NONLINEAR WAVES FOR THE SYSTEM
OF COMPRESSIBLE ADIABATIC FLOW THROUGH POROUS MEDIA

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ABSTRACT

By using the Maximum principle the authors give a fair complete result for the
global existence and for the blow-up phenomena of classical solutions to the Cauchy
problem for the system of compressible adiabatic flow through porous media.

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1. Introduction

The waves considered here are classical solutions, i.e., $C^1$ solutions of the following system

$$
\begin{align*}
\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} &= 0, \\
\frac{\partial u}{\partial t} + \frac{\partial p(v, S)}{\partial x} + 2\alpha u &= 0, \\
\frac{\partial S}{\partial t} &= 0,
\end{align*}
$$

which can be used to model the adiabatic gas flow through a porous medium, where $v$, $u$, $S$ denote the specific volume, the velocity, the entropy respectively, and $p$ denotes the pressure with the state equation

$$
p = p(v, S)
$$

satisfying that on each finite domain of $v > 0$, we have

$$p_v < 0. \quad (1.3)
$$

The system is strictly hyperbolic with eigenvalues

$$
\lambda_1 \triangleq -\sqrt{-p_v}, \quad \lambda_2 \triangleq 0, \quad \lambda_3 \triangleq \sqrt{-p_v}. \quad (1.4)
$$

We prescribe the following initial condition

$$
t = 0 : u = u_0(x), \quad v = v_0(x), \quad S = S_0(x), \quad (1.5)
$$

where $u_0(x)$ and $v_0(x) (> 0)$ are supposed to be $C^1$ functions of $x$ with bounded $C^1$ norm and $S_0(x)$ is a $C^2$ function of $x$ with bounded $C^2$ norm.

For the case $\alpha = 0$, some results on the global existence and the blow-up phenomena of the $C^1$ solutions have been established (cf. [1-3]). For the case that the system has a damping term, i.e., $\alpha \neq 0$, many results on the classical solutions have also been obtained for the case of isentropic flow, namely, $S(t, x) \equiv \text{constant}$ (cf. [4-8]). For the general case, i.e., non-isentropic flow, up to now we only know one result on the global existence of the classical solutions (cf. [9]).

In [9], L. Hsiao and D. Serre considered the following initial data

$$
t = 0 : u = u_0(x), \quad v = \bar{v} + v_0(x), \quad S = \bar{S} + S_0(x), \quad (1.6)
$$

where $\bar{v} (> 0)$ and $\bar{S}$ are constants, $u_0(x)$, $v_0(x)$ are supposed to be $C^1$ functions of $x$ with a compact support, and $S_0(x)$ is a $C^2$ function of $x$ with a compact support. They
proved that there exists a unique globally defined classical solution for the Cauchy problem for the system of the polytropic gases if the $C^1$ norm of $(u_0(x), v_0(x))$ and the $C^2$ norm of $S_0(x)$ are small. This means that the damping dissipation is strong enough to preserve the smoothness of the initial data when it is small.

In this paper, we first improve the above result by means of the so-called maximum principle on nonlinear hyperbolic systems; then we give a fair complete result for the global existence and for the blow-up phenomena of classical solutions for the Cauchy problem for the system of the polytropic gases. Some uniform a priori estimates are given in Section 2 and Section 3. The main results are stated and proved in Section 4 and Section 5, respectively.

2. The uniform a priori estimates I

In what follows, we consider the following special and important case, i.e., the polytropic gases

$$p = (\gamma - 1)\rho^\gamma \exp \left( \frac{S}{c_\nu} \right), \quad (2.1)$$

where $c_\nu > 0$ the specific heat capacity, $\gamma \in (1, 3)$ the adiabatic exponent and $\rho = \frac{1}{v}$ the density of the gas. For simplicity, we may assume that $c_\nu = 1$. Hence, (2.1) simply reduces to the following

$$p = (\gamma - 1)\rho^\gamma \exp S. \quad (2.2)$$

By the third equation of (1.1) and the third initial condition of (1.5), on the existence domain of the $C^1$ solution of the Cauchy problem (1.1) and (1.5) we have

$$S(t, x) = S_0(x). \quad (2.3)$$

Thus, the Cauchy problem (1.1) and (1.5) can be rewritten as

$$\begin{cases}
\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0, \\
\frac{\partial u}{\partial t} + \frac{\partial p(v, S_0(x))}{\partial x} + 2\alpha u = 0,
\end{cases} \quad (2.4)$$

$$t = 0: \ u = u_0(x), \ v = v_0(x), \quad (2.5)$$

where $p$ is given by (2.2).

Let

$$w = u + h(x, v), \quad z = u - h(x, v), \quad (2.6)$$

where

$$h(x, v) = 2\sqrt{\frac{\gamma}{\gamma - 1}} v^{-\frac{\gamma+1}{2}} \exp \left( \frac{S_0(x)}{2} \right). \quad (2.7)$$
The Cauchy problem (2.4)-(2.5) can be rewritten as

\[
\begin{align*}
D^+ w &= \frac{1}{4\gamma} \lambda S'_0(x)(w - z) - \alpha(w + z), \\
D^- z &= \frac{1}{4\gamma} \lambda S'_0(x)(w - z) - \alpha(w + z),
\end{align*}
\]  
\tag{2.8}

\[
t = 0 : w = w_0(x), \ z = z_0(x),
\]  
\tag{2.9}

where

\[
D^+ = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x} \quad \text{and} \quad D^- = \frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x},
\]  
\tag{2.10}

\[
\lambda = \sqrt{\gamma(\gamma - 1)v^{-\frac{\gamma+1}{2}}} \exp \left( \frac{S_0(x)}{2} \right)
\]  
\tag{2.11}

and

\[
w_0 = u_0(x) + h(x, v_0(x)), \ z_0 = u_0(x) - h(x, v_0(x)).
\]  
\tag{2.12}

Let

\[
\bar{w} = w \exp \left( -\frac{S_0(x)}{2} \right), \quad \bar{z} = -z \exp \left( -\frac{S_0(x)}{2} \right),
\]  
\tag{2.13}

then, it follows from (2.8)-(2.9) that

\[
\begin{align*}
D^+ \bar{w} &= \left( \alpha + \frac{\lambda S'_0(x)}{4\gamma} \right) (\bar{z} - \bar{w}), \\
D^- \bar{z} &= \left( \alpha - \frac{\lambda S'_0(x)}{4\gamma} \right) (\bar{w} - \bar{z}),
\end{align*}
\]  
\tag{2.14}

\[
t = 0 : \bar{w} = w_0(x) \exp \left( -\frac{S_0(x)}{2\gamma} \right), \quad \bar{z} = -z_0(x) \exp \left( -\frac{S_0(x)}{2\gamma} \right).
\]  
\tag{2.15}

Assumption (H_1): there are two positive constants \(\varepsilon\) and \(\bar{v}_0\) such that

\[
||u_0(x)||_{C^0} \leq \varepsilon, \quad 0 < \bar{v}_0 - \varepsilon \leq v_0(x) \leq \bar{v}_0 + \varepsilon, \quad ||S'_0(x)||_{C^0} \leq \varepsilon.
\]  
\tag{2.16}

Lemma 2.1. Under the Assumption (H_1), if \(\varepsilon\) is small, then, on any given existence domain of the \(C^1\) solution to the Cauchy problem (1.1) and (1.5), there exist positive constants \(c_i\) \((i = 1, 2, 3)\) independent of \((t, x)\) and \(\varepsilon\), such that the following uniform a priori estimates hold:

\[
-c_1\varepsilon + c_2 \leq \bar{w}(t, x), \quad \bar{z}(t, x) \leq c_1\varepsilon + c_3.
\]  
\tag{2.17}

Proof. By (2.7), (2.12) and (2.16), it follows from (2.15) that

\[-C_1\varepsilon + C_2 \leq \bar{w}_0(x), \quad \bar{z}_0(x) \leq C_1\varepsilon + C_3,
\]  
\tag{2.18}
henceforth $C_j$ ($j = 1, 2, \cdots$) will denote positive constants independent of $\varepsilon$ and $(t, x)$.

According to the local existence and uniqueness theorem, there exists $\tau > 0$ such that the Cauchy problem (1.1) and (1.5) admits a unique $C^1$ solution on $0 \leq t \leq \tau$. By continuity, we choose $\tau > 0$ so small that

$$-C_1 \varepsilon + \frac{1}{2} C_2 \leq \bar{w}(t, x), \quad \tilde{z}(t, x) \leq C_1 \varepsilon + 2C_3. \quad (2.19)$$

In order to prove Lemma 2.1, it suffices to show that we can choose $C_4$, $C_5$ and $C_6$ in such a way that on the existence domain of the $C^1$ solution of the Cauchy problem (1.1) and (1.5) such that

$$-C_4 \varepsilon + \frac{1}{2} C_5 \leq \bar{w}(t, x), \quad \tilde{z}(t, x) \leq C_4 \varepsilon + 2C_6, \quad (2.20)$$

we have

$$-C_4 \varepsilon + C_5 \leq \bar{w}(t, x), \quad \tilde{z}(t, x) \leq C_4 \varepsilon + C_6, \quad (2.21)$$

where

$$C_4 \geq C_1, \quad 0 < C_5 \leq C_2 \quad \text{and} \quad C_6 \geq C_3. \quad (2.22)$$

By (2.6), (2.7), (2.13), (2.20) and the boundedness of the $C^2$ norm of $S(t, x) = S_0(x)$, it follows from (2.11) and (2.16) that

$$\lambda ||S_0'(x)||_{C^0} \leq C_7 \varepsilon \leq 4\alpha \gamma \quad (2.23)$$

provided that $\varepsilon > 0$ is suitably small. Hence we have

$$\alpha - \frac{\lambda S_0'(x)}{4\gamma} \geq 0, \quad \alpha + \frac{\lambda S_0'(x)}{4\gamma} \geq 0. \quad (2.24)$$

Then, using the so-called Maximum principle (cf. [10]) we have that on the existence domain of the $C^1$ solution of the Cauchy problem (1.1) and (1.5)

$$-C_1 \varepsilon + C_2 \leq \bar{w}(t, x), \quad \tilde{z}(t, x) \leq C_1 \varepsilon + C_3. \quad (2.25)$$

Noting (2.22), then we get (2.21). Taking $c_1 \geq C_1$, $0 < c_2 \leq C_2$ and $c_3 \geq C_3$ completes the proof of Lemma 2.1. \hfill {□}

By (2.13) we have

**Corollary 2.1.** Under the Assumption $(H_1)$, if $\varepsilon$ is small, then, on any given existence domain of the $C^1$ solution to the Cauchy problem (1.1) and (1.5), there exist positive constants $c_i$ ($i = 4, 5, 6$) independent of $(t, x)$ and $\varepsilon$, such that the following uniform a priori estimates hold:

$$-c_4 \varepsilon + c_5 \leq w(t, x), \quad -z(t, x) \leq c_4 \varepsilon + c_6. \quad (2.26)$$

Similarly, we can show the following
Lemma 2.2. Under the Assumption \((H_1)\), if \(\varepsilon\) is small and
\[
\|S_0(x) - \bar{S}_0\|_{C^0} \leq \varepsilon,
\]  
(2.27)
where \(\bar{S}_0\) is a constant, then, on any given existence domain of the \(C^1\) solution to the Cauchy problem (1.1) and (1.5), there exist positive constants \(k_i\) \((i = 1, 2, \cdots, 6)\) independent of \((t, x)\) and \(\varepsilon\), such that the following uniform a priori estimates hold:
\[-k_1\varepsilon + k_2 \leq \bar{w}(t, x), \quad \bar{z}(t, x) \leq k_1\varepsilon + k_2,\]
(2.28)
\[-k_3\varepsilon + k_4 \leq w(t, x), \quad -z(t, x) \leq k_3\varepsilon + k_4,\]
(2.29)
moreover,
\[|u(t, x)| \leq k_5\varepsilon, \quad |v(t, x) - \bar{v}_0| \leq k_6\varepsilon.\]
(2.30)

3. The uniform a priori estimates II

Differentiating (2.8) with respect to \(x\) and noting (2.10) we have
\[
\begin{align*}
D^+ w_x &= -\frac{\gamma + 1}{4}{\rho w_x^2} + \frac{\gamma + 1}{4}{\rho w_x z_x} - \left(\alpha - \frac{3\lambda S_0'(x)}{2(\gamma - 1)}\right) w_x - \left(\frac{\lambda S_0'(x)}{2(\gamma - 1)} + \alpha\right) z_x - \frac{\lambda^2 v}{\gamma - 1}\left(\frac{S_0'(x)}{\gamma - 1} - S_0''(x)\right), \\
D^- z_x &= -\frac{\gamma + 1}{4}{\rho z_x^2} + \frac{\gamma + 1}{4}{\rho w_x z_x} - \left(\alpha + \frac{3\lambda S_0'(x)}{2(\gamma - 1)}\right) z_x + \left(\frac{\lambda S_0'(x)}{2(\gamma - 1)} - \alpha\right) w_x - \frac{\lambda^2 v}{\gamma - 1}\left(\frac{S_0'(x)}{\gamma - 1} - S_0''(x)\right).
\end{align*}
\]
(3.1)
Define
\[h = \frac{\gamma + 1}{4}\ln \rho\]
(3.2)
and
\[g_1 = \frac{4\alpha}{3 - \gamma} v^{\frac{3 - \gamma}{4}} + \frac{2\lambda v^{\frac{3 - \gamma}{4}} S_0'(x)}{(3\gamma - 1)(\gamma - 1)}, \quad g_2 = \frac{4\alpha}{3 - \gamma} v^{\frac{3 - \gamma}{4}} - \frac{2\lambda v^{\frac{3 - \gamma}{4}} S_0'(x)}{(3\gamma - 1)(\gamma - 1)}.\]
(3.3)
Then, it follows from (3.1) that
\[
\begin{align*}
D^+ (w_x \exp h) &= -K_1(\rho)(w_x \exp h)^2 - K_2(x, v) w_x \exp h - D^+ g_1 - K_3(x, v), \\
D^- (z_x \exp h) &= -K_1(\rho)(z_x \exp h)^2 - K_4(x, v) z_x \exp h - D^- g_2 - K_5(x, v),
\end{align*}
\]
(3.4)
where

\[
\begin{aligned}
K_1(r) &= \frac{\gamma + 1}{4} \rho \exp(-h), \\
K_2(x,v) &= \alpha - \frac{5 - \gamma}{4(\gamma - 1)} \lambda S_0'(x), \\
K_3(x,v) &= \frac{\lambda^2 v^{\frac{\gamma - 2}{\gamma - 1}}}{(\gamma - 1)(3\gamma - 1)} \left( \frac{5\gamma - 3}{2(\gamma - 1)} (S_0'(x))^2 - 3(\gamma - 1)S_0''(x) \right) - \frac{\alpha}{\gamma - 1} \lambda v^{\frac{\gamma - 2}{\gamma - 1}} S_0'(x), \\
K_4(x,v) &= \alpha + \frac{5 - \gamma}{4(\gamma - 1)} \lambda S_0'(x), \\
K_5(x,v) &= \frac{\lambda^2 v^{\frac{\gamma - 2}{\gamma - 1}}}{(\gamma - 1)(3\gamma - 1)} \left( \frac{5\gamma - 3}{2(\gamma - 1)} (S_0'(x))^2 - 3(\gamma - 1)S_0''(x) \right) + \frac{\alpha}{\gamma - 1} \lambda v^{\frac{\gamma - 2}{\gamma - 1}} S_0'(x).
\end{aligned}
\] (3.5)

Moreover, define

\[
W = w_x \exp h + g_1, \quad Z = z_x \exp h + g_2,
\]

then, it follows from (3.4) that

\[
\begin{aligned}
D^+ W &= -K_1 W^2 + (2K_1 g_1 - K_2) W - K_1 g_1^2 + K_2 g_1 - K_3, \\
D^- Z &= -K_1 Z^2 + (2K_1 g_2 - K_4) Z - K_1 g_2^2 + K_4 g_2 - K_5.
\end{aligned}
\] (3.7)

Let

\[
\Delta_1 = K_2^2 - 4K_1 K_3, \quad \Delta_2 = K_4^2 - 4K_1 K_5
\]

and suppose that on the existence domain of the \(C^1\) solution to the Cauchy problem (1.1) and (1.5)

\[
\Delta_1 \geq 0, \quad \Delta_2 \geq 0.
\] (3.9)

Then (3.7) can be rewritten as follows

\[
\begin{aligned}
D^+ W &= -K_1 (W - W_1) (W - W_2), \\
D^- Z &= -K_1 (Z - Z_1) (Z - Z_2),
\end{aligned}
\] (3.10)

where

\[
\begin{aligned}
W_1 &= \frac{K_2 - 2K_1 g_1 - \sqrt{\Delta_1}}{2K_1}, \\
W_2 &= \frac{K_2 - 2K_1 g_1 + \sqrt{\Delta_1}}{2K_1}, \\
Z_1 &= \frac{K_4 - 2K_1 g_2 - \sqrt{\Delta_2}}{2K_1}, \\
Z_2 &= \frac{K_4 - 2K_1 g_2 + \sqrt{\Delta_2}}{2K_1}.
\end{aligned}
\] (3.11)

On the existence domain \(\mathcal{D}\) of the \(C^1\) solution to the Cauchy problem (1.1) and (1.5), define

\[
\sigma_1 = \inf_{(t,x) \in \mathcal{D}} W_2(t,x) - \sup_{(t,x) \in \mathcal{D}} W_1(t,x), \quad \sigma_2 = \inf_{(t,x) \in \mathcal{D}} Z_2(t,x) - \sup_{(t,x) \in \mathcal{D}} Z_1(t,x).
\] (3.12)
Lemma 3.2. Consider the Cauchy problem for (3.10) with the following initial data

\[ t = 0: W = W_0(x), \quad Z = Z_0(x). \]  

(3.13)

Suppose that, on the existence domain \( \mathcal{D} \) of the \( C^1 \) solution to the Cauchy problem (1.1) and (1.5), \( ||u||_{C^0(\mathcal{D})} \) and \( ||v||_{C^0(\mathcal{D})} \) are bounded and

\[ \sigma_1 > 0, \quad \sigma_2 > 0. \]  

(3.14)

Suppose further that (3.9) holds. If

\[ W_0(x) > \sup_{(t,x)\in \mathcal{D}} W_1(t,x), \quad Z_0(x) > \sup_{(t,x)\in \mathcal{D}} Z_1(t,x), \]  

(3.15)

then, on the existence domain \( \mathcal{D} \) of the \( C^1 \) solution to the Cauchy problem (1.1) and (1.5) we have

\[
\begin{align*}
\min \left\{ \inf_{x \in \mathcal{R}} W_0(x), \inf_{(t,x)\in \mathcal{D}} W_2(t,x) \right\} &\leq W(t,x) \leq \max \left\{ \sup_{x \in \mathcal{R}} W_0(x), \sup_{(t,x)\in \mathcal{D}} W_2(t,x) \right\}, \\
\min \left\{ \inf_{x \in \mathcal{R}} Z_0(x), \inf_{(t,x)\in \mathcal{D}} Z_2(t,x) \right\} &\leq Z(t,x) \leq \max \left\{ \sup_{x \in \mathcal{R}} Z_0(x), \sup_{(t,x)\in \mathcal{D}} Z_2(t,x) \right\}.
\end{align*}
\]

(3.16)

This Lemma follows from the following

Lemma 3.3. Consider the initial value problem

\[
\begin{align*}
\frac{dy}{dt} &= -k(t)(y - y_1(t))(y - y_2(t)), \quad t > 0, \\
y(0) &= y_0,
\end{align*}
\]

(3.17)

where \( k(t), y_1(t) \) and \( y_2(t) \) are \( C^0 \) functions with bounded \( C^0 \) norm and satisfy

\[ k(t) > 0, \quad \max_t y_1(t) < \inf_t y_2(t). \]  

(3.18)

If

\[ y_0 > \max_t y_1(t), \]  

(3.19)

then, on the existence domain of the \( C^1 \) solution to the initial value problem (3.17) we have

\[
\min \left\{ y_0, \inf_t y_2(t) \right\} \leq y(t) \leq \max \left\{ y_0, \sup_t y_2(t) \right\}. \quad \square
\]

(3.20)
In fact, on the existence domain of the $C^1$ solution to the Cauchy problem (1.1) and (1.5) let $x_+ = x_+(t;\beta)$ be the forward characteristic passing through any fixed point $(0;\beta)$

$$\begin{cases} 
\frac{dx_+}{dt} = \lambda, \\
x_+(0) = \beta. 
\end{cases} \quad (3.21)$$

Along this characteristic we consider the initial value problem for the first equation of (3.10) with initial data $W(0) = W_0(\beta)$. Noting the assumptions of Lemma 3.2, we apply Lemma 3.3 to this problem and get

$$\min \left\{ W_0(\beta), \inf_t W_2(t, x_+(t;\beta)) \right\} \leq W(t, x_+(t;\beta)) \leq \max \left\{ W_0(\beta), \sup_t W_2(t, x_+(t;\beta)) \right\}.$$

By the arbitrariness of $\beta$, we easily get the first inequalities of (3.16).

Similarly, we can show the second inequalities of (3.16).

4. The global existence

In this section we shall prove the following Theorem 4.1 and Theorem 4.2.

**Theorem 4.1.** Under the Assumption $(H_1)$, if

$$||u'_0(x)||_{C^0} \leq \varepsilon, \quad ||v'_0(x)||_{C^0} \leq \varepsilon \quad (4.1)$$

and $\varepsilon > 0$ is suitably small, then the Cauchy problem (1.1) and (1.5) admits a unique global $C^1$ solution on $t \geq 0$. 

**Theorem 4.2.** Under the Assumption $(H_1)$, suppose furthermore that

$$||S''_0(x)||_{C^0} \leq \varepsilon, \quad \forall \, x \in \mathbb{R} \quad (4.2)$$

If

$$w'_0(x) \geq -\frac{4\alpha}{3-\gamma} v_0(x) - \frac{2\lambda_0(x)S'_0(x)v_0(x)}{(3\gamma - 1)(\gamma - 1)} + (v_0(x))^{\frac{\gamma + 1}{4}} \max_{(t,x) \in \mathcal{D}} W_1(t,x), \quad (4.3)$$

and

$$\eta'_0(x) \geq -\frac{4\alpha}{3-\gamma} v_0(x) + \frac{2\lambda_0(x)S'_0(x)v_0(x)}{(3\gamma - 1)(\gamma - 1)} + (v_0(x))^{\frac{\gamma + 1}{4}} \max_{(t,x) \in \mathcal{D}} Z_1(t,x), \quad (4.4)$$

then there is a $\varepsilon_0 > 0$ so small that for any $\varepsilon \in [0, \varepsilon_0]$ the Cauchy problem (1.1) and (1.5) admits a unique global $C^1$ solution on $t \geq 0$, where $W_1(t,x)$ and $Z_1(t,x)$ are given by (3.11). 

**Proof of Theorem 4.1.** By the existence and uniqueness theorem of the local classical solutions to Cauchy problem for quasilinear hyperbolic systems, in order to prove...
Theorem 4.1, it suffices to establish a uniform a priori estimate on the $C^1$ norm of the $C^1$ solution $(u(t, x), v(t, x), S(t, x))$ to the Cauchy problem (1.1) and (1.5) on the existence domain of the $C^1$ solution. Noting (2.3) and the hypothesis on $S_0(x)$ we easily get the uniform a priori estimate on the $C^1$ norm of $S(t, x)$ on the existence domain of the $C^1$ solution. Thus, we only need to establish a uniform a priori estimate on the $C^1$ norm of $(u(t, x), v(t, x))$ on the existence domain of the $C^1$ solution.

By (2.6)-(2.7) and (2.3), it follows from Corollary 2.1 that there is a $\varepsilon_0 > 0$ so small that on the existence domain of the $C^1$ solution we have

$$|u(t, x)|, |v(t, x)| \leq C_\gamma. \quad (4.5)$$

In what follows, we shall establish a uniform a priori estimate on the first derivatives of $(u(t, x), v(t, x))$ on the existence domain of the $C^1$ solution.

To do so, let

$$r = -\lambda w_x + \frac{\lambda S_0'(x)}{4\gamma} (w - z), \quad s = \lambda z_x + \frac{\lambda S_0'(x)}{4\gamma} (w - z). \quad (4.6)$$

Then we have

$$\begin{aligned}
D^+ r &= A(t, x)(s - r), \\
D^- s &= B(t, x)(r - s),
\end{aligned} \quad (4.7)$$

$$r = r_0(x) \triangleq -\lambda_0(x)w_0'(x) + \frac{\lambda_0(x)S_0'(x)}{4\gamma} (w_0(x) - z_0(x)), \quad (4.8)$$

$$s = s_0(x) \triangleq \lambda_0(x)z_0'(x) + \frac{\lambda_0(x)s_0'(x)}{4\gamma} (w_0(x) - z_0(x)), \quad (4.8)$$

where

$$\begin{aligned}
A(t, x) &= \alpha + \frac{(\gamma + 1)\lambda w_x}{(\gamma - 1)(w - z)} - \frac{\lambda S_0'(x)}{2(\gamma - 1)}, \\
B(t, x) &= \alpha + \frac{(\gamma + 1)\lambda z_x}{(\gamma - 1)(w - z)} + \frac{\lambda S_0'(x)}{2(\gamma - 1)},
\end{aligned} \quad (4.9)$$

$$\lambda_0(x) = \sqrt{(\gamma - 1)(v_0(x))^{-\frac{\gamma - 1}{2}}} \exp \left( \frac{S_0(x)}{2} \right). \quad (4.10)$$

$w_0(x), z_0(x)$ are given by (2.12) and $\lambda$ is given by (2.11). By (2.16), (4.1), Corollary 2.1 and the definition of $w_0(x)$ and $z_0(x)$, we can take $\varepsilon$ so small that

$$A(0, x) \geq \frac{\alpha}{2}, \quad B(0, x) \geq \frac{\alpha}{2}, \quad \forall x \in \mathbf{R}. \quad (4.11)$$

For the time being, we suppose that on the existence domain of the $C^1$ solution

$$A(t, x) \geq 0, \quad B(t, x) \geq 0. \quad (4.12)$$
We will explain the reasonableness of hypothesis (4.12) at the end of proof.

Noting the hypothesis (4.12) and using the so-called Maximum principle (cf. [10]) we have that on the existence domain of the $C^1$ solution of the Cauchy problem (4.7)-(4.8)

$$\min\left(\inf_{x \in \mathbb{R}} r_0(x), \inf_{x \in \mathbb{R}} s_0(x)\right) \leq r(t, x), s(t, x) \leq \max\left(\sup_{x \in \mathbb{R}} r_0(x), \sup_{x \in \mathbb{R}} s_0(x)\right).$$

(4.13)

Using (2.16), (4.1), (4.13) and the definition of $r(t, x)$ and $s(t, x)$ again, it follows from (4.9) that hypothesis (4.12) is reasonable, provided that $\varepsilon > 0$ is sufficiently small.

By (4.13), we can easily get that there is a $\varepsilon_0 > 0$ so small that on the existence domain of the $C^1$ solution we have

$$|w_x(t, x)|, |z_x(t, x)| \leq C_8. \quad (4.14)$$

By (2.6)-(2.7), it follows from (4.5) and (4.14) that there is a $\varepsilon_0 > 0$ so small that on the existence domain of the $C^1$ solution we have

$$||u||_{C^1}, ||v||_{C^1} \leq C_9. \quad (4.15)$$

Theorem 4.1 is thus proved. $\Box$

Now we prove Theorem 4.2.

Using (3.5), (3.3) and Corollary 2.1, we have

**Lemma 4.1.** Under the assumptions of Theorem 4.2, there is a $\varepsilon_0 > 0$ so small that for any $\varepsilon \in [0, \varepsilon]$, on the existence domain of the $C^1$ solution to the Cauchy problem (1.1) and (1.5), we have

$$M_1 \leq K_1(\rho) \leq M_2, \quad (4.16)$$

$$\alpha - M_3\varepsilon \leq K_2(x, v), K_4(x, v) \leq \alpha + M_3\varepsilon, \quad (4.17)$$

$$|K_3(x, v)|, |K_5(x, v)| \leq M_4\varepsilon, \quad (4.19)$$

$$0 < g_1(t, x), g_2(t, x) \leq M_5, \quad (4.20)$$

henceforth $M_i(i = 1, 2, \cdots)$ denote positive constants independent of $\varepsilon$ and $(t, x)$. $\Box$

**Proof of Theorem 4.2.** Same as the proof of Theorem 4.1, it suffices to establish the uniform a priori estimate on the $C^1$ norm of $(u(t, x), v(t, x))$; moreover, (4.5) still holds. Thus we only need to establish the uniform a priori estimate on the $C^0$ norm of the first order derivatives of $(u(t, x), v(t, x))$.

To do so, noting Lemma 4.1 and the assumptions of Theorem 4.2, we know that all assumptions required by Lemma 3.2 are satisfied, so we can use Lemma 3.2 and easily
establish the uniform a priori estimate on the $C^0$ norm of the first order derivatives of $(u(t,x),v(t,x))$. Thus Theorem 4.2 is easily completed. □

5. The blow-up phenomena

In this Section we shall investigate the blow-up phenomena of the classical solution to the Cauchy problem (1.1) and (1.5). We have

**Theorem 5.1.** Under the Assumption $(H_1)$, if (4.2) holds and there exists a point $\beta \in \mathbb{R}$ such that

$$W_0(\beta) < -\frac{4\alpha}{3-\gamma}v_0(\beta) - \frac{2\lambda_0(\beta)S'_0(\beta)v_0(\beta)}{(3\gamma-1)(\gamma-1)} + (v_0(\beta))^{\frac{2+1}{4}} \inf_{(t,x) \in \mathcal{D}} W_1(t,x) \quad (5.1)$$

or

$$Z_0(\beta) < -\frac{4\alpha}{3-\gamma}v_0(\beta) + \frac{2\lambda_0(\beta)S'_0(\beta)v_0(\beta)}{(3\gamma-1)(\gamma-1)} + (v_0(\beta))^{\frac{2+1}{4}} \inf_{(t,x) \in \mathcal{D}} Z_1(t,x), \quad (5.2)$$

then there is a $\varepsilon_0 > 0$ so small that for any $\varepsilon \in (0,\varepsilon_0]$ the classical solution to the Cauchy problem (1.1) and (1.5) must blow up in a finite time, where $W_1(t,x)$ and $Z_1(t,x)$ are given by (3.11). □

In order to prove Theorem 5.1, we need the following

**Lemma 5.1.** For the Cauchy problem (3.10) and (3.13), suppose that on the existence domain $\mathcal{D}$ of the $C^1$ solution to the Cauchy problem (1.1) and (1.5), $\|u\|_{C^0(\mathcal{D})}$ and $\|v\|_{C^0(\mathcal{D})}$ are bounded, suppose furthermore that (3.9) and (3.14) holds. If there exists a point $\beta \in \mathbb{R}$ such that

$$W_0(\beta) < \inf_{x \in \mathcal{D}} W_1(t,x) \quad (5.3)$$

or

$$Z_0(\beta) < \inf_{x \in \mathcal{D}} Z_1(t,x), \quad (5.4)$$

then the solution to the Cauchy problem (3.10) and (3.13) must blow up in a finite time. □

This Lemma follows from the following

**Lemma 5.2.** Consider the initial value problem (3.17). Suppose that $k(t)$, $y_1(t)$ and $y_2(t)$ are $C^0$ functions with bounded $C^0$ norm and satisfy (3.18). If

$$y_0 < \inf_{t} y_1(t), \quad (5.5)$$
then the solution to the initial value problem (3.17) must blow up in a finite time. □

In fact, without loss of generality, we assume that (5.3) holds. Along the forward characteristic $x_+ = x_+(t; \beta)$ passing through the point $(0, \beta)$ (see (3.21)), we consider the initial value problem for the first equation of (3.10) with initial data $W(0) = W_0(\beta)$. Noting the assumptions of Lemma 5.1, we apply Lemma 5.2 to this problem and easily get Lemma 5.1.

Proof of Theorem 5.1. Under the assumptions of Theorem 5.1, using Corollary 2.1, it is easy to check that all conditions required by Lemma 5.1 are satisfied. Thus, using Lemma 5.1 and noting the definition of $Z_0(x)$ and $W_0(x)$, we can immediately get Theorem 5.1. □

Using Lemma 2.2, it follows from Theorem 5.1 that

Theorem 5.2. Under the Assumption ($H_1$), suppose furthermore that (2.27) and (4.2) hold. If there exists a point $\beta \in \mathbb{R}$ such that

$$w'_0(\beta) < 0$$

or

$$z'_0(\beta) < 0,$$

then there is a $\varepsilon_0 > 0$ so small that for any $\varepsilon \in (0, \varepsilon_0]$ the classical solution to the Cauchy problem (1.1) and (1.5) must blow up in a finite time. □

In fact, noting (2.30), (2.3), (2.27) we have

The right-hand side term of (5.1) ≥ $-C_{10}\varepsilon$ \hspace{1cm} (5.8)

and

The right-hand side term of (5.2) ≥ $-C_{10}\varepsilon$.

Taking $\varepsilon_0$ so small that

$$w'_0(\beta) < -C_{10}\varepsilon_0$$ \hspace{1cm} (5.10)

or

$$z'_0(\beta) < -C_{10}\varepsilon_0,$$ \hspace{1cm} (5.11)

thus, by Theorem 5.1 we immediately obtain Theorem 5.2.

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