FORMATION OF SINGULARITIES
IN ONE-DIMENSIONAL HYDROMAGNETIC FLOW

Kong De-xing
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

Two results on the formation of singularities in solutions to the system of one-dimensional hydromagnetic dynamics under various assumptions on the initial data are presented. In particular, it is shown that a smooth periodic solution will develop shocks in a finite time if the initial amounts of entropy and the “magnetic field” in each period is smaller than that of sound waves. We also give a quantitative estimate of the blow-up time.
1. Introduction

This article deals with the blow-up phenomenon of smooth solution \( u = u(t, x) \) to the following quasilinear hyperbolic system in one space dimension:

\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0
\] (1.1)

which satisfy

\[
t = 0 : u = \phi(x),
\] (1.2)

where \( \phi(x) \) is a \( C^1 \) vector function of \( x \). In (1.1), the Jacobian matrix \( Df(u) \) of the flux function \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is assumed to depend smoothly on the variable \( u \) and possess \( n \) real eigenvalues and a complete system of left (resp. right) eigenvectors.

In the case that the initial data \( \phi(x) \) has compact support (or more generally has certain decay properties as \( |x| \rightarrow \infty \)), the question of whether or not singularities in solutions to (1.1)-(1.2) appear in a finite time is rather well-understood (cf. [1]-[5]). For instance, by introducing the concept of the normalized coordinates and the weak linear degeneracy, Li Ta-tsien, Zhou Yi and Kong De-xing [3-5] gave a complete result on the global existence and the life-span of \( C^1 \) solutions to the Cauchy problem (1.1)-(1.2) with “small” initial data.

Moreover, M.A.Rammaha [6] discussed the formation of singularities in magnetohydrodynamic waves with the “large” initial data which has compact support. Several other results of this kind can be found in [7-9] and the references therein.

However, the case under main consideration in this paper, that the initial data \( \phi(x) \) is a periodic function, has been studied successfully only for scalar equations and 2 \( \times \) 2 systems. The fundamental work by Glimm and Lax [10] on the existence and asymptotic behaviour of solutions to the Cauchy problem for 2 \( \times \) 2 systems with periodic initial data stimulated the development of the theory of qualitative behaviour of solutions for quasilinear hyperbolic conservation laws. For the case that (1.1) is a 2 \( \times \) 2 system and neither genuinely nonlinear nor linearly degenerate in the sense of P.D.Lax, the blow-up phenomenon of \( C^1 \) solutions to the Cauchy problem (1.1)-(1.2) with periodic initial data small in the \( C^1 \) norm was studied by S.Klainerman and A.Majda [11], Li Ta-tsien and Shi Jia-hong [12]. A sharp estimate on the life-span of \( C^1 \) solutions was also obtained in [11]-[12].

Ph.Le Floch and Xin Zhou-ping [13] generalized the method of Glimm and Lax to the case of the system of one-dimensional gas dynamics, and proved a similar result, provided that the initial data satisfy certain smallness hypotheses (see [13] for details).

In this paper we shall investigate the blow-up phenomenon of \( C^1 \) solution to the system of one-dimensional hydromagnetic dynamics with the initial data with certain decay properties as \(|x| \to \infty\), and, particularly, with the periodic initial data. Section 2 contains the statement of our main results. Section 3 gives the proof of main result.

2. Statement of the main results

This section presents our results of formation of singularities for the system of one-dimensional hydromagnetic dynamics (Theorem 2.1 and Theorem 2.2). The proof of the main result (i.e., Theorem 2.2) will be given in Section 3.

We are interested in the following system of one-dimensional hydromagnetic dynamics in Lagrangian representation

\[
\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0, \\
\frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} + \frac{1}{4\pi} \left( H_y \frac{\partial H_y}{\partial x} + H_z \frac{\partial H_z}{\partial x} \right) = 0, \\
\frac{\partial H_y}{\partial t} + \frac{H_y}{v} \frac{\partial u}{\partial x} = 0, \\
\frac{\partial H_z}{\partial t} + \frac{H_z}{v} \frac{\partial u}{\partial x} = 0, \\
\frac{\partial S}{\partial t} = 0.
\]

where \( u, v, H_y, H_z \) and \( S \) are unknowns and denote the velocity, the specific, the \( y \) and \( z \) directional components of magnetic field \( \vec{H} \) (here we suppose that the \( x \) directional com-

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1 Usually, the system of one-dimensional hydromagnetic dynamics in Euler representation reads (cf. [15])

\[
\frac{\partial H_y}{\partial t} + \frac{\partial (uH_y)}{\partial x} = 0, \\
\frac{\partial H_z}{\partial t} + \frac{\partial (uH_z)}{\partial x} = 0, \\
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{4\pi \rho} \left( H_y \frac{\partial H_y}{\partial x} + H_z \frac{\partial H_z}{\partial x} \right) = 0, \\
\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = 0,
\]

where \( \rho = v^{-1} \), denotes the density. Making the following invertible transformation of independent variables

\[
\begin{cases}
\tau = t, \\
m = \int_{x(0,t)}^{x} \rho(t, \xi) \, d\xi,
\end{cases}
\]

where \( x(0, t) \) denotes the trajectory equation of the particle which is at the origin when \( t = 0 \). In the new variables \((\tau, m)\) we can easily obtain the Lagrangian representation (2.1) of the system of one-dimensional hydromagnetic dynamics.
ponent $H_x$ is constant, and that $H_x \equiv 0$) and the entropy respectively; the pressure $P$ is a known function of $v$ and $S$, and the state equation $P = P(v, S)$ satisfies, on each finite domain of $v > 0$

$$P_v < 0, \quad P_{vv} > 0. \quad (2.2)$$

Let

$$H_1 = vH_y, \quad H_z = vH_z, \quad (2.3)$$

then, (2.1) simply reduces to

$$\begin{align*}
\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} &= 0, \\
\frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} + \frac{1}{4\pi} \left( \frac{H_1 (H_1)_x + H_2 (H_2)_x}{v^2} - \frac{H_1^2 + H_2^2}{v^3} v_x \right) &= 0, \\
\frac{\partial P}{\partial t} &= 0, \\
\frac{\partial H_1}{\partial t} &= 0, \\
\frac{\partial H_2}{\partial t} &= 0, \\
\frac{\partial S}{\partial t} &= 0.
\end{align*} \quad (2.4)$$

Hence, in what follows it suffices to discuss the system (2.4).

By (2.2), on each finite domain of $v > 0$, (2.4) is a hyperbolic system with the following real eigenvalues:

$$\lambda_1 \triangleq -\sqrt{-P_v + \frac{H_1^2 + H_2^2}{4\pi v^3}} < \lambda_2 \equiv \lambda_3 \equiv \lambda_4 = 0 < \lambda_5 \triangleq \sqrt{-P_v + \frac{H_1^2 + H_2^2}{4\pi v^3}}. \quad (2.5)$$

Obviously, (2.4) is a quasilinear hyperbolic system with eigenvalues with constant multiplicity, and $\lambda_i$ ($i = 2, 3, 4$) are linear degenerate in the sense of P.D.Lax, then weakly linearly degenerate (cf. [3], [5]), while, by (2.2), both $\lambda_1$ and $\lambda_5$ are genuinely nonlinear in the sense of P.D.Lax.

The corresponding right and left eigenvectors are

$$\begin{align*}
r_1 (U) &\parallel \left( 1, \sqrt{-P_v + \frac{H_1^2 + H_2^2}{4\pi v^3}}, 0, 0, 0 \right) T, \\
r_2 (U) &\parallel \left( \frac{H_1 v}{-4\pi v^3 P_v + H_1^2 + H_2^2}, 0, 1, 0, 0 \right) T, \\
r_3 (U) &\parallel \left( \frac{H_2 v}{-4\pi v^3 P_v + H_1^2 + H_2^2}, 0, 0, 1, 0 \right) T, \\
r_4 (U) &\parallel \left( \frac{4\pi v^3 P_S}{-4\pi v^3 P_v + H_1^2 + H_2^2}, 0, 0, 0, 1 \right) T, \\
r_5 (U) &\parallel \left( 1, -\sqrt{-P_v + \frac{H_1^2 + H_2^2}{4\pi v^3}}, 0, 0, 0 \right) T.
\end{align*} \quad (2.6)$$
and
\[ l_1(U) = \left(1, \sqrt{-P_v + \frac{H_1^2 + H_2^2}{4\pi v^2}}, 0, 0, 0\right), \]
\[ l_2(U) = (0, 0, 1, 0, 0), \]
\[ l_3(U) = (0, 0, 0, 1, 0), \]
\[ l_4(U) = (0, 0, 0, 0, 1), \]
\[ l_5(U) = \left(1, -\sqrt{-P_v + \frac{H_1^2 + H_2^2}{4\pi v^2}}, 0, 0, 0\right) \]

respectively, where
\[ U = (v, u, H_1, H_2, S)^T. \] (2.8)

2.1. The case of the initial data with certain decay properties as \( |x| \to \infty \).

In this case we consider the following initial data
\[ t = 0 : \quad v = \tilde{v}_0 + \varepsilon v_0(x), \quad u = \tilde{u}_0 + \varepsilon u_0(x), \]
\[ H_y = \tilde{H}_y^0 + \varepsilon H_y^0(x), \quad H_z = \tilde{H}_z^0 + \varepsilon H_z^0(x), \quad S = \tilde{S}_0 + \varepsilon S_0(x), \] (2.9)

where \( \varepsilon > 0 \) is a small parameter, \( \tilde{v}_0, \tilde{H}_y^0, \tilde{H}_z^0 \) and \( \tilde{S}_0 \) are real numbers, \( \tilde{v}_0 \) is a positive number and \( U_0(x) \triangleq (v_0(x), u_0(x), H_y^0(x), H_z^0(x), S_0(x))^T \) is a \( C^1 \) vector function of \( x \) and satisfies
\[ \sup_{x \in \mathbb{R}} \left\{ (1 + |x|)^{1+\mu} (|U_0(x)| + |U_0'(x)|) \right\} < \infty, \] (2.10)

where \( \mu \) is a positive constant, \( |U_0(x)| = \max\{|v_0(x)|, |u_0(x)|, |H_y^0(x)|, |H_z^0(x)|, |S_0(x)|\} \) and \( |U_0'(x)| = \max\left\{ \left| \frac{dv_0(x)}{dx} \right|, \left| \frac{du_0(x)}{dx} \right|, \left| \frac{dH_y^0(x)}{dx} \right|, \left| \frac{dH_z^0(x)}{dx} \right|, \left| \frac{dS_0(x)}{dx} \right| \right\} \).

By Theorem 4.2 in [5], we easily get

\textbf{Theorem 2.1.} If \((u_0(x), v_0(x))\) is not identically equal to zero, then there exists \( \varepsilon_0 > 0 \) so small that for any \( \varepsilon \) with \( 0 < \varepsilon \leq \varepsilon_0 \), the \( C^1 \) solution \( U(t, x) \triangleq (v(t, x), u(t, x), H_1(t, x), H_2(t, x), S(t, x))^T \) to Cauchy problem (2.4) and (2.9) (i.e., (2.1) and (2.9)) must blow up in a finite time and there exist two positive constants \( c \) and \( C \) independent of \( \varepsilon \), such that the life-span \( \hat{T}(\varepsilon) \) of \( U = U(t, x) \) satisfies
\[ c\varepsilon^{-1} \leq \hat{T}(\varepsilon) \leq C\varepsilon^{-1}, \] (2.11)

denoted by
\[ \hat{T}(\varepsilon) \approx \varepsilon^{-1}. \quad \square \] (2.12)
2.2. The case of the periodic initial data

In this case we shall consider a smooth solution to the Cauchy problem for (2.4) with smooth periodic initial data

\[ v = v_0(x) > 0, \quad u = u_0(x), \]
\[ t = 0: \quad H_1 = H_{10}(x), \quad H_2 = H_{20}(x), \quad S = S_0(x), \]

where \( v_0, u_0, H_{10}, H_{20} \) and \( S_0 \) are periodic functions of period \( p \). For the smooth solution, by the last three equations in (2.4), we have

\[ H_1(t, x) = H_{10}(x), \quad H_2(t, x) = H_{20}(x), \quad S(t, x) = S_0(x), \quad \text{for all } t \geq 0. \]

Hence, (2.4) reduces to

\[
\begin{cases}
\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0, \\
\frac{\partial u}{\partial t} + \frac{\partial P(v, S_0(x))}{\partial x}
+ \frac{1}{4\pi} \left( \frac{H_{10}(x)(H'_{10}(x))_x + H_{20}(x)(H'_{20}(x))_x}{v^2} - \frac{(H^2_{10}(x) + H^2_{20}(x))}{v^3} v_x \right) = 0.
\end{cases}
\]

Introducing the Riemann invariants

\[
\begin{align*}
\rho &= u + \int_1^v \sqrt{-\frac{\partial P}{\partial v}(v, S_0(x)) + \frac{H^2_{10}(x) + H^2_{20}(x)}{4\pi v^3}} dv, \\
\tau &= u - \int_1^v \sqrt{-\frac{\partial P}{\partial v}(v, S_0(x)) + \frac{H^2_{10}(x) + H^2_{20}(x)}{4\pi v^3}} dv
\end{align*}
\]

as new unknown functions, system (2.5) can be equivalently rewritten as

\[
\begin{align*}
\rho_t - \lambda \rho_x &= \beta_0 S'_0(x) + \beta_1 H'_{10}(x) + \beta_2 H'_{20}(x), \\
\tau_t + \lambda \tau_x &= \beta_0 S'_0(x) + \beta_1 H'_{10}(x) + \beta_2 H'_{20}(x),
\end{align*}
\]

where

\[
\lambda = \sqrt{-\frac{\partial P}{\partial v}(v, S_0(x)) + \frac{H^2_{10}(x) + H^2_{20}(x)}{4\pi v^3}}
\]

and

\[
\beta_0 = -\left\{ \frac{\partial P}{\partial S}(v, S_0(x)) + \lambda \int_1^v \frac{\partial \lambda}{\partial S}(v, H_{10}(x), H_{20}(x), S_0(x)) dv \right\},
\]

\[
\beta_1 = -\left\{ \frac{H_{10}(x)}{4\pi v^2} + \lambda \int_1^v \frac{\partial \lambda}{\partial H_1}(v, H_{10}(x), H_{20}(x), S_0(x)) dv \right\},
\]

\[
\beta_2 = -\left\{ \frac{H_{20}(x)}{4\pi v^2} + \lambda \int_1^v \frac{\partial \lambda}{\partial H_2}(v, H_{10}(x), H_{20}(x), S_0(x)) dv \right\}.
\]

Correspondingly, the initial condition (2.13) is reduced to

\[
\begin{align*}
t = 0: \quad r &= r_0(x) \triangleq r(u_0(x), v_0(x), H_{10}(x), H_{20}(x), S_0(x)), \\
s &= s_0(x) \triangleq s(u_0(x), v_0(x), H_{10}(x), H_{20}(x), S_0(x)).
\end{align*}
\]
Theorem 2.2. Suppose that on the domain under consideration, $P = P(v, S) \in C^2$, $u_0$, $v_0 (> 0) \in C^1$ and $H_{10}$, $H_{20}$, $S_0 \in C^2$, $U_0(x) \triangleq (v_0(x), u_0(x), H_{10}(x), H_{20}(x), S_0(x))^T$ is a periodic vector function with period $p(> 0)$ and

$$TV_0^p(u_0) + TV_0^p(v_0) > 0.$$  \hfill (2.21)

Suppose furthermore that there exists a suitably small positive constant $\varepsilon$ depending only on the $C^1$ norm of $U_0(x)$, such that

$$\sqrt{TV_0^p(H'_{10}) + TV_0^p(H'_{20}) + TV_0^p(S'_0)} \leq \varepsilon (TV_0^p(u_0) + TV_0^p(v_0)),$$ \hfill (2.22)

where $TV_0^p(\cdot)$ denotes the total variation of $\cdot$ on the interval $[0, p]$. Then the life-span $T_{\text{max}}$ of the smooth solution $U(t, x) \triangleq (v, u, H_1, H_2, S)^T$ (in which $u, v \in C^1$ and $H_1, H_2, S \in C^2$) to the Cauchy problem (2.4) and (2.14) satisfies

$$T_{\text{max}} \leq \frac{Cp}{TV_0^p(u_0) + TV_0^p(v_0)},$$ \hfill (2.23)

where $C$ is a positive constant depending only on the $C^1$ norm of $U_0(x)$.

3. Blow-up phenomenon and life-span of $C^1$ solution—proof of Theorem 2.2

By periodicity, it is easy to see that

$$\|H'_{10}\|_{C^0} \leq TV_0^p(H'_{10}), \quad \|H'_{20}\|_{C^0} \leq TV_0^p(H'_{20}), \quad \|S'_0\|_{C^0} \leq TV_0^p(S'_0).$$ \hfill (3.1)

Then, noting (2.2), under hypothesis (2.22) it easily follows from (2.16) that there exist two positive constants $C_*$ and $C^*$ depending only on the $C^1$ norm of $U_0(x)$, such that

$$C_* (TV_0^p(u_0) + TV_0^p(v_0)) \leq TV_0^p(r_0) + TV_0^p(s_0) \leq C^* (TV_0^p(u_0) + TV_0^p(v_0)).$$ \hfill (3.2)

Hence, (2.21) and (2.22) can be rewritten as

$$TV_0^p(v_0) + TV_0^p(s_0) > 0$$ \hfill (3.3)

and

$$\sqrt{TV_0^p(H'_{10}) + TV_0^p(H'_{20}) + TV_0^p(S'_0)} \leq \kappa \max(TV_0^p(r_0), TV_0^p(s_0))$$ \hfill (3.4)

respectively, where $\kappa$ is a suitably small positive constant depending only on the $C^1$ norm of $U_0(x)$. Moreover, in order to get (2.23) it suffices to prove

$$T_{\text{max}} \leq \frac{\bar{C}p}{TV_0^p(r_0) + TV_0^p(s_0)},$$ \hfill (3.5)
where \( \tilde{C} \) is also a positive constant depending only on the \( C^1 \) norm of \( U_0 (x) \).

Noting (2.2) and (2.16), by (2.18) it is easy to see that

\[
\frac{\partial \lambda}{\partial r} < 0, \quad \frac{\partial \lambda}{\partial s} > 0.
\]

Let

\[
\xi = TV_0^p (r_0), \quad \eta = TV_0^p (s_0).
\]

Without loss of generality, we may suppose that

\[
\xi \geq \eta.
\]

Later we will show that there is a positive constant \( \tilde{M} \) depending only on the \( C^1 \) norm of \( U_0 (x) \) such that the existence domain \( 0 < t < T \) of \( C^1 \) solution \((r(t, x), s(t, x))\) to the Cauchy problem (2.17) and (2.20) satisfies

\[
T\xi \leq \tilde{M}.
\]

Noting (3.8), by (3.4) we have

\[
TV_0^p (H_{10}') + TV_0^p (H_{20}') + TV_0^p (S_0') \leq \kappa^2 \xi^2,
\]

where, by assumption, \( \kappa \) is suitably small.

We now prove that

\[
||r(t, \cdot)||_{C^0}, \quad ||s(t, \cdot)||_{C^0} \leq M_0 + 1, \quad \forall t \in [0, T],
\]

where

\[
M_0 = \max \{||r_0(\cdot)||_{C^0}, \quad ||s_0(\cdot)||_{C^0}\}.
\]

In fact, by continuity, (3.11) holds at least on an interval \( 0 \leq t \leq T_0 \), where \( T_0 > 0 \) is suitably small. In order to get (3.11), it suffices to prove that the set of \( T_0 \) (\( 0 < T_0 \leq T \)) such that (3.11) holds on \( 0 \leq t \leq T_0 \) is open on \([0, T]\). For this purpose, we only need to prove that for any fixed \( T_1 \) with \( 0 < T_1 \leq T \), such that

\[
||r(t, \cdot)||_{C^0}, \quad ||s(t, \cdot)||_{C^0} \leq M_0 + 2, \quad \forall t \in [0, T_1],
\]

we have

\[
||r(t, \cdot)||_{C^0}, \quad ||s(t, \cdot)||_{C^0} \leq M_0 + 1, \quad \forall t \in [0, T_1],
\]
Let \( x = x_1(t, y) \) be the slow characteristic passing through any given point \((0, y)\) on the \(x\)-axis. We have

\[
\begin{aligned}
\frac{dx_1(t, y)}{dt} &= -\lambda \left( r(t, x_1(t, y)), s(t, x_1(t, y)), H_{10}(x_1(t, y)), H_{20}(x_1(t, y)), S_0(x_1(t, y)) \right), \\
x_1(0, y) &= y.
\end{aligned}
\]  

(3.15)

Integrating the first equation in (2.17) along \( x = x_1(t, y) \) gives

\[
\begin{aligned}
r(t, x_1(t, y)) &= r_0(y) + \int_0^t \beta_0(\tau, y) S'_0(x_1(\tau, y)) \, d\tau \\
&\quad + \int_0^t \beta_1(\tau, y) H'_{10}(x_1(\tau, y)) \, d\tau \\
&\quad + \int_0^t \beta_2(\tau, y) H'_{20}(x_1(\tau, y)) \, d\tau,
\end{aligned}
\]

(3.16)

where

\[
\beta_i(\tau, y) \equiv \beta_i(r(\tau, x_1(\tau, y)), s(\tau, x_1(\tau, y)), H_{10}(x_1(\tau, y)), H_{20}(x_1(t, y)), S_0(x_1(\tau, y))) \quad (i = 0, 1, 2).
\]

Then, using (3.1) and (3.9)-(3.10) and noting (3.7) and (3.12)-(3.13), we can choose \( \kappa \) so small that for \( t \in [0, T_1] \),

\[
\begin{aligned}
|r(t, x_1(t, y))| &\leq M_0 + M_1 T \cdot \{ TV'_0( H'_{10} ) + TV'_0( H'_{20} ) + TV'_0( S'_0 ) \} \\
&\leq M_0 + M_1 M \kappa \xi \\
&\leq M_0 + M_1 M \kappa \xi \| r'_0(\cdot) \|_{C^0} \\
&\leq M_0 + 1,
\end{aligned}
\]

(3.17)

here and hereafter \( M_i (i = 1, 2, \ldots) \) denote positive constants depending only on the \( C^1 \) norm of \( U_0(x) \) but independent of \( M \). \( s \) can be estimated in a similar manner. Thus, we get (3.14) and then (3.11).

Differentiating both sides of (3.16) with respect to \( y \) yields

\[
\begin{aligned}
r_y &= r'_0(y) + \int_0^t \left( \beta_0 S''_0 + \beta_1 H''_{10} + \beta_2 H''_{20} \right) \frac{\partial x_1(\tau, y)}{\partial y} \, d\tau \\
&\quad + \int_0^t \left[ \frac{\partial \beta_0}{\partial r} r_y + \frac{\partial \beta_0}{\partial s} s_x \frac{\partial x_1}{\partial y} + \left( \frac{\partial \beta_0}{\partial H_1} H'_{10} + \frac{\partial \beta_0}{\partial H_2} H'_{20} + \frac{\partial \beta_0}{\partial S_0} S'_0 \right) \frac{\partial x_1}{\partial y} \right] S'_0 \, d\tau \\
&\quad + \int_0^t \left[ \frac{\partial \beta_1}{\partial r} r_y + \frac{\partial \beta_1}{\partial s} s_x \frac{\partial x_1}{\partial y} + \left( \frac{\partial \beta_1}{\partial H_1} H'_{10} + \frac{\partial \beta_1}{\partial H_2} H'_{20} + \frac{\partial \beta_1}{\partial S_0} S'_0 \right) \frac{\partial x_1}{\partial y} \right] H'_{10} \, d\tau \\
&\quad + \int_0^t \left[ \frac{\partial \beta_2}{\partial r} r_y + \frac{\partial \beta_2}{\partial s} s_x \frac{\partial x_1}{\partial y} + \left( \frac{\partial \beta_2}{\partial H_1} H'_{10} + \frac{\partial \beta_2}{\partial H_2} H'_{20} + \frac{\partial \beta_2}{\partial S_0} S'_0 \right) \frac{\partial x_1}{\partial y} \right] H'_{20} \, d\tau.
\end{aligned}
\]

(3.18)
On the other hand, by differentiation with respect to $y$ we get from (3.15) that

$$
\begin{align*}
\frac{d}{dt} \left( \frac{\partial x_1}{\partial y} \right) &= -\lambda_r r_y - (\lambda_s s_x + \lambda H_1' H_{10} + \lambda H_2' H_{20} + \lambda S S_0') \frac{\partial x_1}{\partial y}, \\
\frac{\partial x_1}{\partial y}(0, y) &= 1.
\end{align*}
$$

(3.19)

By the second equation in (2.17) we have

$$
s_x = \frac{\beta_0 S_0' + \beta_1 H_{10}' + \beta_2 H_{20}' - \frac{ds}{dt}}{2\lambda},
$$

(3.20)

where

$$\frac{ds}{dt} = s_t - \lambda s_x$$

(3.21)

denotes the direction derivative along the backward characteristic. Define $h(r, s, H_{10}, H_{20}, S_0)$ by

$$h = \frac{\lambda_s}{2\lambda}.$$  

(3.22)

Obviously, by (3.11), on the existence domain of $C^1$ solution $(r(t, x), s(t, x))$ we have

$$\|h(r, s, S_0)\| \leq M_2.$$  

(3.23)

Noting (3.20) and (3.22) and using the first equation in (2.17), (3.19) can be rewritten as

$$\begin{align*}
\frac{d}{dt} \left( \frac{\partial x_1}{\partial y} \right) &= -\lambda_r r_y - [\beta_0 (h_r + h_s) + \lambda S - \lambda h S] S_0' \frac{\partial x_1}{\partial y} \\
&\quad + [\beta_1 (h_r + h_s) + \lambda H_1 - \lambda h H_1] H_{10}' \frac{\partial x_1}{\partial y} \\
&\quad + [\beta_2 (h_r + h_s) + \lambda H_2 - \lambda h H_2] H_{20}' \frac{\partial x_1}{\partial y} \\
&\quad - \frac{dh}{dt} \frac{\partial x_1}{\partial y}, \\
\frac{\partial x_1}{\partial y}(0, y) &= 1.
\end{align*}$$

(3.24)

Noting (2.21) and (3.8), $r_0(x)$ is not identically equal to a constant. By periodicity, there exists $y_\ast \in [0, p]$ such that

$$r_0'(y_\ast) = \min_{0 \leq y \leq p} r_0'(y) \triangleq -\delta < 0.$$  

(3.25)

For the time being, we assume that along the backward characteristic $x = x_1(t, y_\ast)$ we have

$$-\frac{3}{2} \delta \leq r_y(t, x_1(t, y_\ast)) \leq -\frac{1}{2} \delta, \quad \forall t \in [0, T].$$

(3.26)
The validity of this assumption will be explained at the end of the proof.

Thus, noting (3.6), it follows from (3.24) that

\[
\left\{ \frac{d}{dt} \left( \frac{\partial x_1}{\partial y} (t, y_*) \right) \right\} \leq -M_3 \delta - [\beta_0 (h_r + h_s) + \lambda_S - \lambda h_S] S_{0}' \frac{\partial x_1}{\partial y} (t, y_*) \\
+ [\beta_1 (h_r + h_s) + \lambda H_1 - \lambda h H_1] H'_{10} \frac{\partial x_1}{\partial y} (t, y_*) \\
+ [\beta_2 (h_r + h_s) + \lambda H_2 - \lambda h H_2] H'_{20} \frac{\partial x_1}{\partial y} (t, y_*) \\
- \frac{dh}{dt} \frac{\partial x_1}{\partial y} (t, y_*),
\]

(3.27)

\[
\frac{\partial x_1}{\partial y} (0, y_*) = 1.
\]

Hence, we have

\[
\frac{\partial x_1}{\partial y} (t, y_*) \leq \exp \left( \int_0^t A(\tau) d\tau \right) \left( 1 - M_3 \delta \int_0^t \exp \left( \int_0^\tau A(\zeta) d\zeta \right) d\tau \right), \quad (3.28)
\]

where

\[
A(\tau) = \left\{ \left[ \beta_0 (h_r + h_s) + \lambda_S - \lambda h_S \right] S_{0}' (\tau, x_1 (\tau, y_*)) \right\}
\]

+ \left\{ \left[ \beta_1 (h_r + h_s) + \lambda H_1 - \lambda h H_1 \right] H'_{10} (\tau, x_1 (\tau, y_*)) \right\}
\]

+ \left\{ \left[ \beta_2 (h_r + h_s) + \lambda H_2 - \lambda h H_2 \right] H'_{20} (\tau, x_1 (\tau, y_*)) \right\}

(3.29)

and then, using (3.1) and (3.9)-(3.11) and noting (3.23), in a way similar to the proof of (3.17) we get

\[
\frac{\partial x_1}{\partial y} (t, y_*) \leq M_4 (1 - M_5 \delta t). \quad (3.30)
\]

Evidently,

\[
\frac{\partial x_1}{\partial y} (0, y_*) = 1 \quad \text{and} \quad \frac{\partial x_1}{\partial y} (t_1, y_*) \leq 0, \quad (3.31)
\]

where

\[
t_1 = \frac{1}{M_5 \delta}. \quad (3.32)
\]

By (3.26) and noting \( r_y = \frac{r_x}{\partial y} \), it is easy to see that \( \frac{\partial r}{\partial x} \) must blow up at a time \( t \) \((t \leq t_1)\), then

\[
T_{\text{max}} \leq \frac{1}{M_5 \delta}. \quad (3.33)
\]

Again by periodicity, we have

\[
\int_{B_1} r'_0(y) dy = \int_{B_2} |r'_0(y)| dy = \frac{1}{2} \xi, \quad (3.34)
\]

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where $B_1 = \{ y \mid y \in [0, p], \ r'_0(y) > 0 \}$ and $B_2 = \{ y \mid y \in [0, p], \ r'_0(y) < 0 \}$. Hence, it follows from (3.25) that

$$\xi < 2\delta p. \quad (3.35)$$

Noting (3.8), the combination of (3.33) and (3.35) yields the desired estimate (3.5) and then (2.23). Moreover, the validity of (3.9) follows from (3.33) and (3.35), provided that we take $M = \frac{2p}{M_5}$.

It remains to show the validity of (3.26). Noting that (3.26) holds at $t = 0$, by continuity it suffices to prove that the set of $T_1 (0 < T_1 \leq T)$ such that (3.26) holds on the interval $[0, T_1]$ is open on $[0, T]$.

Noting that (3.26) is assumed to hold on $[0, T_1]$, by (3.30) we have

$$0 < \frac{\partial x_1}{\partial y} (t, y_*) \leq M_4, \ \forall \ t \in [0, T_1]. \quad (3.36)$$

Then, using (3.25), (3.1), (3.11) and noting that (3.26) holds on $[0, T_1]$ again, it follows from (3.18) that

$$-\delta - M_6 N \leq \frac{\partial r}{\partial y} (t, x_1 (t, y_*)) \leq -\delta + M_6 N, \ \forall \ t \in [0, T_1], \quad (3.37)$$

where

$$N = \int_0^{T_1} \left( |H''_{10} (x_1 (\tau, y_*))| + |H''_{20} (x_1 (\tau, y_*))| + |S''_0 (x_1 (\tau, y_*))| \right) d\tau$$

$$+ T \delta (TV''_0 (H'_{10}) + TV''_0 (H'_{20}) + TV''_0 (S'_0))$$

$$+ (TV'_0 (H'_{10}) + TV'_0 (H'_{20}) + TV'_0 (S'_0)) \int_0^{T_1} |s_x (\tau, x_1 (\tau, y_*))| d\tau$$

$$+ T (TV'_0 (H'_{10}) + TV'_0 (H'_{20}) + TV'_0 (S'_0))^2. \quad (3.38)$$

We first estimate the first term. By periodicity and noting (3.9)-(3.11) and (3.35), it is easy to get

$$\int_0^{T_1} \left( |H''_{10} (x_1 (\tau, y_*))| + |H''_{20} (x_1 (\tau, y_*))| + |S''_0 (x_1 (\tau, y_*))| \right) d\tau$$

$$= - \int_{y_*}^{y^*} \frac{|H'_{10} (x)| + |H'_{20} (x)| + |S'_0 (x)|}{\lambda} dx$$

$$\leq M_T Tp^{-1} (TV'_0 (H'_{10}) + TV'_0 (H'_{20}) + TV'_0 (S'_0))$$

$$\leq M_8 \kappa^2 \delta,$$

where $y^* = x_1 (T_1, y_*)$ and $\kappa$ is given in (3.4).
Similarly, we have

\[
\begin{align*}
&\begin{cases} 
T \delta (TV_0^p (H_{10}')) + TV_0^p (H_{20}') + TV_0^p (S_0') \leq M_9 \kappa^2 \delta, \\
T (TV_0^p (H_{10}') + TV_0^p (H_{20}') + TV_0^p (S_0'))^2 \leq M_{10} \kappa^4 \delta.
\end{cases}
\end{align*}
\]

Finally, we estimate \( \int_0^{T_1} |s_x (\tau, x_1 (\tau, y_*))| d\tau \). Noting (3.1) and (3.11), by (3.20) we have

\[
\int_0^{T_1} |s_x (\tau, x_1 (\tau, y_*))| d\tau 
\leq \int_0^{T_1} \left( (2\lambda)^{-1} \left[ \beta_0 S_0' + \beta_1 H_{10}' + \beta_2 H_{20}' \right] + \frac{1}{2\lambda} \left| \frac{ds (\tau, x_1 (\tau, y_*))}{d\tau} \right| \right) d\tau 
\leq M_{11} \left( T (TV_0^p (H_{10}')) + TV_0^p (H_{20}') + TV_0^p (S_0') \right) + \int_0^{T_1} \left| \frac{ds (\tau, x_1 (\tau, y_*))}{d\tau} \right| d\tau.
\]

Using the second equation in (2.17), it is easy to get

\[
s (\tau, x_1 (\tau, y_*)) = s_0(y(\tau)) + \int_0^{\tau} \left\{ \beta_0 (\mu, y(\tau)) S_0' (x_2 (\mu, y(\tau))) + \beta_1 (\mu, y(\tau)) H_{10}' (x_2 (\mu, y(\tau))) + \beta_2 (\mu, y(\tau)) H_{20}' (x_2 (\mu, y(\tau))) \right\} d\mu,
\]

where

\[
\beta_i (\mu, y(\tau)) = \beta_i (r (\mu, x_2 (\mu, y(\tau))), s(\mu, x_2 (\mu, y(\tau))), H_{10} (x_2 (\mu, y(\tau))), H_{20} (x_2 (\mu, y(\tau))), S_0 (x_2 (\mu, y(\tau)))) \quad (i = 0, 1, 2),
\]

in which \( x = x_2 (\mu, y(\tau)) \) stands for the forward characteristic passing through the point \((0, y(\tau))\) on the \( x \)-axis, where \( y(\tau) \) is defined by

\[
x_1 (\tau, y_*) = x_2 (\tau, y(\tau)).
\]

Since \( \int_0^{T_1} \left| \frac{ds (\tau, x_1 (\tau, y_*))}{d\tau} \right| d\tau \) denotes the total variation of \( s \) along the backward characteristic \( x = x_1 (t, y_*) \) \((0 \leq t \leq T_1)\), noting (3.1), (3.8), (3.11) and (3.35), it is easy to see from (3.42) that

\[
\int_0^{T_1} \left| \frac{ds (\tau, x_1 (\tau, y_*))}{d\tau} \right| d\tau \leq TV_0^{T_1} (s_0(y(\tau))) + TV_0^{T_1} \left( \int_0^{\tau} \left( \frac{T \delta}{p} \xi + T (TV_0^p (H_{10}')) + TV_0^p (H_{20}') + TV_0^p (S_0') \right) d\mu \right)
\leq M_{12} \left( \frac{T \delta}{p} + TV_0^p (H_{10}') + TV_0^p (H_{20}') + TV_0^p (S_0') \right)
\leq M_{13} T (\delta + TV_0^p (H_{10}') + TV_0^p (H_{20}') + TV_0^p (S_0'))
\]

(3.44)
then, noting (3.40) and the smallness of \( \kappa \), it follows from (3.41) that
\[
(TV_0^p (H'_{10}) + TV_0^p (H'_{20}) + TV_0^p (S'_0)) \int_0^{T_1} \left| s_x (\tau, x_1 (\tau, y_*)) \right| d\tau \\
\leq M_{14} \left[ T \delta (TV_0^p (H'_{10}) + TV_0^p (H'_{20}) + TV_0^p (S'_0)) \right] \\
+ M_{15} T (TV_0^p (H'_{10}) + TV_0^p (H'_{20}) + TV_0^p (S'_0))^2 \\
\leq M_{16} \kappa^2 \delta.
\] (3.45)

Substituting (3.39)-(3.40) and (3.45) into (3.38) gives
\[
N \leq M_{17} \kappa^2 \delta.
\] (3.46)

Hence, choosing \( \kappa \) so small that \( M_6 M_{17} \kappa^2 \leq \frac{1}{4} \), we have
\[
M_6 N \leq \frac{\delta}{4},
\] (3.47)
then it comes from (3.37) that
\[
- \frac{5}{4} \delta \leq \frac{\partial r}{\partial y} (t, x_1 (t, y_*)) \leq - \frac{3}{4} \delta, \quad \forall \ t \in [0, T_1].
\] (3.48)

This proves the validity of (3.26). The proof is complete. \( \Box \)

**REFERENCES**


