AND A-NON-DEGENERATE TWO-PHOTON TIME-DEPENDENT JAYNES-CUMMINGS MODEL: AN EXACT ALGEBRAIC SOLUTION

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ABSTRACT

We demonstrate that the dynamics of the Ξ and Λ non-degenerate two-photon time-dependent Jaynes-Cummings models are characterized by the bi-dimensional Fibonacci-like connection between quantal correlations of different orders. The time independent cases are solved and infinite sets of invariants of motion are obtained. The close dynamical relationship between both models is shown.

MIRAMARE – TRIESTE
November 1996

1 Published in Physics Letters A.
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1 Introduction

Over the last decade a lot of theoretical studies has been done in order to understand the dynamical nonlinear effects of an atom in a high-Q cavity. This was mainly due to the large amount of experiments done in the past years revealing the appearance of sub-Poissonian photon statistics among other interesting features of the quantum radiation-matter interaction [1]. Both theoretical and experimental activities concentrate in trying to understand simple nontrivial models of quantum optics involving one atom with a few energy levels and a few modes of the quantized electromagnetic field. The prototype of these models has been proposed by Jaynes and Cummings [2] more than three decades ago and describes a single two-level atom interacting with a single mode of the electromagnetic field.

The experimental realization of a two-photon cascade micromaser [3] stimulates the study of two mode processes. A system composed by a three level atom in Ξ, Λ, and V configurations interacting with two modes of the electromagnetic field was proposed and studied more than ten years ago by Yoo and Eberly [4]. Following these guidelines, Gou [5] investigated the Ξ configuration when the intermediate level can be adiabatically eliminated [6]. This approximation turns the original bilinear photon-level interaction into a three-linear one (usually called "non-linear non-degenerate two-photon" interaction). This model has been broadly used in order to study the time evolution of the atomic and photon operators, the second order coherence function, the one and two-mode squeezing, the atomic-dipole squeezing and the emission spectra [5,7–12]. Usually, the two-level system has been considered initially in the excited state and the two-photons have been chosen initially in two independent coherent states [5,9,10,12], two mode squeezed states [5], pair-coherent states [7], correlated SU(1,1) coherent states [8], etc. The Λ configuration, also called the Raman coupled model, when the intermediate level can be adiabatically eliminated was studied by Abdalla, Ahmed and Obada [13], and independently by Gerry and Eberly [14]. There have been investigations of the atomic inversion, the appearance of antibunched light, the violations of the Cauchy-Schwartz inequality, population trapping, and squeezing [13–17]. Some similarities in the Rabi frequencies of both configurations for some special conditions of the parameters were reported [8], but a deeper analysis was never done.

Recently, an extension of the Jaynes-Cummings Hamiltonian (JCH) to the case of time-dependent couplings has been studied [18]. The important role of quantum correlations in this problem has been analyzed using an infinite set of relevant operators. Also the time-dependent Jaynes-Cummings model with an additional Kerr-like medium has been solved within the same framework [19].

In the present effort, we show the close dynamical relationship that exists between the Ξ and Λ non-degenerate two-photon time dependent Jaynes-Cummings models. Moreover, it is demonstrated that that the dynamical behavior of both models are characterized by the same bi-dimensional Fibonacci-like connection between quantal correlations of different orders. Besides, taking advantage of the underlying semi-Lie algebra structure, exact off-resonance solutions for both models are obtained. From the set of ordinary differential equations that describes the dynamics of both problems we obtained several sets of
invariants of the motion. Finally, the possible use of the results here obtained to similar problems are discussed.

2 Model

The system we consider is the effective two-level atom described, as usual, by the fermionic operators representing the lower and upper states \( \hat{b}_1, \hat{b}_1^\dagger, \hat{b}_2, \hat{b}_2^\dagger \). These levels interact through a two-photon process described by boson operators \( \hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2, \hat{a}_2^\dagger \). Some intermediate levels are involved, but we eliminate them using the adiabatic approximation [6]. So, the effective Hamiltonian in the rotating wave approximation reads \((\hbar \equiv 1) [5]\)

\[
\hat{H} = \sum_{i=1}^{2} E_i \hat{b}_i^\dagger \hat{b}_i + \sum_{i=1}^{2} \omega_i \hat{a}_i^\dagger \hat{a}_i + T(t) (\gamma \hat{a}_1^\dagger \hat{a}_2 \hat{b}_1 \hat{b}_2^\dagger + \gamma^* \hat{b}_2^\dagger \hat{a}_1^\dagger \hat{a}_2 \hat{b}_1^\dagger) \tag{1}
\]

The dynamical equation for the mean value of any operator can be obtained using the Ehrenfest theorem

\[
\frac{d}{dt} \langle \hat{O}_i \rangle = - \sum g_{ij} \langle \hat{O}_i \rangle \tag{2}
\]

where \( g_{ij} \) are the structure constants of a semi-Lie algebra under commutation with the Hamiltonian, i.e.

\[
[ \hat{H}(t), \hat{O}_i ] = i\hbar \sum_j g_{ji}(t) \hat{O}_j \tag{3}
\]

As one is interested in the evolution of the mean values of the populations of the levels and the fields, Eq. (3) selects from all the possible quantal correlations the set that determines their dynamics. The quantal correlations that satisfy Eq. (3) will be called relevant operators (RO). In our case, they are

\[
\hat{N}^{n,m}_1 \equiv (\hat{a}_1^\dagger)^n (\hat{a}_2^\dagger)^m \hat{b}_1^\dagger \hat{b}_1 (\hat{a}_2)^m (\hat{a}_1)^n, \tag{4a}
\]

\[
\hat{N}^{n,m}_2 \equiv (\hat{a}_1^\dagger)^n (\hat{a}_2^\dagger)^m \hat{b}_2^\dagger \hat{b}_2 (\hat{a}_2)^m (\hat{a}_1)^n, \tag{4b}
\]

\[
\hat{\Delta}^{n,m}_1 \equiv (\hat{a}_1^\dagger)^n (\hat{a}_2^\dagger)^m \hat{a}_1^\dagger \hat{a}_1 (\hat{a}_2)^m (\hat{a}_1)^n, \tag{4c}
\]

\[
\hat{\Delta}^{n,m}_2 \equiv (\hat{a}_1^\dagger)^n (\hat{a}_2^\dagger)^m \hat{a}_2^\dagger \hat{a}_2 (\hat{a}_2)^m (\hat{a}_1)^n, \tag{4d}
\]

\[
\hat{I}^{n,m}_1 \equiv (\hat{a}_1^\dagger)^n (\hat{a}_2^\dagger)^m (\gamma \hat{a}_1 \hat{a}_2 \hat{b}_1 \hat{b}_2^\dagger + \gamma^* \hat{b}_2^\dagger \hat{a}_1^\dagger \hat{a}_2 \hat{b}_1^\dagger) (\hat{a}_2)^m (\hat{a}_1)^n, \tag{4e}
\]

\[
\hat{I}^{n,m}_2 \equiv (\hat{a}_1^\dagger)^n (\hat{a}_2^\dagger)^m i(\gamma \hat{a}_1 \hat{a}_2 \hat{b}_1 \hat{b}_2^\dagger - \gamma^* \hat{b}_2^\dagger \hat{a}_1^\dagger \hat{a}_2) (\hat{a}_2)^m (\hat{a}_1)^n, \tag{4f}
\]

\[
\hat{\hat{N}}^{n,m}_{2,1} \equiv (\hat{a}_1^\dagger)^n (\hat{a}_2^\dagger)^m \hat{b}_2^\dagger \hat{b}_2 \hat{b}_1^\dagger \hat{b}_1 (\hat{a}_2)^m (\hat{a}_1)^n, \tag{4g}
\]

where \( n, m = 0, 1, \ldots, \infty \), determine the order of the correlation between a set of basic operators (i.e. those with \( n = m = 0 \)), and both fields. \( \hat{N}^{0,0}_1 \) and \( \hat{\Delta}^{0,0}_1 \), \( l = 1, 2 \), are the population number of the levels and the external fields, respectively. \( \hat{I}^{0,0}_2 \) is the interaction
energy between the levels and the external fields, $\hat{F}^{n,0}$ can be considered as a current of particles and photons and, finally, $N_{2,1}^{0,0}$ as the double occupation number. In terms of these RO the evolution equations (2) read

\[
\frac{d(\hat{N}_{1}^{n,m})}{dt} = T(t)\hat{N}_{1}^{n,m} + nT'(t)\hat{N}_{1}^{(n-1),m} + mT(t)\hat{N}_{1}^{n,(m-1)} + in.mT'(t)\hat{N}_{1}^{(n-1),(m-1)},
\]
\[
\frac{d(\hat{N}_{2}^{n,m})}{dt} = -T(t)\hat{N}_{2}^{n,m},
\]
\[
\frac{d(\hat{\Delta}_{1}^{n,m})}{dt} = (n + 1)T(t)\hat{N}_{1}^{n,m} + mT(t)\hat{N}_{1}^{n,(m-1)} + mT(t)\hat{N}_{1}^{(n+1),(m-1)} + m.mT'(t)\hat{N}_{1}^{n,(m-1)},
\]
\[
\frac{d(\hat{\Delta}_{2}^{n,m})}{dt} = (m + 1)T(t)\hat{N}_{2}^{n,m} + nT(t)\hat{N}_{2}^{(n-1),m} + nT(t)\hat{N}_{2}^{(n-1),(m+1)} + n.mT'(t)\hat{N}_{2}^{(n-1),m},
\]
\[
\frac{d(\hat{N}_{2,1}^{n,m})}{dt} = 0,
\]
\[
\frac{d(\hat{\rho}_{n,m})}{dt} = a\hat{F}^{n,m},
\]
\[
\frac{d(\hat{F}_{n,m})}{dt} = -a\hat{F}_{n,m} + 2|\gamma|^2T(t)\left\{(n + 1)(m + 1)[\hat{N}_{2}^{n,m+1} - \hat{N}_{2}^{n,m+1}] + (n + 1)\hat{N}_{2}^{(n+1),m} + (m + 1)\hat{N}_{2}^{(n+1),m} + \hat{N}_{2}^{(n+1),m}\right\},
\]

\[
\alpha = E_{2} - E_{1} - \omega_{1} - \omega_{2}.
\]

Equations (5) are the exact evolution equations for the non-degenerate time-dependent two-photon $\Xi$ model. From these equations different sets of constants of the motion can be obtained, i.e.

\[
\left\{\langle \hat{\Delta}_{1}^{n,m} \rangle + (n + 1)[\langle \hat{N}_{1}^{n,m} \rangle + m\langle \hat{N}_{1}^{(n-1),m} \rangle] + m\langle \hat{N}_{1}^{(n+1),(m-1)} \rangle\right\}_{n=0}^{\infty},
\]
\[
\left\{\langle \hat{\Delta}_{2}^{n,m} \rangle + (m + 1)[\langle \hat{N}_{2}^{n,m} \rangle + n\langle \hat{N}_{2}^{(n-1),m} \rangle] + n\langle \hat{N}_{2}^{(n-1),(m+1)} \rangle\right\}_{n=0}^{\infty},
\]
\[
\left\{\sum_{i=1}^{2}\langle \hat{N}_{i}^{n,m} \rangle + m.m\left[\frac{\langle \hat{N}_{2}^{(n-1),m} \rangle}{m} + \frac{\langle \hat{N}_{2}^{(m-1)} \rangle}{n} + \langle \hat{N}_{2}^{(n-1),(m-1)} \rangle\right]\right\}_{n=0}^{\infty},
\]

for any function of time $T'(t)$, and

\[
\left\{\langle \hat{\rho}_{n,m} \rangle + m\langle \hat{\rho}_{2,n,m} \rangle\right\}_{n=0}^{\infty},
\]

for $T(t) = 1$. They include the excitation number operators of modes 1 and 2, and the exchange constant operator [12]. Using the RO set, a maximum entropy principle density matrix can be used in order to evaluate initial mean values. This density matrix can be
diagonalized following the guidelines developed in Refs [18,19]. This detailed calculation will be presented elsewhere due to lack of space.

Now, in order to show the benefit of Eqs. (5) we consider the Hamiltonian (1) in the time-independent case \([\mathcal{H}(t) = 1]\). The evolution of the mean values of the RO can be obtained via the temporal series expansion

\[
\langle \hat{O} \rangle_t = \langle \hat{O} \rangle_0 + \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\hbar}{i} \right)^n \langle [\ldots, \hat{O}, \mathcal{H}, \ldots, \hat{O}] \rangle_0 ,
\]

by taking advantage of the semi-Lie structure [Eq. (3)]. As an example we evaluate the evolution of \(\langle \hat{N}_1^{(0)} \rangle_t\). The linear and quadratic temporal terms in Eq. (7) are obtained using Eq. (5a) and Eq. (5g), respectively, with \(n = 0, m = 0\). In evaluating higher order commutation relations one must realize the driving role of operator \(\hat{F}^{n,m}\). The double commutation of this operator with the Hamiltonian reads

\[
[[ \hat{F}^{n,m}, \hat{H} ], \hat{H} ] = \Omega^2_{n,m} \hat{F}^{n,m} + (\Omega^2_{n,m+1} - \Omega^2_{n,m}) \hat{F}^{n,m+1} + (\Omega^2_{n+1,m} - \Omega^2_{n,m}) \hat{F}^{n+1,m} + 4|\gamma|^2 \hat{F}^{n+1,m+1}.
\]

So, the different orders of the correlations of \(\hat{F}^{n,m}\) are connected by functions of the generalized Rabi frequency \(\Omega^2_{n,m} = \alpha^2 + 4|\gamma|^2(n + 1)(m + 1)\). A similar behavior has been reported for the one-photon Jaynes-Cummings model (see Refs [18,19]). In order to fully understand Eq. (8) we represent each correlation \(\hat{F}^{n,m}\) with a level (see Fig. 1).

In the JCII, \(\mathcal{L}^{n,m}(y)\) reduces to a sum over terms depending on all the frequencies \(\Omega_{0,0}, \Omega_{1,0}, \ldots, \Omega_{n,0}\) [see Eq. (9) and path \(A\) in Fig. 1] because the Hamiltonian is linear. In the Kerr-JCII, as the Hamiltonian has bilinear and quadratic terms in field operators, the links between RO can only consist of those corresponding to one or two level jumps with \(m = 0\) [see Eq. (8)]. In this two-boson JCII we have linear and three-linear terms in field operators and the possible links are bidimensional. Before going into the general formula of the functional we give its explicit form for \(n = 1, m = 2\) in order to fix ideas. So \(\mathcal{L}^{1,2}(y)\) reads

\[
\mathcal{L}^{1,2}(y) = \frac{(\Omega^2_{1,2} - \Omega^2_{1,1})(\Omega^2_{1,1} - \Omega^2_{1,0})(\Omega^2_{1,0} - \Omega^2_{0,0})}{\text{path } A} \times
\]

\[
\sum_{k=0}^{3} \tilde{\alpha}_{3,k}(\Omega_{0,0}, \Omega_{1,0}, \Omega_{1,1}, \Omega_{1,2}) \left[ \frac{\cos(\Omega_{i,k} t) - 1}{\Omega^2_{i,k}} \right] \times
\]

\[
(\Omega^2_{1,2} - \Omega^2_{1,1})(\Omega^2_{1,1} - \Omega^2_{0,1})(\Omega^2_{0,1} - \Omega^2_{0,0}) \times \text{path } B
\]
\[
\sum_{k=0}^{3} \tilde{a}_{3,k}(\Omega_{0,0}, \Omega_{0,1}, \Omega_{1,1}, \Omega_{1,2}) \frac{[\cos(\Omega_{0,k} t) - 1]}{\Omega_{0,k}^2} \\
+ \left( \Omega_{1,2}^2 - \Omega_{0,2}^2 \right) (\Omega_{0,2}^2 - \Omega_{0,1}^2) (\Omega_{0,1}^2 - \Omega_{0,0}^2) \times \\
\sum_{k=0}^{3} \tilde{a}_{3,k}(\Omega_{0,0}, \Omega_{0,1}, \Omega_{0,2}, \Omega_{1,2}) \frac{[\cos(\Omega_{0,k} t) - 1]}{\Omega_{0,k}^2} \\
+ 4|\gamma|^2 (\Omega_{1,2}^2 - \Omega_{0,1}^2) \sum_{k=0}^{3} \tilde{a}_{2,k}(\Omega_{0,0}, \Omega_{1,1}, \Omega_{1,2}) \frac{[\cos(\Omega_{0,k} t) - 1]}{\Omega_{0,k}^2} \\
+ 4|\gamma|^2 (\Omega_{1,1}^2 - \Omega_{0,0}^2) \sum_{k=0}^{3} \tilde{a}_{2,k}(\Omega_{0,0}, \Omega_{0,1}, \Omega_{1,2}) \frac{[\cos(\Omega_{0,k} t) - 1]}{\Omega_{0,k}^2}, \\
\tag{9}
\]

where \(i_k\) is the \(k\)-th component of \(\tilde{a}\), and \(\tilde{a}_{3,k}(\Omega_{0,0}, \Omega_{1,0}, \Omega_{1,1}, \Omega_{1,2}) = (-1)^{3+k+1}|\Psi_{3,k}|/|\Psi|\), \(\Psi_{n,k}\) being the reduced \(n,k\) matrix of the Vandermonde matrix \(\Psi\) [20],

\[
\Psi = \begin{pmatrix}
1 & 1 & 1 & 1 \\
\Omega_{1,0}^2 & \Omega_{1,0}^2 & \Omega_{1,1}^2 & \Omega_{1,2}^2 \\
\Omega_{0,0}^2 & \Omega_{0,1}^2 & \Omega_{1,1}^2 & \Omega_{1,2}^2 \\
\Omega_{0,0}^6 & \Omega_{0,1}^6 & \Omega_{1,1}^6 & \Omega_{1,2}^6
\end{pmatrix},
\tag{10}
\]

It is important to notice that the weight [paths \(A, \cdots, E\) in Eq. (9)] multiplying each sum in Eq. (9) depends on the path. In the JCI II the weights are path independent because \((\Omega_{n+1,0}^2 - \Omega_{n,0}^2)\) (and the determinant of \(\Psi\)) is a \(n\)-independent constant. In the general case the number of different paths (or links) \(F(n,m)\) of order \(n,m\) are obtained from a bi-dimensional Fibonacci-like sequence [21], \(F(n,m) = F(n-1,m) + F(n,m-1) + F(n-1,m-1)\). Each path can be represented by an \(n+1, m+1\)-matrix, \(\tilde{a}_{n,m,k}\), of zeros and ones, where a 1 (0) appears in each frequency that is (not) included in this path. Thus, for example, it is obtained
etc. So, all the different paths, or links of order \( n, m \) are

\[
\{ \tilde{x}_{n,m} \} = \left\{ \left( \begin{array}{c} \{ \tilde{x}_{n-1,m} \} 0 \\ 1 \\ 1 \end{array} \right) \right\} \cup \left\{ \left( \begin{array}{c} \{ \tilde{x}_{n,m-1} \} 1 \\ 0 \\ 1 \end{array} \right) \right\} \cup \left\{ \left( \begin{array}{c} \{ \tilde{x}_{n-1,m-1} \} 0 \\ 0 \\ 1 \end{array} \right) \right\},
\]

(12)

with \( n > 0 \) and \( m > 0 \), \( \{ \tilde{x}_{n,m} \} \equiv \{ x_{n,m,l} \mid l = 0, 1, \ldots, F(n,m) \} \). The \( n = 0 \) or \( m = 0 \) links are the ones obtained for the JCH problem. Now, the general form of \( \mathcal{L}^{n,m}(y) \) reads

\[
\mathcal{L}^{n,m}(y) = \sum_{l=1}^{F(n,m)} \left( \prod_{j=1}^{n} \prod_{k=1}^{m} \phi_{n,m,l}(j,k) \right) \left[ \prod_{j'=1}^{m} \prod_{k'=1}^{m} \psi_{n,m,l}(j',k') \right] \sum_{r=0}^{s_{n,m,l}-1} \delta_{s_{n,m,l}-1,r}(\Omega_{y}, \ldots, \Omega_{x_{n,m,l-1}}) y_{r},
\]

(13)

where \( s_{n,m,l} = \sum_{j=0}^{n} \sum_{k=0}^{m} x_{n,m,l}(j,k) \) gives the number of frequencies associated with a given path, \( n \) and \( m \) determine the correlation's order, \( I \) the path, \( j \) and \( k \) the frequency,

\[
\phi_{n,m,l}(j,k) = \begin{cases} (\Omega_{j,k}^{2} - \Omega_{j-1,k}^{2}) & \text{if } x_{n,m,l}(j,k) = 1 \text{ and } x_{n,m,l}(j-1,k) = 1 \\ (\Omega_{j,k}^{2} - \Omega_{j,k-1}^{2}) & \text{if } x_{n,m,l}(j,k) = 1 \text{ and } x_{n,m,l}(j,k-1) = 1 \\ 1 & \text{otherwise}, \end{cases}
\]
\[
\psi_{n,m,l}(j, k) = \begin{cases} 
4|\gamma|^2 & \text{if } x_{n,m,l}(j, k - 1) = 0 \text{ and } x_{n,m,l}(j - 1, k) = 0 \\
1 & \text{otherwise,}
\end{cases}
\]
and \( y_{ir} = [\cos(\Omega_{ir}t) - 1]/\Omega^2_i \), \( y_{ir} = \sin(\Omega_{ir}t)/\Omega_i \) are the components of \( y \equiv C/\Omega^2 \) \((y \equiv S/\Omega)\).

So, the exact solution for \( \langle \hat{N}_1^{0,0} \rangle_t \) reads

\[
\langle \hat{N}_1^{0,0} \rangle_t = \langle \hat{N}_1^{0,0} \rangle_0 - \sum_{n=0}^\infty \sum_{m=1}^\infty \left[ \langle \hat{F}_{n,m} \rangle_0 \mathcal{L}^{n,m}(S/\Omega) + \langle \hat{A}_{n,m} \rangle_0 \mathcal{L}^{n,m} \left( C/\Omega^2 \right) \right] \\
+ 2|\gamma|^2 \sum_{n=0}^\infty \sum_{m=1}^\infty (n + 1)(\langle \hat{N}_2^{n,m} \rangle_0 - \langle \hat{N}_2^{n,m} \rangle_0) \mathcal{L}^{n,m-1} \left( C/\Omega^2 \right) \\
+ 2|\gamma|^2 \sum_{n=0}^\infty \sum_{m=0}^\infty (m + 1)(\langle \hat{N}_2^{n,m} \rangle_0 - \langle \hat{N}_2^{n,m} \rangle_0) \mathcal{L}^{n-1,m} \left( C/\Omega^2 \right) \\
+ 2|\gamma|^2 \sum_{n=0}^\infty \sum_{m=0}^\infty \langle \hat{N}_2^{n,m} \rangle_0 \mathcal{L}^{n-1,m-1} \left( C/\Omega^2 \right)
\]

(14)

where \( \langle \hat{A}_{n,m} \rangle_0 = \alpha \langle \hat{f}_{n,m} \rangle_0 + 2|\gamma|^2 \langle \hat{N}_1^{n+1,m+1} \rangle_0 - (n + 1)(m + 1)(\langle \hat{N}_1^{n,m} \rangle_0 - \langle \hat{N}_1^{n,m} \rangle_0) \). Equivalent results can be obtained for all the RO.

As can be seen, the three-linear nature of this problem brings nontrivial behavior since the number of different sums multiplying the RO of order \( n, m \) in the exact solution, corresponding to all the possible paths between levels 0, 0 and \( n, m \), is given by \( \mathcal{F}(n, m) \).

Equation (11) can be simplified if one notices the particular Fibonacci-like properties of the frequencies \( \Omega_{n,m} \), \( \Omega_{n,m}^2 = \Omega_{n-1,m}^2 + 4|\gamma|^2(m + 1), \Omega_{n,m}^2 = \Omega_{n,m-1}^2 + 4|\gamma|^2(n + 1), \) or \( \Omega_{n,m}^2 - \Omega_{j,k}^2 = 4|\gamma|^2[(n + 1)(m + 1) - (j + 1)(k + 1)] \). We obtain

\[
\langle \hat{N}_1^{0,0} \rangle_t = \langle \hat{N}_1^{0,0} \rangle_0 + 2|\gamma|^2 \left[ \sum_{n=0}^\infty \sum_{m=1}^\infty \left( \langle \hat{N}_2^{n,m} \rangle_0 - \langle \hat{N}_2^{n,m} \rangle_0 \right) \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} a_{n-j,m-k}^0 C_{j,k} \frac{\Omega_{j,k}}{\Omega_{j,k}} \right] \\
+ 2|\gamma|^2 \sum_{n=0}^\infty \sum_{m=0}^\infty \left( n + 1 \right) \left( \langle \hat{N}_2^{n,m} \rangle_0 - \langle \hat{N}_2^{n,m} \rangle_0 \right) \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} a_{n-j,m-k}^0 C_{j,k} \frac{\Omega_{j,k}}{\Omega_{j,k}} \\
+ 2|\gamma|^2 \sum_{n=0}^\infty \sum_{m=0}^\infty \left( m + 1 \right) \left( \langle \hat{N}_2^{n,m} \rangle_0 - \langle \hat{N}_2^{n,m} \rangle_0 \right) \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} a_{n-j,m-k}^0 C_{j,k} \frac{\Omega_{j,k}}{\Omega_{j,k}} \\
- \sum_{n=0}^\infty \sum_{m=0}^\infty \langle \hat{F}_{n,m} \rangle_0 \sum_{j=0}^{n} \sum_{k=0}^{m} a_{j,k}^0 \frac{S_{j,k}}{\Omega_{j,k}} + \langle \hat{A}_{n,m} \rangle_0 \sum_{j=0}^{n} \sum_{k=0}^{m} a_{j,k}^0 \frac{C_{j,k}}{\Omega_{j,k}}
\]

(15)

where \( a_{n,m}^0 = (-1)^{n+m+j+k+1}/[(n-j)!j!(m-k)!k!] \). Due to the fact that Eq. (15) has been obtained by adding up a temporal series expansion it is specially convenient for numerical calculations. For the JCI [18] the temporal function multiplying the correlation \( \langle \hat{O}^a \rangle \) \((\sum_{k=0}^n a_{n,k} y_k) \) is proportional to \( t^{2n+1} \) (i.e. the first 2n terms of the Taylor's expansion vanish). This is not the case in the two-photon problem due to the fact that shorter paths exist (i.e. the path with three-linear transitions will link the states 0, 0 and \( n, n \) in \( n \) steps;
see path $D$ in Fig. 1). We want to mention than even though double infinite sums are involved in Eq. (13) only a finite number of correlations (those with $n + m \leq N_{max}$, as shown by Fig. 1) substantially contribute to the evolution up to a given time and each correlation is weighted by functions which depend on a finite number of generalized Rabi frequencies. The order of the sums in Eq. (15) can be exchanged if one wants to compare with the expressions usually shown in the literature. We get

$$\langle \hat{N}_1^{0,0} \rangle_t = \langle \hat{N}_1^{0,0} \rangle_0 - \sum_{j,k=0}^{\infty} \frac{S_{j,k}}{\Omega_{j,k}} \sum_{n=j}^{\infty} \sum_{m=k}^{\infty} a_{j,k}^{n,m} \langle \hat{F}^{n,m} \rangle_0$$

$$- \sum_{j,k=0}^{\infty} \frac{C_{j,k}}{\Omega_{j,k}} \sum_{n=j}^{\infty} \sum_{m=k}^{\infty} a_{j,k}^{n,m} \left\{ \alpha \langle \hat{I}^{n,m} \rangle_0 + 2|\gamma|^2 \left[ \langle \hat{N}_1^{n+1,m+1} \rangle_0 - \langle \hat{N}_2^{n+1,m+1} \rangle_0 \right] \right. $$

$$- (n+1)(m+1) \left\langle \langle \hat{N}_1^{n,m} \rangle_0 - \langle \hat{N}_2^{n,m} \rangle_0 \right\rangle $$

$$\left. - (m+1) \left\langle \langle \hat{N}_2^{n+1,m} \rangle_0 - \langle \hat{N}_2^{n,m} \rangle_0 \right\rangle - (n+1) \left\langle \langle \hat{N}_2^{n,m+1} \rangle_0 - \langle \hat{N}_2^{n+1,m+1} \rangle_0 \right\rangle \right\} , \quad (16)$$

These results are valid off-resonance and for any initial conditions. It is possible to obtain simplified expressions when the double sums over the correlations can be explicitly evaluated. As an example we consider the particular case of having one particle in level one, zero particles in level two, and both bosonic fields initially in coherent states. We obtain

$$\langle \hat{N}_1^{0,0} \rangle_t = 1 + 2|\gamma|^2 \sum_{j,k=0}^{\infty} \frac{1}{jk! \Omega_{j,k}^2} \langle \hat{\Delta}_1^{0,0} \rangle_0 \langle \hat{\Delta}_2^{0,0} \rangle_0 e^{-(\hat{\Delta}_1^{0,0})_t} - (\hat{\Delta}_2^{0,0})_t \right\} . \quad (17)$$

If we consider the resonant case (i.e. $\alpha = 0$) then we obtain the expression given in Ref. [12]. We want to stress the fact that in this reference only the resonant case has been considered due to difficulties in obtaining some mean values. As it was shown this is not the case in our approach. As a final remark we want to mention that the mathematical details of how to handle the double infinite sums over the initial correlations in Eqs. (16)-(17) has been considered in Refs. [7,8,15,16]. However, this problem is avoided using Eq. (15) because only the correlations which contribute to the solution up to a given time must be included. Let us study in the following section the A model.

3 A model

We consider now the effective two-level atom interacting through a two-photon process in $A$ configuration. The effective Hamiltonian in the adiabatic and rotating wave approximations reads ($\hbar \equiv 1$) [13,14]

$$\hat{H} = \sum_{i=1}^{2} E_i \hat{b}_i \hat{b}_i + \sum_{i=1}^{2} \omega_i \hat{a}_i \hat{a}_i + T(t)(\gamma \hat{a}_1 \hat{b}_1 \hat{b}_1 + \gamma^* \hat{b}_2 \hat{b}_2 \hat{a}_2 \hat{a}_2) \quad (18)$$
The RO for this model are \( \hat{X}_1^{n,m} \), \( \hat{X}_2^{n,m} \), \( \hat{A}_1^{n,m} \), \( \hat{A}_2^{n,m} \), and

\[
\hat{F}^{n,m} \equiv (\hat{a}_1^+)^m (\hat{a}_2^+)^m (\gamma \hat{a}_1 \hat{a}_2 b_1 b_2 + \gamma^* b_1 \hat{a}_1^\dagger \hat{a}_2^\dagger) (\hat{a}_2^\dagger)^m (\hat{a}_1^\dagger)^n) \quad (19a),
\]

\[
\hat{F}^{n,m} \equiv (\hat{a}_1^+)^m (\hat{a}_2^+)^m i(\gamma \hat{a}_1 \hat{a}_2 b_1 b_2 - \gamma^* b_1 \hat{a}_1^\dagger \hat{a}_2^\dagger) (\hat{a}_2^\dagger)^m (\hat{a}_1^\dagger)^n) \quad (19b),
\]

\( n, m = 0, 1, \ldots, \infty \). In terms of these RO the evolution Eqs. (2) read

\[
\frac{d(\hat{X}_1^{n,m})}{dt} = T(t) \hat{F}^{n,m} + n T(t) \hat{F}^{(n-1),m} \quad (20a),
\]

\[
\frac{d(\hat{X}_2^{n,m})}{dt} = -T(t) \hat{F}^{n,m} - m T(t) \hat{F}^{(m-1),n} \quad (20b),
\]

\[
\frac{d(\hat{A}_1^{n,m})}{dt} = (n + 1) T(t) \hat{F}^{n,m} - m T(t) \hat{F}^{(m+1),n} \quad (20c),
\]

\[
\frac{d(\hat{A}_2^{n,m})}{dt} = -(m + 1) T(t) \hat{F}^{n,m} + n T(t) \hat{F}^{(n+1),m} \quad (20d),
\]

\[
\frac{d(\hat{F}^{n,m})}{dt} = (n + 1) \hat{F}^{n,m} - \alpha \hat{F}^{n,m} \quad (20e),
\]

\[
\frac{d(\hat{F}^{n,m})}{dt} = -\alpha' \hat{F}^{n,m} - 2|\gamma|^2 T(t) \left[ (m+1) \left( \hat{X}_1^{(n+1),m} - \hat{X}_2^{(n+1),m} \right) + (n+1) \left( \hat{A}_2^{(n+1),m} - \hat{A}_1^{(n+1),m} \right) \right] \quad (20f).
\]

\( \alpha' \equiv F_2 - F_1 + \omega_2 - \omega_1 \). Equations (20) are the exact evolution equations for the non-degenerate time-dependent two-photon \( \Lambda \) model. Also for this model different sets of constants of the motion can be obtained, i.e.

\[
\left\{ \langle \hat{X}_1^{(n-1),m} \rangle, \langle \hat{X}_2^{(n-1),m} \rangle \right\}_{n=0}^{\infty}, \quad \left\{ \langle \hat{A}_1^{(n-1),m} \rangle, \langle \hat{A}_2^{(n-1),m} \rangle \right\}_{n=0}^{\infty}. \quad (21)
\]

Let us analyze now the time independent case [\( T(t) = 1 \)]. The evolution of the mean values can be obtained using Eq. (7). The dynamics in both models will be different because Eqs. (20) and (5) are not the same. Nevertheless, the main features of the dynamical evolution of the \( \Lambda \) and \( \Xi \) models will be the same because the driving operator \( \hat{F}^{n,m} \) in the \( \Lambda \) model obeys exactly the same Eq. (8) which is fulfilled by \( \hat{F}^{n,m} \) in the \( \Xi \) model provided \( \alpha \) is replaced by \( \alpha' \). This is due to the fact that the bosonic part of the \( \Xi \) (\( \Lambda \)) model is the two-mode representation of the \( \mathfrak{su}(1,1) \) [\( \mathfrak{su}(2) \)] Lie algebra [22]. The commutation relations of the \( \mathfrak{su}(1,1) \) and \( \mathfrak{su}(2) \) algebras differ only in a sign which cancels out in the double commutator Eq. (8). So, all the analysis given in terms of paths for the \( \Xi \) model applies for the \( \Lambda \) model and we can obtain the exact solution for \( \langle \hat{X}_1^{0,0} \rangle_t \),

\[
\langle \hat{X}_1^{0,0} \rangle_t = \langle \hat{X}_1^{0,0} \rangle_0 - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ \langle \hat{F}^{n,m} \rangle_0 \mathcal{L}^{n,m} (S/\Omega) + \langle \hat{A}^{n,m} \rangle_0 \mathcal{L}^{m,n} (C/\Omega^2) \right]
\]
\[-2\gamma^2 \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (n+1)(|\hat{A}_{n,m}^n|_0 - |\hat{A}_{n,m}^{n-1}|_0) \mathcal{L}^{n,m-1} \left( C/\Omega^2 \right) \]
\[-2\gamma^2 \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (m+1)(|\hat{A}_{n,m}^n|_0 - |\hat{A}_{n,m}^{n-1}|_0) \mathcal{L}^{n-1,m} \left( C/\Omega^2 \right) \quad (22)\]

where \( |\hat{A}_{n,m}^n|_0 = a'(\hat{B}_{n,m}^n)_0 + 2|\gamma|^2(|\hat{B}_{n+1,m+1}^{n+1}|_0 - |\hat{B}_{n,m}^n|_0) \). Equivalent results can be obtained for all the RO. Simplified expressions similar to Eqs. (15) and (16) can also be obtained for this model because the generalized Rabi frequencies has the same Fibonacci-like properties. We leave all this derivation for an extended article.

4 Conclusions

In the present letter we have analyzed the non-degenerate two-photon time-dependent Jaynes-Cummings model in the \( \Xi \) and \( \Lambda \) configurations. We have obtained the sets of RO for both problems and we have written down their exact evolution equations. All order field correlations are included in our set of relevant operators (e.g. \( g_{1,1}, g_{2,2}, g_{1,2}, \) etc [5,7,8,12,14,16]). So, it is possible to study the production of nonclassical effects such as antibunched light or violations of the Cauchy-Schwartz inequality. We also obtained different sets of constants of the motion for both models. The time independent case has been explicitly solved. We have shown the very rich dynamical behavior which underlies both models. This behavior, characterized by the Fibonacci-like connection between quantal correlations of different orders can not be seen in other approaches. We have also shown the deep connection between the \( \Xi \) and \( \Lambda \) models. This fact has remained mainly unnoticed in the literature.

Finally, we want to mention that, as it was previously reported both for the \( \Xi \) [23,24] and \( \Lambda \) [25,26] models, when doing the adiabatic approximation the appearance of Stark shifts must be taken into account. We have neglected the Stark shifts in this first approach in order to separate their contribution to the nontrivial dynamics here shown. The presence of Stark shifts increment the number of possible paths, as it was reported for the one-photon JCII [19]. We will study the dynamical evolution of the non-degenerate two-photon time-dependent Jaynes-Cummings and Raman models with Stark shifts and in the presence of a Kerr-like medium in the near future.

Acknowledgments The authors acknowledge support from CONICET, Argentina. JA acknowledges the support and hospitality of the International Centre for Theoretical Physics (ICTP), Trieste, Italy.

References


Fig. 1. Correlations with $n + m = 0, 1, 2,$ and $3,$ and all possible paths for $n = 1, m = 2.$