EQUIVARIANT HIGHER K-THEORY
FOR COMPACT LIE GROUP ACTIONS

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MIRAMARE – TRIESTE
May 1997
Introduction

The aim of this paper is to construct an equivariant higher $K$-theory for compact Lie group actions in a way analogous to the ones constructed in [2], [3] for finite groups and [6] for profinite groups.

In Sec. 1 we discuss the category $\mathcal{A}(G)$ of homogeneous spaces on which Mackey functors are defined. In Sec. 2 we define the higher $K$-groups $K_n^G(G/H, \mathcal{C})$ $n \geq 0$ for any exact category $\mathcal{C}$ and show that $K_n^G(\cdot, \mathcal{C}) : \mathcal{A}(G) \to \mathbb{Z}$-mod is a Mackey functor and that $K_n^G(\cdot, \mathcal{C})$ are $K_n^G(\cdot, \mathcal{C})$-modules. In Sec. 3, we explore induction techniques in the style of [4] to show that $K_n^G(G/G, M(\mathcal{C}))$ (all $n \geq 0$) are hyperelementary computable where $M(\mathcal{C})$ is the category of finite dimensional vector spaces over the complex numbers.

In a final section, we briefly discuss possible generalisations of the foregoing to the category of $G$-spaces of $G$-homotopy type of $G$-CW complexes.

1 Mackey Functors on the Category $\mathcal{A}(G)$ of Homogeneous Spaces

1.1 Let $G$ be a compact Lie group, $X$ a $G$-space. The component category $\pi_0(G, X)$ is defined as follows: Objects of $\pi_0(G, X)$ are homotopy classes of maps $\alpha : G/H \to X$ where $H$ is a closed subgroup of $G$. A morphism from $[\alpha] : G/H \to X$ to $[\beta] : G/K \to X$ is a $G$-map $\sigma : G/H \to G/K$ such that $\beta \sigma$ is $G$-homotopic to $\alpha$.

Note that since $\text{Hom}(G/H, X) \simeq X^H$ where $\varphi \to \varphi(eH)$, we could consider objects of $\pi_0(G, H)$ as pairs $(H, c)$ where $c \in \pi_0(X^H) = \text{the set of path components of } X^H$.

1.2 A G-ENR (Euclidean Neighbourhood Retract) is a $G$-space that is $G$-homeomorphic to a $G$-retract of some open $G$-subset of some $G$-module $V$. Let $Z$ be a compact $G$-ENR, $f : Z \to X$ a $G$-map. For $\alpha : G/H \to X$ in $\pi_0(G, X)$, we identify $\alpha$ with the path component $X^H_\alpha$ into which $G/H$ is mapped by $\alpha$.

Put $Z(f, \alpha) = Z^H \cap f^{-1}(X^H_\alpha) := \text{subspace of } Z^H \text{ mapped under } f_H \text{ into } X^H_\alpha$. The action of $N_\alpha H/H$ on $Z^H$ induces an action of $N_\alpha H/H$ on $Z(f, \alpha)$ i.e. $Z(f, \alpha)$ is an $\text{Aut}(\alpha)$-space (see [4]).

1.3 Let $\{Z_i\}$ be a collection of $G$-ENR, $f_i : Z_i \to X$. Say that $f_i : Z_i \to X$ is equivalent to $f_j : Z_j \to X$ if and only if for each $\alpha : G/H \to X$ in $\pi_0(G, X)$, the Euler characteristic
\[ \chi(Z(f_i, \alpha)/\text{Aut}(\alpha)) = \chi(Z(f_j, \alpha_j)/\text{Aut}(\alpha)). \]

Let \( U(G, X) \) be the set of equivalence classes \([f : Z \to X]\) where addition is given by \([f_0 : Z_0 \to X] + [f_1 : Z_1 \to X] = [f_0 + f_1 : Z_0 + Z_1 \to X]\); the identity element is \( \phi \to X \); and the additive inverse of \([f : Z \to A]\) is \([f \circ \phi : Z \times A \to Z \to X]\), where \( A \) is a compact \( G \)-ENR with trivial \( G \)-action and \( \chi(A) = -1 \) (see [4]).

Then \( U(G, X) \) is the free Abelian group generated by \([\alpha], \alpha \in \pi_0(G, X)\) i.e. \([f : Z \to X] = \Sigma n(\alpha)[\alpha]\), where \( G/H \times E^n \subset Z \) is an open \( n \)-cell of \( Z \), the restriction of \( f \) to \( G/H \times E^n \) defines an element \([\alpha]\) of \( U(G, X)\).

The cell is called an \( n \)-cell of type \( \alpha \). Let \( n(\alpha) = \Sigma(-1)^i n(\alpha, i) \) where \( n(\alpha, i) \) is number of \( i \)-cells of type \( \alpha \) (see [4]).

If \( X \) is a point, write \( U(G) \) for \( U(G, X) \).

1.4 For a compact Lie group \( G \), the category \( \mathcal{A}(G) \) is defined as follows: \( \text{ob} \mathcal{A}(G) := \) homogeneous spaces \( G/H \); The morphisms in \( \mathcal{A}(G)(G/H, G/K) \) are the elements of the Abelian group \( U(G, G/H \times G/K) \) and have the form \( \alpha : G/L \to G/H \times G/K \) which can be represented by diagram \( \{G/H \xrightarrow{\alpha} G/L \xrightarrow{\beta} G/K\} \), so that \( U(G, G/H \times G/K) = \) free Abelian group on the equivalence classes of diagrams \( G/H \xrightarrow{\alpha} G/L \xrightarrow{\beta} G/K \) where two such diagrams are equivalent if there exists an isomorphism \( \sigma : G/L \to G/L' \) such that the diagram

\[
\begin{array}{ccc}
G/L & \xrightarrow{\alpha} & G/L' \\
\downarrow & & \downarrow \\
G/H & \xrightarrow{\sigma} & G/K \\
\downarrow & & \downarrow \\
& & \\
G/H & & G/K \\
\end{array}
\]

commutes up to homotopy.

Composition of morphisms are given by a bilinear map

\[ U(G, G/H_1 \times G/H_2) \times U(G, G/H_2 \times G/H_3) \to U(G, G/H_1 \times G/H_3) \]

where the composition of \((\alpha, \beta_1) : A \to G/H_1 \times G/H_2 \) and \((\beta_2, \gamma) : B \to G/H_2 \times G/H_3 \) yields a \( G \)-map \((\alpha\bar{\alpha}, \gamma \bar{\gamma}) : C \to G/H_1 \times G/H_3 \), where \( \bar{\gamma}, \bar{\alpha} \) are maps \( \bar{\gamma} : C \to B \) and \( \bar{\alpha} : C \to A \).
Remarks  (i) Each morphism $G/H \overset{\alpha}{\longrightarrow} G/L \overset{\beta}{\longrightarrow} G/K$ is the composition of special types of morphisms

$$G/H \overset{\alpha}{\leftarrow} G/L \overset{\text{id}}{\rightarrow} G/L \text{ and } G/L \overset{\text{id}}{\leftarrow} G/L \overset{\beta}{\rightarrow} G/K.$$  

(ii) If $\pi_0$ (or $G$) is the homotopy category of the orbit category or $(G)$, that is, the objects of $\pi_0$ (or $G$) are the homogeneous $G$-spaces $G/H$ and morphisms are homotopy classes $[G/L \rightarrow G/K]$ of $G$-maps $G/L \rightarrow G/K$. We have a covariant functor $\pi_0$ (or $G$) $\rightarrow \mathcal{A}(G)$ given by $[G/L \overset{\beta}{\rightarrow} G/K] \rightarrow (G/L \overset{\text{id}}{\leftarrow} G/L \overset{\beta}{\rightarrow} G/K)$ and a contravariant functor $\pi_0$ (or $G$)

$$[G/H \rightarrow G/L] \rightarrow (G/H \leftarrow G/L \overset{\text{id}}{\rightarrow} G/L).$$

(iii) Addition is defined in $\mathcal{A}(G)(G/H, G/K) = \mathcal{U}(G, G/H \times G/K)$ by

$$(G/H \leftarrow G/L \rightarrow G/K) + (G/H \leftarrow G/L' \rightarrow G/K)$$

$$= (G/H \leftarrow (G/L) \cup (G/L') \rightarrow G/K)$$

where $(G/L) \cup (G/L')$ is the topological sum of $G/L$ and $G/L'.$

1.6 Let $R$ be a commutative ring with identity. A Mackey functor $M$ from $\mathcal{A}(G)$ to $R$-$\text{mod}$ is a contravariant additive functor. Note that $M$ is additive if

$$M : \mathcal{A}(G)(G/H, G/K) \rightarrow R$-$\text{mod}(M(G/K), M(G/H))$$

is an Abelian group homomorphism.

Remarks 1.7 $M$ comprises of two types of induced morphisms

(i) If $\alpha : G/H \rightarrow G/K$ is a $G$-map, regarded as an ordinary morphism $\alpha_1 : G/H \overset{\text{id}}{\leftarrow} G/H \overset{\alpha}{\rightarrow} G/K$ of $\mathcal{A}(G)$, we have an induced morphism

$$M(\alpha_1) = M^*(\alpha) =: \alpha^* : M(G/K) \rightarrow M(G/H).$$

(ii) If $\alpha$ in (i) is induced from $H \subset K$, i.e. $\alpha(gH) = gK$, call $\alpha^*$ the restriction morphism.

(iii) If we consider $\alpha$ as a transfer morphism $\alpha_1 : G/H \rightarrow G/K \rightarrow G/K$ in $\mathcal{A}(G)$, then we have

$$M(\alpha_1) =: M_*(\alpha) =: \alpha_* : M(G/H) \rightarrow M(G/K)$$

and call $\alpha_*$ the induced homomorphism associated to $\alpha.$
1.6 Let $M, N, L$ be Mackey functors on $\mathcal{A}(G)$. A pairing $M \times N \rightarrow L$ is a family of bilinear maps $M(S) \times N(S) \rightarrow L(S) (x, y) \rightarrow x \cdot y (S \in \mathcal{A}(G))$ such that for each $G$-map $f : G/H = S \rightarrow T = G/K$ we have $L*f(x, y) = (M*f x) \cdot (N*f y) (X \in M(T) y \in N(T)$.

$x \cdot (N*fy) = L*f((M*f x) - y)$, $x \in M(T), y \in N(S)$.

$M*fx) \cdot y = L*f(x \cdot (N*fy)), x \in N(S), y \in N(T)$.

A Green functor $V : \mathcal{A}(G) \rightarrow R$-mod is a Mackey functor together with a pairing $V \times V \rightarrow V$ such that for each object $S$, the map $V(S) \times V(S) \rightarrow V(S)$ turns $V(S)$ into an associative $R$-algebra such that $f^*$ preserves units.

If $V$ is a Green functor, a left $V$-module is a Mackey functor $M$ together with a pairing $V \times M \rightarrow M$ such that $M(S)$ is a left $V(S)$-module for every $S \in \mathcal{A}(G)$.

1.7 Remarks The Mackey functor as defined in 1.5 is equivalent to the earlier definitions in [2], [3], [7] defined for finite and profinite groups $G$ as functors from the category $\hat{G}$ of $G$-sets to $R$-mod. Observe that if $(M_*, M^*) - M$ is a Mackey functor (bifunctor) $\hat{G} \rightarrow R$-mod (see [7]) we can get $\bar{M} : \mathcal{A}(G) \rightarrow R$-mod by putting $\bar{M}(G/H) = M_*(G/H) = M^*(G/H)$ on objects while a morphism $G/H \leftarrow^{\alpha} G/L \rightarrow^{\beta} G/K$ in $\mathcal{A}(G)$ is mapped onto $M(G/H) \leftarrow^{M_*(\alpha)} M(G/L) \rightarrow^{M_*(\beta)} M(G/K)$ in $R$-mod. Then $M$ is compatible with composition of morphisms.

Conversely, let $\bar{M} : \mathcal{A}(G) \rightarrow R$-mod be given and for $(\alpha, \beta) \in \mathcal{A}(G)(G/H, G/K)$ $\bar{M}^*(\alpha)$ and $M_*(\beta)$ as defined in 1.7. Then, we can extend $\bar{M}$ additively to finite $G$-sets to obtain Mackey functors as defined in [2], [3], [7].

1.8 Universal Example of a Green Functor Define $\bar{V}(G/H) := U(G, G/H) = U(H, G/G) := \bar{V}(H)$. Now, consider $U(G, G/H) = U(G, G/H \times G/H)$, as a morphism set in $\mathcal{A}(G)$. Then, the composition of morphisms

$U(G, G/H \times G/K) \times U(G, G/H \times G/K) \rightarrow U(G, G/G \times G/H)$

defines an action of $\bar{V}$ on morphisms.

Theorem 1.9 [4] $\bar{V} : \mathcal{A}(G) \rightarrow \mathbb{Z}$-mod is a Green functor, and any Mackey functor $\bar{V} : \mathcal{A}(G) \rightarrow \mathbb{Z}$-mod is a $\bar{V}$-module.

2 An Equivariant Higher $K$-Theory for $G$-Actions

2.1 Let $G$ be a compact Lie group, $X$ a $G$-space. We can regard $X$ as a category $\mathcal{X}$ as follows. The objects of $\mathcal{X}$ are elements of $X$ and for $x, x' \in X, X(x, x') = \{ g \in G | gx = x' \}$.
2.2 Let $X$ be a $G$-space, $\mathcal{C}$ an exact category in the sense of Quillen \cite{1}. i.e. $\mathcal{C}$ is an additive category embeddable as a full subcategory of an Abelian category $\mathcal{O}$ such that $\mathcal{C}$ is equipped with a class $\mathcal{E}$ of exact sequences

$$0 \to M' \to M \to M'' \to 0 \quad (I)$$

such that (i) $\mathcal{E}$ is the class of sequences (I) in the $\mathcal{C}$ that are exact in $\mathcal{O}$.

(ii) $\mathcal{C}$ is closed under extensions in $\mathcal{O}$ that is, if (I) is an exact sequence in $\mathcal{O}$ and $M', M'' \in \mathcal{C}$ then $M \in \mathcal{C}$.

Let $[X, \mathcal{C}]$ be the category of functors $X \to \mathcal{C}$. Then $[X, \mathcal{C}]$ is an exact category where a sequence $0 \to \zeta' \to \zeta \to \zeta'' \to i$ is exact in $[X, \mathcal{C}]$ if and only if

$$0 \to \zeta'(x) \to \zeta(x) \to \zeta''(x) \to 0$$

is exact in $\mathcal{C}$. In particular for $X = G/H$ in $\mathcal{A}(G)$, $[G/H, \mathcal{C}]$ is an exact category.

Example 2.3 The most important example of $G/H, \mathcal{C}]$ is when $\mathcal{C}$ is the category $\mathcal{M}(\mathbb{C})$ of finite dimensional vector spaces over the field $\mathbb{C}$ of complex numbers. Here, the category $[G/H, \mathcal{M}(\mathbb{C})]$ can be identified with the category of $G$-vector bundles on the compact $G$-space $G/H$ where for any $\zeta \in [G/H, \mathcal{M}(\mathbb{C})]$, $x \in G/H$, $\zeta(x) \in \mathcal{M}(\mathbb{C})$ is the fibre of $\tilde{\zeta}$ of the vector bundle $\tilde{\zeta}$ associated with $\zeta$. Indeed, $\tilde{\zeta}$ is completely determined by $\zeta_0$ where $\tilde{e} = eH$ (see \cite{10}).

Definition 2.4 For $X = G/H$ and all $n \geq 0$, define $K_n^G(X, \mathcal{C})$ as the $n^{th}$ algebraic $K$-group of the exact category $[X, \mathcal{C}]$ with respect to fibre-wise exact sequence introduced in 2.2.

Theorem 2.5 (i) For all $n \geq 0$, $K_n^G(-, \mathcal{C}) : \mathcal{A}(G) \to \mathbb{Z}$-mod is a Mackey functor

(ii) $K_0^G(-, \mathcal{C}) : \mathcal{A}(G) \to \mathbb{Z}$-mod is a Green functor and $K_n^G(-, \mathcal{C})$ is a $K_0^G(-, \mathcal{C})$-module for all $n \geq 0$.

Before proving 2.5, we first briefly discuss pairings and module structures on higher $K$-theory of exact categories.

2.6 Let $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ be three exact categories and $\mathcal{E}_1 \times \mathcal{E}_2$ the product category. An exact pairing $\mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E} : (M_1, M_2) \to M_1 \circ M_2$ is a covariant functor from $\mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}$ such
that \( \mathcal{E}_1 \times \mathcal{E}_2((M_1, M_2), (M'_1, M'_2)) = \mathcal{E}_1(M_1, M'_1) \times \mathcal{E}_2(M_2, M'_2) \to \mathcal{E}(M_1 \circ M_2, M'_1 \circ M'_2) \) is bi-additive and bi-exact, that is, for a fixed \( M_2 \), the functor \( \mathcal{E}_1 \to \mathcal{E} \) given by \( M_1 \to M_1 \circ M_2 \) is additive and exact and for fixed \( M_1 \), the functor \( \mathcal{E}_2 \to \mathcal{E} : M_2 \to M_1 \circ M_2 \) is additive and exact. It follows from [12] that such a pairing gives rise to a \( K \) theoretic product \( K_1(\mathcal{E}_1) \times K_2(\mathcal{E}_2) \to K_{i+j}(\mathcal{E}) \) and in particular to natural pairing \( K_0(\mathcal{E}_1) \times K_n(\mathcal{E}_2) \to K_n(\mathcal{E}) \) which could be defined as follows.

Any object \( M_1 \in \mathcal{E} \) induces an exact functor \( M_1 : \mathcal{E}_2 \to \mathcal{E} : M_2 \to M_1 \circ M_2 \) and hence a map \( K_n(M_1) : K_n(\mathcal{E}_2) \to K_n(\mathcal{E}) \). If \( M'_1 \to M_1 \to M''_1 \) is an exact sequence in \( \mathcal{E} \), then we have an exact sequence of exact functors \( M'_1^* \to M_1^* \to M''_1^* \) from \( \mathcal{E}_2 \) to \( \mathcal{E} \) such that for each object \( M_2 \in \mathcal{E}_2 \) the sequence \( M'_1^*(M_2) \to M_1^*(M_2) \to M''_1^*(M_2) \) is exact in \( \mathcal{E} \) and hence by a result of Quillen [9] induces a relation \( K_n(M'_1^*) + K_n(M''_1^*) - K_n(M_1^*) \) held. So the map \( M_1 \to K_n(M_1) \in \text{Hom}(K_n(\mathcal{E}_2), K_n(\mathcal{E})) \) induces a homomorphism \( K_0(\mathcal{E}_1) \to \text{Hom}(K_n(\mathcal{E}), K_n(\mathcal{E})) \) and hence a pairing \( K_0(\mathcal{E}_1) \times K_n(\mathcal{E}) \to K_n(\mathcal{E}) \).

If \( \mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E} \) and the pairing \( \mathcal{E} \times \mathcal{E} \) is naturally associative (and commutative), then the associated pairing \( K_0(\mathcal{E}) \times K_0(\mathcal{E}) \to K_0(\mathcal{E}) \) turns \( K_0(\mathcal{E}) \) into an associative (and commutative) ring which may not contain the identity.

Now, suppose that there is a pairing \( \mathcal{E} \circ \mathcal{E}_1 \to \mathcal{E}_1 \) which is naturally associative with respect to the pairing \( \mathcal{E} \circ \mathcal{E} \to \mathcal{E} \), then the pairing \( K_0(\mathcal{E}) \times K_n(\mathcal{E}_1) \to K_n(\mathcal{E}_1) \) turns \( K_n(\mathcal{E}_1) \) into a \( K_0(\mathcal{E}) \)-module which may or may not be unitary. However, if \( \mathcal{E} \) contains a unit i.e. an object \( E \) such that \( E \circ M = M \circ \mathcal{E} \) are naturally isomorphic to \( M \) for each \( \mathcal{E} \)-object \( M \), then the pairing \( K_0(\mathcal{E}) \times K_n(\mathcal{E}_1) \to K_n(\mathcal{E}_1) \) turns \( K_n(\mathcal{E}_1) \) into a unitary \( K_0(\mathcal{E}) \)-module.

**Proof of 2.5(i)** It is clear from the definition of \( K_n^G(G/H, \mathcal{C}) \) that for any \( G/H \in \mathcal{A}(G) \),

\[
K_n^G(G/H, \mathcal{C}) \in \mathbb{Z}\text{-mod.}
\]

Now suppose that \( G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K \)

\[
= K_n^G(G/H, \mathcal{C}) \xleftarrow{} K_n^G(G/L, \mathcal{C}) \xrightarrow{} K_n^G(G/K, \mathcal{C})
\]

in \( R\text{-mod}(K_n^G(G/K, \mathcal{C}), K_n^G(G/H, \mathcal{C})) \) and that if we write \( K_n^G \) for \( K_n^G(-, \mathcal{C}) \), then

\[
K_n^G(G/H \xleftarrow{} L \cup L' \to G/K)
\]

\[
= K_n^G(G/H \xleftarrow{} G/L \to G/K) + K_n^G(G/H \xleftarrow{} G/L' \to G/K).
\]

Hence \( K_n^G(-, \mathcal{C}) \) is a Mackey functor

(ii) From the discussion in 2.6, it is clear that if we put \( \mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E} = [G/H, \mathcal{C}] \). Then

\[
K_0^G(G/H, \mathcal{C}) \times K_0^G(G/H, \mathcal{C}) \to K_0^G(G/H, \mathcal{C})
\]

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turns $K_0^G(G/H, \mathcal{C})$ into a commutative ring with identity. Also,

$$K_0^G(G/H, \mathcal{C}) \times K_n^G(G/H, \mathcal{C}) \rightarrow K_n^G(G/H, \mathcal{C})$$

turns

$$K_n^G(G/H, \mathcal{C})$$

into $K_0^G(G/H, \mathcal{C})$-modules

**Examples 2.7**

(i) In general $[G/H, \mathcal{C}] = \text{category of } H\text{-representations in } \mathcal{C}$. Hence $[G/G, \mathcal{C}] = \text{category of } G\text{-representations in } \mathcal{C}$. If $\mathcal{C} = \mathcal{M}(\mathbb{C})$, the category of finite dimensional vector spaces over the complex numbers $\mathbb{C}$, $K_0^G(G/G, \mathcal{M}(\mathbb{C}))$ is the complex representation ring denoted by $R_{\mathbb{C}}(G)$ or simply $R(G)$ in the literature.

(ii) If $\mathcal{C} = \mathcal{M}(R) := \text{category of finitely generated } R\text{-modules, where } R\text{ is a Noetherian ring compatible with the topological structure of } G$, then $K_n^G(G/H, \mathcal{M}(R)) \simeq G_n(RH)$.

(iii) If $\mathcal{C} = \mathcal{P}(R) = \text{category of finitely generated projective } R\text{-modules}$, we have

$$K_n^G(G/H, \mathcal{P}(R)) = G_n(H, R)$$

when $R$ is regular, $G_n(R, H) \simeq G_n(RH)$.

### 3 Induction Theory

In this section, we discuss the induction properties of the Mackey functors constructed in Sec. 2.

**Definition 3.1**

Let $G$ be a compact Lie group. A finite family $\Sigma = ((G/H)_{j \in J})$ is called an inductive system. Such a system yields two homomorphisms $p(\Sigma)$ (induction map) and $i(\Sigma)$ (restriction maps) defined by

$$p(\Sigma) : \bigoplus_{j \in J} M(G/H_j) \rightarrow M(G/G)$$

$$(x_j)_{j \in J} \mapsto \sigma_j \sigma_i h_j x_j$$

$$i(\Sigma) : M(G/G) \rightarrow \bigoplus_{j \in J} M(G/H_j)$$

$$x \mapsto (p(H_j)^* x | j \in J)$$

$\Sigma$ is said to be projective if $p(\Sigma)$ is surjective and $\Sigma$ is said to be injective if $i(\Sigma)$ is injective.

Note that the identity $[\text{id}]$ of $\mathcal{U}(G, G/K \times G/H)$ has the form

$$[\text{id}] = \Sigma_\alpha n_\alpha [\alpha : G/L_\alpha \rightarrow G/K \times G/H] \quad (I)$$

see 1.3.
3.2 Let $S(K, H)$ be the set of $\alpha$ over which the summation (I) is taken and let $\alpha = (\alpha(1), \alpha(2))$ be the component of $\alpha$.

Define induction map

$$p(\Sigma, G/H) : \bigoplus_{j \in J} (\bigoplus_{\alpha \in S(H_j, H)} M(G/L_\alpha)) \to M(G/H)$$

by

$$(x(j, \alpha)) \mapsto (\Sigma_{j \in J} (\Sigma_{\alpha \in S(H_j, H)} n_\alpha \alpha(2)) x(j, \alpha)) .$$

and restriction maps

$$i(\Sigma, G/H) : M(G/H) \to \bigoplus_{j \in J} (\bigoplus_{\alpha \in S(H_j, H)} M(G/L_\alpha))$$

by

$$x \to \alpha(2)^* x \ (\alpha \in (S(H_j, H)) j \in J) .$$

3.3 Let $V : A(G) \to \mathbb{Z}\text{-mod}$ be a Green functor and $M : A(G) \to \mathbb{Z}\text{-mod}$ a left $V$-module (Take the $K$-theoretic functors defined in Sec. 2), i.e. $M = K^G_\mathbb{Z}(-, \mathcal{C}), V = K^G_0(-, \mathcal{C})$. If $\Sigma$ is projective for $V$, then for each homogeneous space $G/H$, the induction map $p(\Sigma, G/H)$ is split surjective and the restriction map is split injective. So induction theorems for $K^G_0(-, \mathcal{C})$ implies an induction theorem for

$$K^G_n(-, \mathcal{C})$$

Proof Since $p(\Sigma)$ is surjective for $V$, there exists elements $x_j \in V(G/H_j)$ such that $\Sigma(H_j), x_j = 1$.

Define a homomorphism

$$q(\Sigma, G/H) : M(G/H) \to \bigoplus_j (\bigoplus_{\alpha} M(G/L_\alpha))$$

by

$$q(\Sigma, G/H)/x = \alpha(1)^* x_j \cdot \alpha(2)^* x \text{ such that } \alpha \in S(H_j, h)$$

$j \in J$. Then $p(\Sigma, G/H)q(\Sigma, G/H)$

$$= \Sigma_j (\Sigma_{\alpha} n_\alpha \alpha(2), (\alpha(1)^* x_j \cdot \alpha(2)^* x))$$

$$= \Sigma_j (\Sigma_{\alpha} n_\alpha \alpha(2), (\alpha(1)^* x_j)) x$$

$$= \Sigma_j (\Sigma_{\alpha} n_\alpha \alpha(2), \alpha(1)^* x_j) x$$

$$= \Sigma_j p(H)^* p(H_j, x_j) \cdot x$$

$$= \Sigma_j p(H)^* (\Sigma_j p(H_j, x_j)) x$$

$$= p(H)^* (1 \cdot x = x) .$$
So, \( p(\Sigma, G/H)q(\Sigma, G/H) \) is the identity. Hence \( q(\Sigma, G/H) \) is a splitting for \( p(\Sigma, G/H) \).

We can also define a splitting \( j(\Sigma, G/H) \) for \( i(\Sigma, G/H) \) by \( j(\Sigma, G/H) : \oplus_j (\oplus_n (MG/L_n)) \rightarrow M(G|H) \) by

\[
x(j, \alpha) \mapsto \Sigma_j (\Sigma_n \alpha_n(2)_* \alpha(1)^* x_j \circ x(j, \alpha) .
\]

**Definition 3.4** A finite set \( E \) of conjugacy classes \((H)\) is an induction set for a Green functor \( V \) if \( \oplus V(G/H) \rightarrow V(G/G) \) given by

\[
(x(H)) \mapsto \Sigma_V(H)_* x(H) \text{ is surjective}.
\]

Define \( E \leq F \) iff for each \((H) \in E\), there exists \((K) \in F\) such that \((H) \leq (K)\) i.e. \( H \) is subconjugate to \( K \).

3.5 Every Green functor \( V \) possesses a minimal induction set \( D(V) \)-called the defect set of \( V \). Hence \( K_0^G(\rightarrow, C) \) have defect sets. For proof see [4].

3.6 Let \( M \) be a \( V \)-module. Define homomorphism \( p_1, p_2 : \oplus_{\alpha \in S(i)} M(G/K_\alpha) \rightarrow \oplus_{k \in J} M(G/H_k) \) by \((x(i, j, \alpha) = \Sigma_{i \in J} \sigma_{\alpha \in S(i)} \eta_\alpha(2)_* x(i, j, \alpha) \) where \( S(i, j) \in S(H_i, H_j), [\alpha] \in U(G, G/H \times G/K) \).

3.7 Let \( M = K_\alpha^G(\rightarrow, C) \). Then there exists an exact sequence

\[
\oplus_{i \in J} (\oplus_{\alpha \in S(i)} M(G/L_\alpha))^{p_2-p_1} \oplus_{k \in J} M(G/H_k) \xrightarrow{p} M(G/G) \rightarrow 0
\]

**Proof** We have seen in 3.3 that \( p \) is surjective through the construction of a splitting homomorphism \( q \) such that \( pq=\text{identity} \). We now construct a homomorphism \( q_1 \) such that \( (p_2 - p_1)q_1 + qp = \text{id} \) from which exactness follows.

Since \( p_2 \) is defined as \( \oplus_{k \in J} p(\Sigma G/H_k) \) we define \( q_1 = \oplus_{k \in J} q(\Sigma G/H_k) \) and obtain as in the proof of 3.3 that \( p_2q_1 = \text{identity} \). One can also show that \( p_1q_1 = qp \). Hence the result.

**Definition 3.8** A subgroup \( C \) of \( G \) is said to be cyclic if powers of a generator of \( G \) are dense in \( G \). A subgroup \( K \) of \( G \) is called \( p \)-hyerelementary if there exists an exact sequence \( \rightarrow C \rightarrow K \rightarrow P \rightarrow 1 \) where \( P \) is a finite \( p \)-group and \( C \) a cyclic group such that the order of \( K/C \) is prime to \( p \). It is called hyperelementary if it is \( p \)-hyper-elementary for some \( p \). Let \( \mathcal{H} \) be the set of hyperelementary subgroups of \( G \). We now have the following result which typifies results that can be obtained.
Theorem 3.9  Let $M = K^G_n(-, m(C))$. Then $\oplus_H M(G/H) \to M(G/G)$ is surjective (i.e. $M$ satisfies hyper-elementary induction) i.e. $M(G/G)$ can be computed in terms of $p$-hyperelementary subgroups of $C$).

Proof  It suffices to show that if $V = K^G_n(-, m(C))$ then $\oplus_H V(G/H) \to V(G/C)$ is surjective (since $M$ is a $V$-module). To do this it suffices to show that $\Sigma_H V(G/H)_{(p)} \to V(G/G)_{(p)}$ is surjective. Now $V$ is an algebra over the universal Green functor $V$ and every torsion element in $V(G/G)$ is nilpotent (see [4])

It is also known that $V(G/G) \otimes Q \to \prod V(G/H) \otimes Q$ is surjective. Hence the induction map $\oplus_{K \in H(p)} V(G/K)_{(p)} \to V(G/G)_{(p)}$ is surjective (see [11]).

4 Remarks on Possible Generalizations

4.1 Let $\mathcal{B}$ be a category with finite sums, a final object and finite pullbacks (and hence finite products).

A Mackey functor $M : \mathcal{B} \to \mathcal{Z}$-mod is a bifunctor $M = (M_*, M^*)$, $M_*$ covariant, $M^*$ contravariant such that $M(X) = M_*(X) = M(X)$ for all $x \in \mathcal{B}$ and (i) For any pullback diagram

\[ \begin{array}{ccc}
A' & \xrightarrow{p_2} & A_2 \\
\downarrow p_1 & & \downarrow f_2 \\
A_1 & \xrightarrow{f_1} & A \\
\end{array} \]

in $\mathcal{B}$

the diagram

\[ \begin{array}{ccc}
M(A') & \xrightarrow{p_2^*} & M(A_2) \\
\uparrow p_1^* & & \uparrow f_2^* \\
M(A_1) & \xrightarrow{f_1^*} & M(A) \\
\end{array} \]

(ii) $M^*$ transforms finite coproducts in $\mathcal{B}$ over finite products in $\mathcal{Z}$-mod.

Example 4.2  Now suppose that $G$ is a compact Lie group. Let $\mathcal{B}$ be the category of $G$-spaces of the $G$-homotopy type of $G$-CW-complexes (e.g. $G$-ENR spaces see [4] or [8]). Then $\mathcal{B}$ is a category with finite coproduct (topological sums) final object and finite pullbacks (fibred products) (see [1]). Hence a Mackey functor is defined on $\mathcal{B}$ along the lines of 4.1.

Hence in a way analogous to what was done in [2] or [3] we could define for $X, Y \in \mathcal{B}$, the notion of $Y$-exact sequences in the exact category $[X, C]$ (where $C$ is an exact category) and obtain $K^G_n(X, C, Y)$ as the $n^{th}$ algebraic $K$-group of $[X, C]$ with respect to $Y$-exact sequences.
We could also have the notion of an element $\zeta \in [X, C]$ being $Y$ projective and obtain a full subcategory $[X, C]_Y$ of $Y$-projective functors in $[X, C]$ so that we could obtain $P_n^G(X, C, Y)$ as the $n^{th}$ algebraic $K$-group of $[X, C]_Y$ with respect to split exact sequences and then show that $K_n^G(\_, C, Y)$, $P_n^G(\_, C, Y)$; $B \to \mathbb{Z}$-mod are Mackey functors and that $K_0^G(\_, C, Y) : B \to \mathbb{Z}$-mod is a Green functor and $K_n^G(\_, C, Y)$, $P_n^G(\_, C, Y)$ are $K_0^G(\_, C, T)$-modules in a way analogous to what was done in [2], [3].

It is hoped to explore these possibilities for further results in a future paper.
References


