the form:

\[
\delta_\omega A_\mu = D_\mu \omega \quad \text{and} \quad \delta_\omega \Psi = [\omega, \Psi],
\]

where \( \omega \) is a Lie-algebra valued function. The supersymmetry transformations are given by

\[
\delta_\varepsilon A_\mu = i \varepsilon \hat{\Gamma}_\mu \Psi \quad \text{and} \quad \delta_\varepsilon \Psi = \frac{1}{2} F^{\mu\nu} \hat{\Gamma}_{\mu\nu} \varepsilon,
\]

where \( \varepsilon \) is a constant negative-chirality Majorana–Weyl spinor and \( \hat{\Gamma}_{\mu\nu} = \frac{1}{2} (\hat{\Gamma}_\mu \hat{\Gamma}_\nu - \hat{\Gamma}_\nu \hat{\Gamma}_\mu). \) There are eight bosonic and eight fermionic real physical degrees of freedom, but because the bosonic and fermionic off-shell degrees of freedom do not match in this formulation, the supersymmetry algebra will only close on shell and, indeed, up to gauge transformations.

### 3 Eight dimensions

In this section we derive a supersymmetric Yang–Mills theory in \( \mathbb{R}^8 \) by dimensional reduction from \( \mathbb{R}^{9+1} \) and then we will extend it to a riemannian manifold \( M \) of Spin(7) holonomy. The resulting action defines a cohomological theory whose observables are topological invariants of the moduli space of instantons on \( M. \)

#### 3.1 Properties of spinors

In order to facilitate the dimensional reduction we will first choose a convenient realisation of the \( \Gamma \)-matrices in ten dimensions. The Clifford algebra isomorphism \( \text{Cl}(9, 1) \cong \text{Cl}(8, 0) \otimes \text{Cl}(1, 1) \) suggests one such realisation. We let the Clifford algebra \( \text{Cl}(1, 1) \) be generated by \( \sigma_1 \) and \(-i \sigma_2. \) The chirality operator is given by \( \sigma_3. \) Notice that these matrices are real and the chirality operator is diagonal, indicative of the existence of Majorana–Weyl spinors in 1+1 dimensions. Let \( \Gamma_i \) for \( i = 1, \ldots, 8 \) denote the \( \Gamma \)-matrices in eight dimensions. They are 16 \( \times \) 16 matrices which can be chosen to be real and antisymmetric.
Moreover since Majorana-Weyl spinors also exist in 8+0 dimensions, we can take $\Gamma_9 \equiv \Gamma_1 \cdots \Gamma_8$ to be diagonal. Now consider the following expressions:

$$
\begin{align*}
\hat{\Gamma}_0 &= 1 \otimes \sigma_1 \\
\hat{\Gamma}_i &= \Gamma_i \otimes \sigma_3 \\
\hat{\Gamma}_9 &= 1 \otimes (-i\sigma_2) .
\end{align*}
$$

These are $\Gamma$-matrices for $O(9,1)$. Notice that they are real and antisymmetric, except for $\hat{\Gamma}_0$ which is real and symmetric. In this realisation, the charge conjugation matrix $\hat{C}$ given by (1) becomes $\hat{C} = \Gamma_9 \otimes (-i\sigma_2)$. Notice that $\Gamma_9$ can be identified with the charge conjugation matrix $\hat{C}$ in 8+0 dimensions, since it is symmetric and anticommutes with the $\Gamma_i$. In this realisation, the chirality operator in 9+1 dimensions is given by $\hat{\Gamma}_{11} = \Gamma_9 \otimes \sigma_3$.

Now let $\Psi$ be a 32-component spinor in 9+1 dimensions. In terms of the above decomposition of $O(9,1)$ we can write it as follows: $\Psi = (\psi_1, \psi_2)^t$ where $\psi_A$ are 16-component spinors acted on by the $\Gamma_i$. Suppose that $\Psi$ is chiral: $\hat{\Gamma}_{11}\Psi = \pm \Psi$. Then the components $\psi_A$ are chiral with respect with $\Gamma_9$. Indeed $\Gamma_9\psi_1 = \pm \psi_1$ and $\Gamma_9\psi_2 = \mp \psi_2$. On the other hand, if $\Psi$ is Majorana: $\Psi \equiv \Psi^\dagger \hat{\Gamma}_0 = \Psi^\dagger \hat{C}$, then $\psi_A$ satisfy the following reality conditions: $\psi_1^* = -\Gamma_9 \psi_1$ and $\psi_2^* = \Gamma_9 \psi_2$. Therefore, if $\Psi$ is Majorana-Weyl then $\psi_A$ are either both real or both imaginary according to whether $\Psi$ has negative or positive chirality, respectively. In our case, we have chosen the spinor $\Psi$ in (2) to have negative chirality, whence $\psi_A$ are real.

### 3.2 Dimensional Reduction

We now dimensionally reduce to $E^8$ by simply dropping the dependence on $x^0$ and $x^9$. In other words, we let $\delta_0 = \delta_9 = 0$. The Lorentz symmetry of the theory is therefore broken down to $SO(8) \times SO(1,1)$, which suggests the following decomposition of the fields: $A_\mu = (A_1, A_9, A_0)$ and $\Psi = (\psi_1, \psi_2)^t$, where $\psi_1$ and $\psi_2$ are respectively negative and positive chirality spinors of Spin(8), and $A_9$ and $A_0$ are scalars. Of course, all fields remain Lie algebra valued.

We first tackle the bosonic part of the action. It is enough to realise that $F_{0i} = -D_i A_0$, $F_{9i} = -D_i A_9$ and $F_{09} = [A_9, A_0]$, to to derive

$$
\frac{1}{4} (F_{\mu\nu}, F^{\mu\nu}) = \frac{1}{4} ||F_{ij}||^2 + \frac{1}{2} ||D_i A_9||^2 - \frac{1}{2} ||D_i A_0||^2 - \frac{1}{2} ||[A_9, A_0]||^2 .
$$

Similarly, notice that $D_9\Psi = -[A_0, \Psi]$ and $D_9\Psi = -[A_9, \Psi]$. Using this and the explicit realisation of the $\hat{\Gamma}$-matrices given by (5) one obtains (with $\epsilon^{12} = -\epsilon^{21} = 1$)

$$
\frac{i}{2} (\bar{\Psi}, \hat{\Gamma}^\mu D_\mu \Psi) = -\frac{i}{2} \epsilon^{AB} (\psi_A^t, \Gamma_i D_i \psi_B) \\
- \frac{i}{2} (\psi_1^t, [A_9 - A_0, \psi_1]) + \frac{i}{2} (\psi_2^t, [A_9 + A_0, \psi_2]) .
$$
This suggests that we define fields $\phi_{\pm} \equiv A_9 \pm A_0$. In terms of these fields the dimensionally reduced lagrangian density becomes:

$$\mathcal{L} = \frac{1}{4} \| F_{ij} \|^2 + \frac{1}{2} (D_i \phi_+ + D_i \phi_-) - \frac{1}{2} \| [\phi_+, \phi_-] \|^2$$

$$- \frac{1}{2} e^{AB} (\psi^A, \Gamma_i D_i \psi^B) - \frac{i}{2} (\psi^I, [\phi_-, \psi_I]) + \frac{i}{2} (\psi^I, [\phi_+, \psi_I]) . \quad (6)$$

The R-symmetry $SO(1, 1)$ acts diagonally on the fields $(A_i, \phi_+, \phi_-, \psi_1, \psi_2)$ with weights $(0, +1, -1, +\frac{1}{2}, -\frac{1}{2})$. Finally, using:

$$\hat{\Gamma}_{ij} = \Gamma_{ij} \otimes 1 \quad \hat{\Gamma}_{0i} = \Gamma_i \otimes (-i \sigma_2) \quad \hat{\Gamma}_{9i} = \Gamma_i \otimes \sigma_1 \quad \hat{\Gamma}_{00} = 1 \otimes \sigma_3 ,$$

the supersymmetry transformations take the form:

$$\delta \epsilon A_i = -i e^{AB} A^A_i \Gamma_i \psi_B$$
$$\delta \epsilon \phi_+ = 2i \epsilon^I \psi_I$$
$$\delta \epsilon \phi_- = -2i \epsilon^I \psi_I$$
$$\delta \epsilon \psi_1 = \frac{1}{2} (F_{ij} \Gamma_{ij} + [\phi_+, \phi_-]) \epsilon_1 - D_i \phi_+ \Gamma_i \epsilon_2$$
$$\delta \epsilon \psi_2 = \frac{1}{2} (F_{ij} \Gamma_{ij} - [\phi_+, \phi_-]) \epsilon_2 - D_i \phi_- \Gamma_i \epsilon_1 . \quad (7)$$

### 3.3 Manifolds of exceptional holonomy

A natural question to ask is whether this theory can be defined on 8-dimensional manifolds other than $\mathbb{E}^8$. The action defined by (6) makes sense on an arbitrary spin manifold provided that we redefine the covariant derivative on the fermions to include the spin connection. However it will not be invariant under the supersymmetry transformations unless the spinor parameters $\epsilon_A$ are covariantly constant with respect to the spin connection. We are therefore led to ask on which 8-dimensional spin manifolds do parallel spinors exist. Of necessity such manifolds must have reduced holonomy $H \subset SO(8)$, since the spinor representations of $Spin(8)$ must contain a singlet when decomposed under $H$.

The subgroups of $Spin(8)$ which leave invariant a spinor are all subgroups of a $Spin(7)$ subgroup. Those under which the manifold remains irreducible are $Spin(7) \supset SU(4) \supset Sp(2)$. The latter two groups correspond to Calabi–Yau 4-folds and hyperkähler manifolds respectively. Manifolds of $Spin(7)$ holonomy possess one parallel spinor, whose chirality depends on which $Spin(7)$ subgroup of $SO(8)$ we choose. There are three inequivalent $Spin(7)$ subgroups of $SO(8)$ related by triality. One of these conjugacy classes does not lead to parallel spinors, but the other two do. Calabi–Yau 4-folds possess two parallel spinors of the same chirality, determined by the conjugacy class of the $SU(4)$ subgroup of $SO(8)$ or equivalently by the conjugacy class of the $Spin(7)$ subgroup to which the $SU(4)$ belongs. A similar story holds for hyperkähler manifolds, which possess three parallel spinors of the same chirality.
Although similar results to those we are about to describe hold on Calabi–Yau 4-folds and hyperkähler manifolds, we will focus in this paper on the theory defined by the lagrangian (6) on manifolds of Spin(7) holonomy.

3.4 Super Yang–Mills on manifolds of Spin(7) holonomy

We will fix once and for all a conjugacy class of Spin(7) subgroups of SO(8). We choose the one for which the positive chirality spinor representation $\Delta_+$ of Spin(8) decomposes as $\Lambda^1 \oplus \Lambda^0$, where $\Lambda^1$ is the vector representation of Spin(7) and $\Lambda^0$ is the trivial 1-dimensional representation. For this Spin(7) subgroup, the vector representation of SO(8) and the negative chirality spinor representation $\Delta_-$ remain irreducible and isomorphic to the spinor representation $\Delta$. We want to write down the theory in terms of fields transforming according to irreducible representations of Spin(7). For this we will need to introduce some explicit projectors onto the irreducible representations of Spin(7) which occur in the theory. Moreover since the fields in the theory all come from irreducible representations of Spin(8), it will prove convenient to define these projectors directly in terms of $\Gamma$-matrices.

Let us introduce a commuting spinor $\theta \in \Delta_+$ invariant under Spin(7) and normalised to $\theta^* \theta = 1$. Then $\theta^* \theta$ is the projector onto the space of Spin(7) invariants in $\Delta_+$. The Fierz rearrangement formula yields:

$$\theta^* \theta = \frac{1}{16} (1 + \Gamma_9) + \frac{1}{16} \Gamma^{ijkl} \theta \Gamma_{ijkl}.$$  

The only supersymmetry transformation which will remain a symmetry of the action on a spin manifold $M$ of Spin(7) holonomy is the one which has the parallel spinor as a parameter. In other words we must set $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon \theta$, where $\varepsilon$ is the anticommuting parameter of the symmetry. Setting $\varepsilon_1 = 0$ already means that $\delta \phi_+= 0$. The variation of the gauge field is given by $\delta \phi A_i = i \varepsilon^2 \Gamma_i \psi_1 = i \varepsilon \theta \Gamma_i \psi_1$. Defining

$$\psi_1 \equiv \theta^* \Gamma_i \psi_1,$$

we can write it as $\delta \phi A_i = i \varepsilon \psi_1$. Notice that $\psi_1$ can be interpreted now as a 1-form in $M$. Its variation can be readily calculated to give $\delta \varepsilon \psi_1 = \theta^* \Gamma_i \delta \varepsilon \psi_1 = D_i \phi_\varepsilon \varepsilon$, where we have used that $\theta^* \Gamma_i \theta = 0$.

Under Spin(7) the spinor $\psi_2 \in \Delta_+$ decomposes into a singlet and a vector. As fields on $M$, the singlet $\eta$ is a fermionic scalar whereas the vector can be understood as a fermionic 2-form $\chi_{ij}$ satisfying an ‘anti-self-duality’ condition to be defined presently. We first turn to the scalar. Clearly, $\eta = \theta^* \psi_2 = \psi_2^* \theta$, from where it follows that $\delta \varepsilon \phi_- = -2i \varepsilon \eta$. In turn its variation can be readily calculated: $\delta \varepsilon \eta = \theta^* \delta \varepsilon \psi_2 = -\frac{1}{2} [\phi_+, \phi_-] \varepsilon$. Finally we arrive at the vector
component of $\psi_2$. Consider $\Gamma_{ij}\theta$. Since $\theta$ is a Spin(7)-singlet, $\Gamma_{ij}\theta$ transforms under $\Lambda^2 \oplus \Lambda^1$ of Spin(7). Since $\psi_2$ transforms according to $\Lambda^0 \oplus \Lambda^1$, their inner product will pick out only the $\Lambda^1$ component: $\chi_{ij} = \frac{1}{2} \psi^1 \Gamma_{ij} \psi_2 = -\frac{1}{2} \psi^1 \Gamma_{ij} \theta$. Not all the components of $\chi_{ij}$ are independent. Indeed, $\chi_{ij}$ satisfies the following 'anti-self-duality' condition:

$$\chi_{ij} = -\frac{1}{6} \Omega_{ijkl} \chi_{kl} \ ,$$

where $\Omega_{ijkl}$ is a self-dual Spin(7)-invariant 4-form on $M$ defined by:

$$\Omega_{ijkl} \equiv \theta^l \Gamma_{ijkl} \theta \ .$$

It follows from this expression that

$$\Omega_{ijmn} \Omega_{mnpkl} = 6 (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) - 4 \Omega_{ijkl} \ .$$

This allows us to define complementary projectors to decompose the adjoint representation $\Lambda^2$ of Spin(8) into irreducible representations $\Lambda^1 \oplus \Lambda^2$ of Spin(7). Indeed let,

$$P_{ij} = -\frac{1}{8} \theta^l \Gamma_{ij} \Gamma_{kl} \theta$$

$$= \frac{1}{8} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} - \Omega_{ijkl})$$

$P$ is the projector onto $\Lambda^1$, whereas $1 - P$ is the complementary projector onto $\Lambda^2$. We can verify that the fact that $\chi_{ij}$ belongs to $\Lambda^1$, equivalently $P \chi = \chi$, implies (8). It follows from these expressions that a 2-form $F_{ij}$ in $\Lambda^2$ of Spin(8) belongs to the vector representation $\Lambda^1$ of Spin(7) if and only if

$$\frac{1}{2} \Omega_{ijkl} F_{kl} = -3 F_{ij} \ ;$$

whereas it belongs to the adjoint representation $\Lambda^2$ of Spin(7) if and only if

$$\frac{1}{2} \Omega_{ijkl} F_{kl} = F_{ij} \ .$$

For $F_{ij}$ the Yang–Mills curvature, these equations are the instanton equations in eight dimensions studied for the first time in [13,21].

The variation of $\chi_{ij}$ can now be computed as follows: $\delta_{\varepsilon} \chi_{ij} = \frac{1}{2} \theta^l \Gamma_{ij} \delta_{\varepsilon} \psi_2 = \frac{1}{4} F_{kl} \theta^l \Gamma_{ij} \Gamma_{kl} \theta \varepsilon$. From (10) one sees that $\delta_{\varepsilon} \chi_{ij} = -2 (PF)_{ij} \varepsilon$ or explicitly as

$$\delta_{\varepsilon} \chi_{ij} = -\frac{1}{2} \left( F_{ij} - \frac{1}{2} \Omega_{ijkl} F_{kl} \right) \varepsilon \ .$$

Finally we can now rewrite the action in terms of the new fields $A_i$, $\psi_i$, $\phi_\pm$, $\chi_{ij}$ and $\eta$. First notice that we can invert the definitions of the spinor fields: $\psi_1 = -\psi_1 \Gamma_i \theta$ and $\psi_2 = \theta \eta - \frac{1}{4} \chi_{ij} \Gamma_{ij} \theta$. Plugging these expressions into (6) we
\[ \mathcal{L} = \frac{1}{4} \| F_{ij} \|^2 + \frac{1}{2} (D_i \phi_+ + D_i \phi_-) - \frac{1}{8} \| [\phi_+, \phi_-] \|^2 + i (\psi_1, [\phi_+, \psi_1]) + 2i ( \chi_{ij}, D_i \psi_j ) + \frac{1}{2} ( \chi_{ij}, [\phi_+, \chi_{ij}] ) + i ( \eta, D_i \psi_i ) + \frac{1}{2} ( \eta, [\phi_+ \eta] ) , \quad (11) \]

where \( D_i \) now includes not just the gauge field but also the reduction to Spin(7) of the spin connection. This action is invariant under the following fermionic transformation

\[
\begin{align*}
\delta A_i &= i \psi_i \\
\delta \phi_+ &= 0 \\
\delta \phi_- &= -2i \eta \\
\delta \eta &= -\frac{1}{2} [\phi_+, \phi_-] \\
\delta \chi_{ij} &= -\frac{1}{2} ( F_{ij} - \frac{1}{2} \Omega_{ijkl} F_{kl} ) , \\
\end{align*}
\]

which up to gauge transformations and field equations obeys \( \delta^2 = 0 \). We finish with the observation that this action agrees with the action \( S_1 \) given by equation (12) in \cite{2} under the dictionary in Table 1.

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<td>( \lambda )</td>
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<td>( \chi_{\alpha \beta} )</td>
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<td>( \phi_{\alpha \beta \gamma \delta} )</td>
<td>(- \Omega_{ijkl} )</td>
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Table 1
Comparison with \cite{2}.

### 3.4.1 The BRST transformations

The fermionic symmetry given by equation (12) only squares to zero on shell and up to gauge transformations. In fact, we find the following:

\[
\begin{align*}
\delta^2 A_i &= i D_i \phi_+ \\
\delta^2 \phi_+ &= 0 \\
\delta^2 \phi_- &= i [\phi_+, \phi_-] \\
\delta^2 \eta &= i [\phi_+, \eta] \\
\delta^2 \chi_{ij} &= -2i \mathcal{P} ( D_i \psi_j - D_j \psi_i ) ;
\end{align*}
\]

but the equation of motion which follows from varying the action with respect to \( \chi_{ij} \) is given by

\[
\mathcal{P} ( D_i \psi_j - D_j \psi_i ) = -\frac{1}{2} [\phi_+, \chi_{ij} ] ,
\]

whence \( \delta^2 \chi_{ij} = i [\phi_+, \chi_{ij}] \) on shell.
It is possible to modify the fermionic symmetry in order to obtain something which does square to zero on the nose. The problem that \( \delta \) squares to zero up to gauge transformations is routine to solve and not terribly relevant for our present purposes. Nevertheless its solution will be described below for completeness. The problem that \( \chi_{ij} \) has to be put on shell is more relevant. It is solved by introducing a Lie algebra valued bosonic auxiliary field \( A_{ij} \) satisfying the same ‘anti-self-duality’ condition (8) as \( \chi_{ij} \) and redefining:
\[
\delta \chi_{ij} = A_{ij} \quad \text{and} \quad \delta A_{ij} = i[\phi_+, \chi_{ij}].
\]

Now \( \delta \) is manifestly nilpotent off shell but only up to a gauge transformation with parameter \( i\phi_+ \). We must now write the action in terms of the auxiliary field. We consider first the following ‘gauge fermion’
\[
\Psi = -\frac{1}{4}(\chi_{ij}, A_{ij} + 4(PF)_{ij}).
\]

It is clearly gauge invariant, whence \( \delta^2 \Psi = 0 \). In other words,
\[
\delta \Psi = 2i(\chi_{ij}D_i\psi_j) + \frac{1}{4}(\chi_{ij}, [\phi_+, \chi_{ij}]) - \frac{1}{4}(A_{ij}, A_{ij} + 4(PF)_{ij}), \quad (13)
\]
is invariant under \( \delta \). Solving for \( A_{ij} \) we see that \( A_{ij} = -2(PF)_{ij} \) and plugging it back into the last term in (13) we obtain:
\[
\|(PF)_{ij}\|^2 = \frac{1}{4}||F_{ij}||^2 - \frac{1}{8}\Omega_{ijkl}(F_{ij}, F_{kl}).
\]

Calculating a little further we see that the lagrangian (11) can be rewritten as follows:
\[
\mathcal{L} = \frac{1}{8}\Omega_{ijkl}(F_{ij}, F_{kl}) + \delta \left(\Psi - \frac{1}{2}(\phi_-, D_i\psi_i + \frac{1}{2}[\phi_+, \eta])\right),
\]
which shows that \( \mathcal{L} \) is obtained up to a BRST exact term from a ‘topological’ term as befits a cohomological theory.

This result agrees with [4] modulo the fact that their action is missing some of these terms while at the same time containing some more fields necessary to fix the gauge. These are the fields which solve the other problem with \( \delta \): that it squares to zero only up to gauge transformations. In order to solve this problem we introduce a ghost (a Lie algebra valued fermionic field) \( c \). We now define a new transformation \( \delta' \), defined on all fields but \( c \) as a gauge transformation with parameter \( c \):
\[
\begin{align*}
\delta' A_i &= D_i c \\
\delta' \phi_+ &= [c, \phi_+] \\
\delta' \phi_- &= [c, \phi_-] \\
\delta' \eta &= [c, \eta] \\
\delta' \chi_{ij} &= [c, \chi_{ij}],
\end{align*}
\]

and then defined on \( c \) in such a way that \( \delta'^2 = 0 \): \( \delta' c = \frac{1}{2}[c, c] \). On the other hand, \( \delta \delta' + \delta' \delta \) is a gauge transformation with parameter \( \delta c \). Therefore if we
define $\delta c = -i\phi_+$, then the combination $d \equiv \delta + \delta'$ squares to zero off shell on all fields, including c. In order to give dynamics to the ghost it is necessary to also introduce an antighost $b$ and in order for $d^2 b = 0$ we need to introduce a so-called Nakanishi–Lautrup auxiliary field. This field also serves the dual purpose of fixing the gauge so that the propagator of the gauge field is well-defined and thus allowing in principle for the perturbative treatment of the theory. If we follow this procedure for the gauge fixing $\nabla_i A_i = 0$ one recovers the action in [4] up to the terms $\frac{1}{4} \delta (\phi_- | [\phi_+, \eta])$ which they omit. These terms are presumably not important in that the theory gives the same observables with or without them; but are unavoidable if one reduces from 10-dimensional super Yang–Mills as advocated here.

4 Seven dimensions

In this section we describe the reduction of supersymmetric Yang–Mills theory from $\mathbb{R}^{10+1}$ to $\mathbb{E}^7$ and to 7-dimensional manifolds of $G_2$ holonomy.

4.1 Properties of spinors

Unlike the 8-dimensional case, there is no isomorphism relating $\text{Cl}(9,1)$ and $\text{Cl}(7,0)$. However there exists an isomorphism $\text{Cl}(7,0) \cong \text{Cl}(8,0)^{\text{even}}$, where this last algebra refers to the subalgebra of $\text{Cl}(8,0)$ generated by even products of $\Gamma$-matrices. This isomorphism induces an embedding $\text{Spin}(7) \subset \text{Spin}(8)$ under which the half-spin representations $\Delta_\pm$ of $\text{Spin}(8)$ remain irreducible and isomorphic to the half-spin representation $\Delta$ of $\text{Spin}(7)$, whereas the vector representation $\Lambda^0$ of $\text{Spin}(8)$ decomposes into a vector and a scalar $\Lambda^0 \oplus \Lambda^1$. We will identify $\Delta$ with $\Delta_+$ once and for all. In other words, we think of $\text{Cl}(7,0)$ as $\text{Cl}(8,0)^{\text{even}}$ and of $\Delta$ as $\Delta_+$.

Explicitly, the isomorphism $\text{Cl}(7,0) \cong \text{Cl}(8,0)^{\text{even}}$ is given as follows: let $i$ run from 1 to 7 and let $\Gamma_i$ and $\Gamma_8$ denote the $\Gamma$-matrices in eight dimensions. Define $\tilde{\Gamma}_i = \Gamma_i \Gamma_8$. A moment’s reflection shows that they generate $\text{Cl}(8,0)^{\text{even}}$, whereas it is evident that they are $\Gamma$-matrices for $\text{Cl}(7,0)$. This defines an embedding $\text{Spin}(7) \subset \text{Spin}(8)$, given infinitesimally by: $\frac{1}{2} \tilde{\Gamma}_{ij} = \frac{1}{2} \Gamma_{ij}$ but where $i, j$ only run from 1 to 7. The $\text{Spin}(7)$-isomorphism $\Delta_- \cong \Delta \equiv \Delta_+$ is then given by $\Gamma_8$.

This course of action allows us to use the results of Section 3.1 concerning the reduction from $\text{Cl}(9,1)$ to $\text{Cl}(8,0)$, and sets the stage for a further reduction to $\text{Cl}(7,0)$, to which we now turn.
4.2 Dimensional reduction

We now dimensionally reduce from \( M^{10+1} \) to \( E^7 \) by dropping the dependence on \( x^8, x^9 \) and \( x^{10} \); that is \( \partial_8 = \partial_9 = \partial_{10} = 0 \). This choice of privileged coordinates breaks the Lorentz symmetry down to \( \text{SO}(7) \times \text{SO}(2,1) \). This suggests that we arrange the fields into irreducible representations of this group or its spin cover. The gauge field will break up into an \( \text{SO}(2,1) \) singlet and \( \text{SO}(7) \) vector \( A_i \) and an \( \text{SO}(7) \) singlet and \( \text{SO}(2,1) \) triplet \( \phi_\alpha \) where \( (\phi_0, \phi_1, \phi_2) = (A_0, A_9, A_8) \). The bosonic part of the action defined by (2) can therefore be written as

\[
\mathcal{L}_B = \frac{1}{4} \| F_{ij} \|^2 + \frac{1}{2} \eta^{\alpha\beta} (D_i \phi_\alpha, D_i \phi_\beta) + \frac{1}{4} \eta^{0\gamma} \eta^{\beta\delta} ([\phi_\alpha, \phi_\beta], [\phi_\gamma, \phi_\delta]),
\]

where \( \eta = (-++++) \) is the \( \text{SO}(2,1) \) invariant metric.

As for the fermions, \( \Psi = (\lambda_1 \lambda_2)^t \) where \( \lambda_1 \in \Delta_- \) and \( \lambda_2 \in \Delta_+ \) of \( \text{Spin}(8) \). We want to decompose them into representations of \( \text{Spin}(7) \times \text{SU}(2) \), where \( \text{SU}(2) \) is the spin cover of \( \text{SO}(2,1) \). Let \( \mathbf{O}^p \) denote the \( p \)-th symmetric tensor product of the defining (real 2-dimensional) representation of \( \text{SU}(2) \). For example, \( \mathbf{O}^0 \) is the singlet, \( \mathbf{O}^1 \) is the defining representation and \( \mathbf{O}^2 \) is the adjoint representation.

Let us therefore introduce a doublet of spinors \( \psi_A \in \Delta \otimes \mathbf{O}^1 \), defined by: \( \psi_1 = \Gamma_8 \lambda_1 \) and \( \psi_2 = \lambda_2 \). In terms of these fields, the fermionic action becomes

\[
\mathcal{L}_F = \frac{i}{2} \epsilon^{AB} (\psi_A^t, \Gamma_i D_i \psi_B) + \frac{i}{2} \eta^{\alpha\beta} M^{AB}_\beta (\phi_\alpha, [\psi_A^t, \psi_B]), \tag{15}
\]

where the matrices \( M^{AB}_\alpha \) are such that

\[
\phi^{AB}_\alpha \equiv \eta^{\alpha\beta} \phi_\beta M^{AB}_\beta = \begin{pmatrix} \phi_1 - \phi_0 & \phi_2 \\ \phi_2 & - (\phi_1 + \phi_0) \end{pmatrix}.
\]

Our conventions for \( \epsilon_{AB} \) are as follows. We take \( \epsilon^{12} = -\epsilon^{21} = \epsilon_{12} = -\epsilon_{21} = +1 \). We raise and lower indices using the “northeast” convention:

\[
O_A = \epsilon_{AB} O^B \quad \text{and} \quad O^A = O_B \epsilon^{BA} = -\epsilon^{AB} O_B.
\]

Therefore \( \epsilon_A^B = \epsilon_{AC} \epsilon^{CB} = -\delta_A^B \).

The action inherits the supersymmetry from ten dimensions and in particular from (4). We let \( \varepsilon = (\xi_1, \xi_2)^t \) where \( \xi_1 \in \Delta_- \) and \( \xi_2 \in \Delta_+ \). As before we introduce a doublet of spinors \( \varepsilon_A \in \Delta \otimes \mathbf{O}^1 \) by

\[
\varepsilon_1 = \Gamma_8 \xi_1 \quad \text{and} \quad \varepsilon_2 = \xi_2.
\]

Decomposing (4) in terms of irreducible representations of \( \text{Spin}(7) \times \text{SU}(2) \) we
find:
\[
\begin{align*}
\delta_x A_i &= i \epsilon^{AB} \epsilon_A^i \tilde{\Gamma}^i \psi_B \\
\delta_x \phi_A &= i M^{AB} \epsilon_A^i \psi_B \\
\delta_x \phi_A &= \frac{1}{2} F_{ij} \tilde{\Gamma}^i \epsilon_A - D_i \phi_A D^i \epsilon_B + \frac{1}{2} [\phi, \phi]^{AB} \epsilon_B ,
\end{align*}
\]
where
\[
\frac{1}{2} [\phi, \phi]^{AB} \equiv \frac{1}{2} \epsilon_{CD} [\phi^{AC}, \phi^{DB}] = \begin{pmatrix} [\phi_2, \phi_1 - \phi_0] & [\phi_0, \phi_1] \\ [\phi_0, \phi_1] & [\phi_2, \phi_1 + \phi_0] \end{pmatrix} .
\]

The action is then the sum of \( \mathcal{L}_F \) and the bosonic action \( \mathcal{L}_B \) which can be rewritten in terms of \( \phi^{AB} \) as
\[
\mathcal{L}_B = \frac{1}{2} \| F_{ij} \|^2 + \frac{1}{4} \text{Tr}(D_i \phi, D_j \phi) + \frac{1}{2} \text{Det} \left( \frac{1}{2} [\phi, \phi]^{AB} \right) ,
\]
where \( \text{Tr}(D_i \phi, D_j \phi) = (D_i \phi_A, D_j \phi_A) \), and the last term is the 'determinant' of the matrix \( \frac{1}{2} [\phi, \phi]^{AB} \) defined above:
\[
\text{Det} \left( \frac{1}{2} [\phi, \phi] \right) = \left( \frac{1}{2} [\phi, \phi]^1 , \frac{1}{2} [\phi, \phi]^2 \right) - \left( \frac{1}{2} [\phi, \phi]^1 , \frac{1}{2} [\phi, \phi]^1 \right) .
\]

4.3 Reduction of the holonomy group

In order to define the action \( \mathcal{L} = \mathcal{L}_B + \mathcal{L}_F \) on a 7-dimensional spin manifold \( M \) instead of \( \mathbb{F}^7 \) we need only covariantise the derivatives to include the spin connection. As in eight dimensions, the supersymmetry transformations in (16) will not be a symmetry of the action unless the parameters are parallel spinors, which reduces the holonomy of \( M \) from \( SO(7) \) to (a subgroup of) \( G_2 \). It therefore pays to rewrite the theory in terms of irreducible representations of \( G_2 \subset SO(7) \). Under \( G_2 \) the spinor representation breaks up as \( \Delta \cong \varrho \oplus \tau \) where \( \tau \) is a singlet of \( G_2 \) and \( \varrho \) is in the 7-dimensional irreducible representation of \( G_2 \). On the other hand, the vector representation \( \Lambda^1 \) of \( SO(7) \) remains irreducible and goes over to \( \varrho \).

Let \( \theta \) denote a commuting parallel spinor in \( M \) normalised pointwise to \( \theta^\dagger \theta = 1 \). The parallel spinor will allow us to decompose \( \psi_A \) into its \( G_2 \) irreducible components. In other words, we introduce fields \( \chi_{iA} \) and \( \eta_A \) which transform under \( G_2 \times SL(2) \) as \( \varrho \otimes \mathbb{O}^1 \) and \( \tau \otimes \mathbb{O}^1 \) respectively. These fields are defined as follows: \( \chi_{iA} = \theta^i \tilde{\Gamma}^i \psi_A \) and \( \eta_A = \theta^A \psi_A \). Using the fact that \( \theta^i \tilde{\Gamma}^i \theta = 0 \), we can invert the above definitions and write \( \psi_A \) as follows:
\[
\psi_A = \eta_A \theta - \chi_{iA} \tilde{\Gamma}^i \theta .
\]
In terms of these new fields, the fermionic action $\mathcal{L}_F$ in (15) is given by:

$$\mathcal{L}_F = i e^{AB} (\eta_A, D_i \chi_B) - \frac{i}{2} e^{AB} \varphi_{ijk}(\chi_{iA}, D_j \chi_{kB})$$

$$+ \frac{i}{2} (\phi^{AB}, [\eta_A, \eta_B]) + \frac{i}{2} (\phi^{AB}, [\chi_i A, \chi_i B]), \quad (18)$$

where we have used that $\theta^t \tilde{\Gamma}_i \theta = \theta^t \tilde{\Gamma}_{ij} \theta = 0$ and where we have defined

$$\varphi_{ijk} \equiv \theta^t \tilde{\Gamma}_{ijk} \theta .$$

The action is invariant under two fermionic symmetries, obtained from (16) by choosing the spinors $\varepsilon_A$ to be parallel. In other words we let $\varepsilon_A = \lambda_A \theta$ where $\lambda_A$ are anticommuting parameters. Rewriting (16) in terms of the new fields and for this choice of supersymmetry parameters, we find:

$$\delta \lambda_A = i e^{AB} \lambda_A \chi_{iB}$$

$$\delta \epsilon_A = i M^A \lambda_A \eta_B$$

$$\delta \lambda_A = \frac{1}{2} [\phi, \phi] A^B \lambda_B$$

$$\delta \lambda_A = \frac{1}{2} \varphi_{ijk} F_{jk} \lambda_A + D_i \phi_{A} B \lambda_B , \quad (19)$$

where we have again used that $\theta^t \tilde{\Gamma}_i \theta = \theta^t \tilde{\Gamma}_{ij} \theta = 0$.

The 3-form $\varphi_{ijk}$ defined above is a $G_2$ singlet in $\wedge^3 \mathcal{Q}$. We can relate this to the Spin(7)-invariant 4-form $\Omega_{ijkl}$ defined in Section 3.4. Under the identification $\Delta \equiv \Delta_+$, the normalised $G_2$-invariant spinor $\theta$ is (up to a sign) the Spin(7)-invariant spinor introduced in Section 3.4 and bearing the same name. Therefore,

$$\varphi_{ijk} = \theta^t \tilde{\Gamma}_{ijk} \theta = \theta^t \tilde{\Gamma}_{ijk} \theta = \Omega_{ijkl} .$$

This 3-form obeys an identity similar to the relation (9) obeyed by $\Omega_{ijkl}$ and derivable in the same way:

$$\varphi_{ijm} \varphi_{mkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} - \varphi_{ijkl} , \quad (20)$$

where the $G_2$-invariant 4-form $\tilde{\varphi}$ is simply the restriction of $\Omega$:

$$\tilde{\varphi}_{ijkl} = \theta^t \tilde{\Gamma}_{ijkl} \theta = \theta^t \tilde{\Gamma}_{ijkl} \theta = \Omega_{ijkl} .$$

One can check that $\tilde{\varphi}$ is the 7-dimensional Hodge dual of $\varphi$. Two more identities relate $\tilde{\varphi}$ and $\varphi$:

$$\tilde{\varphi}_{ijmn} \tilde{\varphi}_{mkl} = 4 (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) - 2 \tilde{\varphi}_{ijkl}$$

$$\tilde{\varphi}_{ijmn} \varphi_{mnl} = -4 \varphi_{ijk} . \quad (21)$$

These identities are consistent with (9) and indeed, together with (20), imply it.

Just as in the case of Spin(7) holonomy it is possible to use the 4-form $\tilde{\varphi}$ in order to construct projectors onto the $G_2$ irreducible subspaces of the adjoint

15
representation \( \Lambda^2 \) of \( \text{SO}(7) \). Under \( G_2 \), this representation breaks up as \( \mathfrak{g} \oplus \mathfrak{g}_2 \), where \( \mathfrak{g}_2 \) denotes the (fourteen-dimensional) adjoint representation of \( G_2 \).

The 3-form \( \varphi \) defines a \( G_2 \)-equivariant map \( \Lambda^1 \to \Lambda^2 \) by \( \nu_i \mapsto \varphi_{ijk} v_k \) which sets up an isomorphism onto the subrepresentation \( \mathfrak{g} \subset \Lambda^2 \). It also allows us to define another \( G_2 \)-equivariant map \( \Lambda^2 \to \Lambda^1 \) by \( u_{ij} \mapsto \frac{1}{2} \varphi_{ijk} u_{jk} \) whose kernel coincides with the subrepresentation \( \mathfrak{g}_2 \subset \Lambda^2 \). Using these facts and the identities (20) and (21) the expressions for the projectors follow easily. For the projector \( P \) onto \( \mathfrak{g} \) we find

\[
P_{ijkl} = \frac{1}{6} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} - \tilde{\varphi}_{ijkl}) \; ,
\]

and naturally \( 1 - P \) for the projector onto \( \mathfrak{g}_2 \). It follows from these expressions that a 2-form \( F_{ij} \in \Lambda^2 \) belongs to the subrepresentation \( \mathfrak{g} \) if and only if

\[
\frac{1}{2} \tilde{\varphi}_{ijkl} F_{kl} = -2 F_{ij} ;
\]

whereas it belongs to the subrepresentation \( \mathfrak{g}_2 \) if and only if

\[
\frac{1}{2} \tilde{\varphi}_{ijkl} F_{kl} = F_{ij} .
\]

As in eight dimensions, for \( F_{ij} \) the Yang–Mills curvature, these equations are the instanton equations in seven dimensions defined for the first time in [13,21].

4.4 A cohomological theory for instantons

The theory described in Section 4.3 has two BRST symmetries: one for each parameter \( \lambda_A \). In order to fix a BRST operator, we set \( \lambda_1 = 0 \) and let \( \lambda_2 = -\lambda \). Any other choice will be related to this one by an \( \text{SO}(1,1) \subset \text{SL}(2) \) transformation, but this is an automorphism of the resulting theory. We let \( \phi = \phi_2 \) and \( \phi_\pm = \phi_1 \pm \phi_0 \). We further define \( \psi_i = \chi_{i1} \) and \( \chi_i = \chi_{i2} \). In terms of these fields the fermionic symmetry in (19) (omitting the parameter \( \lambda \)) becomes:

\[
\begin{align*}
\delta A_i &= i \psi_i \\
\delta \phi &= -i \eta_1 \\
\delta \phi_- &= 2i \eta_2 \\
\delta \phi_+ &= 0 \\
\delta \psi_i &= D_i \phi_+ \\
\delta \eta_1 &= [\phi, \phi_+] \\
\delta \eta_2 &= \frac{1}{2} [\phi_+, \phi_-] \\
\delta \chi_i &= D_i \phi - \frac{1}{2} \varphi_{ijk} F_{jk} .
\end{align*}
\] (22)

The action is given by \( \mathcal{L}_B + \mathcal{L}_F \), which are in turn given by equations (17) and (18).

As before \( \delta^2 \) only squares to zero on the \( \chi_i \) shell and modulo a gauge transformation with parameter \( i \phi_+ \). We can remedy this situation as we did in Section 3.4.1. In order to lift the on-shell condition we introduce a bosonic Lie algebra
valued auxiliary field $A_i$ and define $\delta \chi_i = \Lambda_i$ and $\delta A_i = i[\phi_i, \chi_i]$. It is now clear that $\delta$ squares off-shell to a gauge transformation with parameter $i\phi_+$. Introducing as before a ghost $c$ with $\delta c = -i\phi_+$ and a second fermionic symmetry $\delta'$ defined as a gauge transformation with parameter $c$ on all fields but $c$ itself and by $\delta' c = \frac{1}{2}[c, c]$, guarantees that the combined fermionic symmetry $d = \delta + \delta'$ now squares to zero off shell. At the end of the day the lagrangian can be written as the gauge fixing of a topological term:

$$\mathcal{L} = \frac{i}{8} \tilde{\phi}_{ijkl} (F_{ij}, F_{kl}) + \frac{i}{2} \varphi_{ijk}(D_i \phi, F_{jk}) + \partial \Psi$$

where the gauge fermion $\Psi$ is now given by

$$\Psi = -\frac{1}{2}(\chi_i, A_i - 2D_i \phi + \varphi_{ijk} F_{jk}) - \frac{1}{2}(\phi_-, D_i \psi_i + [\phi, \eta_i] - \frac{1}{2}[\phi_+, \eta_2])$$

At first sight the above theory seems to describe the topology of what could be termed the monopole moduli space on a 7-manifold of $G_2$ holonomy; but as it stands it computes invariants of the moduli space of instantons. To see this we discuss first some extra structure that this theory possesses when we take both BRST symmetries into consideration.

### 4.5 A balanced cohomological field theory

The theory described in Section 4.3 actually has a richer structure than the one just described. If instead of focusing on one of the BRST operators we take both into account we are led to a structure which the authors of [14] call a balanced topological field theory. As shown in [12] balanced topological field theories are in fact equivalent to a class of topological theories possessing two topological charges [11,10]. Balanced topological field theories are characterised by having two BRST operators and a global $SL(2)$ symmetry. It is remarkable that this is precisely the structure present in the dimensional reduction of 10-dimensional super Yang–Mills theory on a riemannian 7-manifold of $G_2$ holonomy. One may be tempted to think that this example is paradigmatic: giving as it does a geometric origin for the $SL(2)$ symmetry.

Let us define an $SL(2)$ doublet of fermionic transformations $\delta_A$ by $\delta_A = \epsilon^{AB} \lambda_A \delta_B$, where $\lambda_A$ is defined by equation (19). From this equation we can then read off the action of $\delta_A$ on the fields:

$$\delta_A A_i = i\chi_i A$$
$$\delta_A \eta_B = -\frac{1}{2}[\phi, \phi]_{AB}$$

$$\delta_A \phi_{BC} = i(\epsilon_{AB} \eta_C + \epsilon_{AC} \eta_B)$$
$$\delta_A \chi_i B = -\frac{1}{2} \varphi_{ijk} F_{jk} \epsilon_{AB} - D_i \phi_{AB}$$

It is routine to prove that up to equations of motion, $\{\delta_A, \delta_B\}$ equals a gauge transformation with parameter $-2i\phi_{AB}$. As usual, this is off shell for all fields
but $\chi_{iA}$. In order to lift this restriction we will introduce an auxiliary field $\Lambda_i$ and redefine

$$
\delta_A \chi_{iB} = \varepsilon_{AB} \Lambda_i - D_i \phi_{AB} \quad \text{and} \quad \delta_A \Lambda_i = -i[A_B, \chi_{iB}] + i D_i \eta_A.
$$

Notice that $\{\delta_A, \delta_B\} = -2i[A_{AB}, -]$ on both $\chi_{iA}$ and $\Lambda_i$. An important feature of balanced topological field theories is that the action is given in terms of a gauge invariant potential. In this case we have the following expression for the action:

$$
\mathcal{L} = \mathcal{L}_{\text{top}} + \frac{1}{2} \varepsilon^{AB} \delta_A \delta_B \mathcal{V}, \quad (26)
$$

where

$$
\mathcal{L}_{\text{top}} = \frac{1}{2} \varepsilon_{ijkl}(F_{ij}, F_{kl}) + \frac{1}{2} \varphi_{ijk}(D_i \phi, F_{jk}) \quad (27)
$$

and

$$
\mathcal{V} = \frac{i}{4} \text{CS}(A) + \frac{1}{4} \varepsilon^{AB}(\chi_{iA}, \chi_{iB}) - \frac{1}{4} \varepsilon^{AB}(\eta_A, \eta_B), \quad (28)
$$

where $\text{CS}(A)$, reminiscent of the Chern–Simons form, is given by:

$$
\text{CS}(A) = \varphi_{ijk} \left( (A_i, \partial_j A_k) - \frac{1}{3}(A_i, [A_j, A_k]) \right)
= \frac{1}{2} \varphi_{ijk} \left( (A_i, F_{jk}) + \frac{1}{3}(A_i, [A_j, A_k]) \right). \quad (29)
$$

As with the genuine Chern–Simons form, this $\text{CS}(A)$ is invariant under infinitesimal gauge transformations, which is all that is required for $\mathcal{L}$ to be BRST-invariant. Balanced topological field theories have the property that the path integral localises on the critical points of the potential. In this case, the critical points of the potential $\mathcal{V}$ are configurations for which $\chi_{iA} = \eta_A = 0$ and for which the Yang–Mills curvature defines an instanton: $\varphi_{ijk} F_{jk} = 0$. It follows that this theory computes topological invariants of the instanton moduli space. Its partition function, for example, computes the Euler characteristic.

### 4.6 Comparing with [2]

Now we compare the action given by (17) and (18) with the one in [2], or equivalently with the action in Section 3.4 reduced to seven dimensions and to holonomy $G_2$. In order to do this we will have to truncate the theory and at the same time rewrite it in terms of a slightly different set of fields. We first set $\phi_2$ (that is, $A_8$) to zero. Setting its variation to zero demands that we take $\lambda_1 \eta_2 = -\lambda_2 \eta_1$. Without loss of generality we will take $\lambda_1 = \eta_1 = 0$ and $\lambda_2 \equiv -\lambda$ and $\eta_2 \equiv -\eta$. Any other choice will be related to this one by an $\text{SO}(1,1) \subset \text{SL}(2)$ transformation, but this is an automorphism of the resulting
theory. This leaves only one remaining fermionic symmetry which, omitting
the parameter $\lambda$, is given by:

$$\begin{align*}
\delta A_i &= i\chi_{i1} \\
\delta \phi_0 &= i\eta \\
\delta \phi_1 &= -i\eta
\end{align*}$$

$$\begin{align*}
\delta \eta &= -[\phi_0, \phi_1] \\
\delta \chi_{i1} &= D_i(\phi_0 + \phi_1) \\
\delta \chi_{i2} &= -\frac{1}{2}\varphi_{ijk} F_{jk} .
\end{align*}$$

(30)

By inspection, we find that under the following dictionary:

$$\begin{align*}
\phi_\pm &= \phi_1 \pm \phi_0 , \\
\psi_i &= \chi_{i1} \quad \text{and} \quad \chi_{ij} = \frac{1}{2}\varphi_{ijk} \chi_{k2} ,
\end{align*}$$

(31)

and after using identity (20), the two sets of transformation laws (12) and
(30) agree. Moreover, a little calculation shows that in terms of these fields
the action agrees with the dimensional reduction of (11) and the truncation
$A_8 = \psi_8 = 0$. The only subtlety in this calculation is the fact that in (11),
$\chi$ has components $\chi_{ij}$ and $\chi_{is}$. But using the ‘anti-self-duality’ condition (8),
we can solve for $\chi_{is}$ in terms of $\chi_{ij}$: $\chi_{is} = -\frac{1}{4}\varphi_{ijk} \chi_{jk}$. In addition $\chi_{ij}$
also obeys its own ‘anti-self-duality’ condition $\chi_{ij} = -\frac{1}{4}\varphi_{i}^{\,ijkl} \chi_{kl}$, as can be read
from the second identity in (21) and definition of $\chi_{ij}$ in (31). This is to be
expected since by construction $\chi_{ij}$ belongs to the subrepresentation $\mathfrak{g}$. Finally
to make contact with the action in [2] it is necessary to further rescale $\chi_{ij}$:

$$\chi_{ij}^{[2]} = -\frac{4}{3}\chi_{ij} .$$

It should be mentioned, however, that this truncation results in only a partial
gauge-fixing of the ‘topological’ symmetries of the original theory, since the
field $\eta$ is no longer propagating. Therefore the theory is not well-defined. The
correct cohomological theory describing the topology of the instanton moduli
space for 7-manifolds of $G_2$ holonomy is the untruncated theory given by the
sum of (17) and (18), or equivalently by (26).

4.7 Monopoles in seven dimensions

The BRST transformation law for $\chi_i$ in (22) is reminiscent, when set to zero,
of an equation of Bogomol’nyi type. In fact, this equation has certain parallels
with the more familiar case of 3-dimensional monopoles, which we would
like to exploit. Let us then first review the 3-dimensional case. A monopole
configuration is specified by a gauge field $A_i$ and an adjoint Higgs $\phi$ satisfying
the Bogomol’nyi equation

$$D_i \phi = \pm \frac{1}{2}\epsilon_{ijk} F_{jk} ,$$

(32)

where the sign differentiates the monopole from the antimonopole. We can
understand this equation as a (anti-)self-duality equation in four dimensions by
simply thinking of the Higgs field as the fourth component of a 4-dimensional
gauge field $A_i = (A_i, \phi)$. Of course, the fourth dimension is fake and $A_i$ does not depend on it. Nevertheless the self-duality equation

$$F_{ij} = \pm \frac{1}{2} \varepsilon_{ijkl} F_{kl},$$

is clearly seen to be equivalent to the Bogomol'nyi equation (32).

Now let us consider the 7-dimensional case. In order to guess the form of the Bogomol'nyi equation, we play the same game and dimensionally reduce the 8-dimensional instanton equations. Again we have a gauge field $A_i$ and a Higgs field $\phi$, but where $i$ runs from 1 to 7, and we will understand $\phi$ as $A_8$ but with the tacit understanding that $A_i = (A_i, \phi)$ does not depend on $x^8$. The 8-dimensional curvature has components $F_{ij}$ and $F_{i8} = D_i \phi$. As we saw above there are two possible instanton equations in eight dimensions. We can demand that $F$ belongs either to $\Lambda^1$ or to $\Lambda^2$ of $SO(7)$. It is the latter equation with which this paper is concerned; although it would be interesting to investigate the other equations and in particular its possible supersymmetric origins.\(^9\)

Demanding that $F$ belongs to $\Lambda^2$ is equivalent to

$$\frac{1}{2} \Omega_{ijkl} F_{kl} = F_{ij}.$$

Remarkably, as in three dimensions, this equation is equivalent to the equation of Bogomol'nyi type:

$$D_i \phi = \frac{1}{2} \varepsilon_{ijk} F_{jk}. \quad (33)$$

Notice parenthetically that this equation reduces to one of the instanton equations when the 7-manifold is compact: just as in three dimensions, using the Bianchi identity, the Bogomol'nyi equation implies the equation $D_i D_i \phi = 0$, which in a compact space implies $D_i \phi = 0$.

Now let $X \in \mathfrak{g}$ be a fixed element of the Lie algebra, and consider the gauge-fixing condition: $\nabla_i A_i = [X, \phi]$. We can incorporate this condition on the action in the usual way. We introduce the antighost $b$ and the Nakanishi-Lautrup auxiliary field $\Sigma$, fermionic and bosonic respectively, and both Lie algebra valued. Their transformation laws are given by:

$$\delta b = \Sigma \quad \text{and} \quad \delta \Sigma = i[\phi_+, b],$$

which supplement equation (22). $\delta'$ on these fields is defined as gauge transformations with a ghost parameter:

$$\delta' b = [c, b] \quad \text{and} \quad \delta' \Sigma = [c, \Sigma].$$

\(^9\) Notice that the other 8-dimensional instanton equation is equivalent to the equation: $F_{ij} = -\varepsilon_{ijk} D_k \phi$; which is in turn equivalent to one of the instanton equations in seven dimensions. This is to be expected since both instanton equations restrict the curvature to the $\Lambda^1$ of $Spin(7)$ which remains irreducible under $G_2$.\(^9\)
The new action is again given by equation (23), but where the gauge fermion $\Psi$ in equation (24) receives an extra term: $(b, \Sigma - \nabla_1 A_1 + [X, \phi])$. By analogy with the 3-dimensional case, the resulting theory presumably localises on the solutions of equation (33). However, this being a seven-dimensional theory makes an explicit verification difficult.

5 Conclusion

In this paper we have proven the conjectures made in [4] and [2] concerning the supersymmetric origin of the cohomological field theories appearing in those papers. This gives further evidence of the fact that the effective world-volume theories of curved euclidean D-branes of type II superstring theory are cohomological. This holds both for D-branes wrapping around calibrated submanifolds and for those wrapping around the whole manifold (the whole manifold of course is a trivial case of calibrated submanifold for which the calibration is the volume form). Furthermore, for the higher-dimensional theories defined on six-, seven- and eight-manifolds, the calibrations (the forms, $\Omega, \varphi, \tilde{\varphi}$ discussed in the present paper, and also the holomorphic three-form for the Calabi-Yau three-fold) play an integral role in the definition. This leads one to expect that these higher-dimensional theories will be important in the understanding of the submanifolds and wrapped D-branes, as well as the for the study as presented here of a D-brane wrapping the entire manifold.

In Tables 2 and 3 we present the euclidean curved D-brane scan for type II superstrings. The proofs that most of these resulting theories are cohomological can be found in the papers [7,12,9] and, of course, in this one.

<table>
<thead>
<tr>
<th>dim $M$</th>
<th>Holonomy</th>
<th>Calibrated submanifolds (dimension)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$\text{Spin}(7)$</td>
<td>Cayley (4)</td>
</tr>
<tr>
<td>8</td>
<td>$\text{SU}(4)$</td>
<td>complex (2,4,6), special lagrangian (4)</td>
</tr>
<tr>
<td>8</td>
<td>$\text{Sp}(2)$</td>
<td>complex (2,4,6)</td>
</tr>
<tr>
<td>7</td>
<td>$G_2$</td>
<td>associative (3), coassociative (4)</td>
</tr>
<tr>
<td>6</td>
<td>$\text{SU}(3)$</td>
<td>complex (2,4), special lagrangian (3)</td>
</tr>
<tr>
<td>4</td>
<td>$\text{SU}(2)$</td>
<td>complex (2)</td>
</tr>
</tbody>
</table>

Table 2
Calibrated submanifolds of manifolds $M$.

We end by remarking that formula (26) relates the three topological quantities associated to seven-dimensional gauge theory: the instanton charge, the monopole charge and the Chern–Simons form. In particular the appearance of the Chern–Simons form gives further evidence (see [15] for another approach) of the existence of a higher-dimensional Floer theory. We hope to return to this point in a future publication.
Table 3
Euclidean curved Dp-brane scan organised by holonomy group. This table includes the manifold and its calibrated submanifolds. There are two kinds of 4-dimensional calibrated submanifolds in a Calabi-Yau 4-fold: complex and special lagrangian.

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References


