Groups of diffeomorphisms of infinite-dimensional Hilbert manifolds are defined. Their structure is studied. Quasi-invariant measures on them, relative to dense subgroups, are constructed. Possible applications of measures are discussed for studying unitary representations. Moreover, irreducible unitary representations of a group of diffeomorphisms associated with quasi-invariant measures on a Hilbert manifold are constructed.
1 Introduction.

For compact Riemannian manifolds (finite-dimensional) measures on groups of diffeomorphisms [?], [?] were constructed, such that measures were quasi-invariant relative to dense subgroups. Such groups are not locally compact, therefore, they can not possess measures quasi-invariant relative to entire groups [?].

On the other hand, groups of diffeomorphisms appear naturally in partial differential equations and mathematical physics, for example, in quantum mechanics [?], [?], [?], [?]. In such theories, weighted Sobolev, Besov and Hölder spaces play a very important role [?], [?], [?], [?]. They are actively used in theories of elliptic equations on manifolds Euclidean at infinity. Few articles were devoted to investigations of connections and curvatures of groups of diffeomorphisms $Diff^t(M)$ of locally compact manifolds $M$ over $\mathbb{R}$ of classes of smoothness $t$ (see also notations below). Such groups $Diff^t(M)$ are infinite-dimensional manifolds themselves [?], [?]. They are not locally compact, hence they can’t have quasi-invariant measures relative to the entire group, but only relative to a subgroup $G' \neq Diff^t(M)$ (see [?]). Quasi-invariant measures produce unitary regular representations that can be decomposed into irreducible components (see [?] and references in [?]). Then irreducible unitary representations and quasi-invariant measures can be used to study a manifold $Diff^t(M)$ itself and $M$.

In [?] a possibility for construction of a differentiable measure was outlined, but it wasn’t given precisely. The approach proposed in [?] apart from [?] was applicable also to non-flat manifolds, but uses the Lebesgue measures on $\mathbb{R}^m$, where $m > dim_{\mathbb{R}} M$, $dim_{\mathbb{R}} M$ is a dimension of $M$ over $\mathbb{R}$, so it is of no use for infinite-dimensional $M$. Moreover, formulas in [?], [?] can’t be transferred automatically to Hilbert manifolds $M$.

In the papers [?] and [?] quasi-invariant measures $\mu$ on $Diff^t(M)$ for infinite-dimensional Banach manifolds over non-Archimedean locally compact fields were constructed. The structure of such groups and their irreducible representations was studied. Nevertheless, cases of $Diff^t(M)$ of infinite-dimensional $M$ over $\mathbb{R}$ were not considered yet.

This paper is devoted to the construction of quasi-invariant measures $\mu$ on $Diff^t(M)$ for infinite-dimensional Hilbert manifolds $M$ (over the field $\mathbb{R}$). The constructed measures $\mu$ can be chosen in addition infinitely many times differentiable relative to one-parameter subgroups $< g >$ of dense subgroups $G'$.

In §2 notations and definitions are given. §3 contains results about the structure of a group of diffeomorphisms. Quasi-invariant measures on a group of diffeomorphisms are produced in §4, §5 contains the theorem about the existence of non-measurable representations of $Diff^t(M)$ relative to $\mu$. This develops my previous results [?], [?], [?] for locally compact and infinite-dimensional Banach-Lie groups and $Diff^t(M)$ for locally compact $M$ (the last two cases were considered only briefly in preceding papers).
Irreducible unitary representations of a group of diffeomorphisms associated with a quasi-invariant measure on a Hilbert manifold are described in §6. The main results of the present paper are deduced for the first time and given below in theorems 3.4, 4.1, 5.1, 6.1, 6.17, 6.18.

2 Notations and definitions.

To avoid misunderstandings, we first present our notations and terminology.

2.1. Definition. Let $U$ and $V$ be open subsets in $l_2$. We consider a space of all infinitely many times Frechet (strongly) differentiable functions $f, g : U \to V$ fulfilling (i, ii) and with a finite metric $\rho^{\beta, \gamma}_{t}(f, h) < \infty$, where $h$ is some fixed smooth mapping $h : U \to V$ (that is of class $C^\infty$);

\[
(i) \quad \rho^{\beta, \gamma}_{t}(f, g) := \sup_{x \in U, \ y \neq x, \ y \in U} \left( \sum_{n=0}^{\infty} \left( d^{n}_{t, \beta, \gamma}(f, g) \right)^{2} \right)^{1/2} < \infty;
\]

\[
d^{n}_{t, \beta, \gamma}(f, g) := \| x >^{\beta} (f(x) - g(x)) \|_{l_{2, \gamma}}, \quad (d^{n}_{t, \beta, \gamma}(f, g))^{2} := \sum_{|\alpha| \leq t, \ \alpha = (\alpha^1, \ldots, \alpha^n)} \| n^{\alpha} \| < x >^{\beta + |\alpha|} D^{n}_{x, \gamma}(f(x) - g(x)) \|_{l_{2, \gamma}} + \sum_{|\alpha| = t} \| n^{\alpha} \| < x >^{\beta + |\alpha| + b} |D^{n}_{x, \gamma}(f(x) - g(x)) - D^{n}_{y, \gamma}(f(y) - g(y))| \|_{l_{2, \gamma}} / |x^n - y^n|^{b} \text{ for } n \in \mathbb{N} := \{1, 2, 3, \ldots\}, \quad d^{t}_{n, \beta, \gamma}(f, g) = d^{t}_{n, \beta, \gamma}(f, g)(x, y),
\]

such that

\[
(ii) \quad \lim_{R \to \infty} \rho^{t}_{\beta, \gamma}(f|_{U, 0}) = 0,
\]

where $x = (x^j : j \in \mathbb{N}, \ x^j \in \mathbb{R}) \in l_{2, \gamma}$ that is $\| x \|_{l_{2, \gamma}} = \left\{ \sum_{j=1}^{\infty} (x^j)^{2} \right\}^{1/2} < \infty$, $\infty > \gamma \geq 0$, $l_2 = l_{2,0}$ is the standard separable Hilbert space over $\mathbb{R}$ with the orthonormal base $\{e_n : n \in \mathbb{N}\}$, $U_{R} := \{x \in U : \| x \|_{l_2} > R\}$, $f(x) = (f^j(x) : j \in \mathbb{N}, f^j(x) \in \mathbb{R})$, $t \geq 0$, $|t|$ is the integral part of $t$ (the largest integer such that $|t| \leq t$, $b = \{t\} := t - |t|$, $0 \leq b < 1$ (for $b = 0$ the last term in the definition of $d^{t}_{n, \beta, \gamma}$ is omitted), $D^{n}_{x, \gamma} := \partial/\partial x^n := \partial_j, \quad D^{n}_{x, \gamma}(f(x)) := D^{n}_{j, \gamma}(f^j(x)), \quad e_j = (0, \ldots, 0, 1, 0, \ldots) \text{ with } 1 \text{ in the } j\text{-th place}, \quad \alpha = (\alpha^1, \ldots, \alpha^n), \quad \alpha^j \in \mathbb{N} \cup 0 =: \mathbb{N}_0, \ |\alpha| = \alpha^1 + \ldots + \alpha^n, \ \beta \in \mathbb{R}, \quad < x > := \min(< x >, < y >), \quad < x > := (1 + \| x \|_{l_2}^{2})^{1/2}, \quad f(x) \in l_2, \ f|A \text{ denotes a restriction of } f \text{ on a subset } A \subset U, \quad n^{\alpha} := 1 \omega^1 \ldots n^{\alpha^n} \text{ for } n \in \mathbb{N}.$

We denote by $E^{t, h}_{\beta, \gamma}(U, V)$ the completion of such metric space, $E^{\infty}_{\beta, \gamma} := \bigcap_{t=1}^{\infty} E^{t}_{\beta, \gamma}(U, V)$ with the topology given by the family $\{\rho^{t}_{\beta, \gamma} : j \in \mathbb{N}\}$ in the latter case. For $V = l_2$ and $h(u) = 0$ it is the Banach space with $\| f - g \|_{E^{t, h}_{\beta, \gamma}(U, l_2)} := \rho^{t}_{\beta, \gamma}(f, g) = \rho^{t}_{\beta, \gamma}(f - g, 0)$ that is, the infinite-dimensional separable analog of the weighted Hölder space $C^{a}_{\beta}(U', \mathbb{R}^m)$ (compare with [7]) for open $U' \subset \mathbb{R}^k$, $k$ and $m \in \mathbb{N}$. When $\gamma = 0$ or $h(U) = 0$ we omit $\gamma$ or $h$ respectively. It is evident that each cylindrical function $g(P_k x)$ is in $E^{t}_{\beta}(U, l_2)$ if $g \in C^{t}_{\beta}(U', \mathbb{R}^m)$, $P_k : l_2 \to \mathbb{R}^k$ is the orthogonal projection, $U = (P_k)^{-1}(U')$, $g(P_k x) := (g^1(P_k x), \ldots, g^m(P_k x), 0, 0, \ldots)$. The spaces $E^{t}_{\beta}(U, V)$ differ from $E^{0}_{\beta}(U, V) := E^{t}(U, V)$ for unbounded $U$ if $\beta > 0$. 

3
2.2. Definition. Let $M$ be a manifold modelled on $l_2$ and fulfilling conditions (i-vi) below:

(i) an atlas $\mathcal{A}(M) = [(U_j, \phi_j) : j = 1, \ldots, k]$ is finite, $k \in \mathbb{N}$ (or countable, $k = \infty$), $\phi_j : U_j \to l_2$ are homeomorphisms of $U_j$ onto $\phi_j(U_j) \ni 0$, $U_j$ and $\phi_j(U_j)$ are open in $M$ and $l_2$ respectively, $(\phi_j \circ \phi_j^{-1} - id) \in E^{\omega, 0}_{\omega, \delta} (\phi(U_i \cap U_j), l_2)$ for each $U_i \cap U_j \neq \emptyset$, where $\omega > 0$, $\gamma > 0$, $id$ is the identity mapping $id(x) = x$ for each $x$;

(ii) $TM$ is a Riemannian vector bundle with a projection $\pi : TM \to M$ and a metric $g_x$ in $T_x M$ induced by $|| * ||_{l_2}$ with a RMZ-structure. This means that a connector $K$ and $g$ are such that $g(c(X, Y))$ is constant for each $C^\infty$-curve $c : [0, 1] \to M$, $\Xi(M) := \Xi(TM(M))$ is the algebra of infinitely differentiable vector fields on $M$ (see 3.7 in [?]);

(iii) $(M, g)$ is geodesically complete and supplied with the Levi-Civita connection and the corresponding covariant differentiation $\nabla$ (see 1.1, 2.1 and 5.1 in [?]);

(iv) the charts $(U_j, \phi_j)$ are natural with the natural (Gaussian) coordinates with locally convex $\phi_j(U_j)$ and the exponential mapping $exp : V_p \to M$ corresponding to $\nabla$, where $V_p$ is open in $T_p M$ for each $p \in M$, each restriction $exp|_{V_p}$ is the local homeomorphism (see §III.8 in [?], §6, 7 in [?]) such that $r_{\text{inj}} := \inf_{x \in M} r_{\text{inj}}(x) > 0$, where $r_{\text{inj}}(x)$ is a radius of injectivity for $exp_x$, $r_{\text{inj}}$ is for entire $M$;

(v) $M$ is Hilbertian at infinity, that is, there exists $\tilde{M}_R \subset M$ with $M \setminus \tilde{M}_R := M_R$ equal to finite (or countable) disjoint union of connected open components $\Omega_a$, $a = 1, \ldots, p$, such that $\phi_a^{-1}(\Omega_a) = l_2 \setminus B_a$, where $B_a$ are closed balls in $l_2$, each $\Omega_a$ is with a metric $\tilde{e}$ induced by $\phi_a^{-1}$ and the standard metric in $l_2$. Let a metric $g$ for $M$ be elliptic, that is, there exists $\lambda > 0$ such that $\lambda \tilde{e}_x(\xi, \xi) \leq g_x(\xi, \xi)$ for each $\xi \in T_x M$ and $x \in M$, where $\tilde{M}_R := [x \in M : d_M(x, x_0) \leq R]$, $x_0$ is some fixed point in $M$, $d_M$ is the distance function on $M$ induced by $g$, $\infty > R > 0$ (see for comparison the finite-dimensional case of $M$ in [?]);

(vi) $M$ contains a sequence of $M_k$ and $N_k$. They are supposed to be closed $E^{\omega, 0}_{\omega, \gamma}$-submanifolds with finite dimensions $\dim_R M = k$ for $M_k$ and codimensions $\text{codim}_R N_k = k$ for $N_k$, $k = k(n) \in \mathbb{N}$, $k(n) < k(n + 1)$ for each $n$, $M_k \subset M_l$ and $N_k \supset N_l$ for each $k < l$, $M = M_k \cup N_k$, $M_k \cap N_k = \partial M_k \cap \partial N_k$ for each $k$ such that $\bigcup_k M_k$ is dense in $M$, $\mathcal{A}(M)$ and $M$ are foliated in accordance with this decompositions. These means that (a) $\phi_{i,j} := \phi_i \circ \phi_j^{-1}|_{\phi_j(U_i \cap U_j)} \to l_2$ are of the form $\phi_{i,j}(x^l : l \in \mathbb{N}) = (\alpha_{i,j,k}(x_1, \ldots, x_k), \gamma_{i,j,k}(x^l : l > k))$ for each $n \in \mathbb{N}$, $k = k(n)$, when $M$ is without boundary, $\partial M = \emptyset$. If $\partial M \neq \emptyset$ there is the following additional condition: (b) for each boundary component $M_0$ of $M$ and $U_i \cap M_0 \neq \emptyset$ we have $\phi_i : U_i \cap M_0 \to H_i$, where $H_1 = \{(x^j : j \in \mathbb{N}) | x^j \geq 0\}$. If $U_i \cap M_0 \neq \emptyset$ and $U_j \cap M_0 \neq \emptyset$ we have both images in $H_1$ (or in $H_l$ with $l > 1$), then the foliation is called transverse (tangent respectively) to $M_0$. Then the equivalence relation of $E^{\omega, 0}_{\omega, \gamma}$-atlases that produces foliated $M$ (see also [?]) for finite-dimensional $C^r$-manifolds is as usually considered.
2.3. Definition. Let $M$ and $\tilde{M}$ be two manifolds as in 2.2 with a smooth mapping (for example, an embedding) $\theta : \tilde{M} \to M$, $\omega$ and $\omega' \geq \max(0, \beta)$, $\beta \in \mathbb{R}$, $t \in \mathbb{R}_+ := [0, \infty)$, $\infty > \gamma \geq 0$, $\delta$ and $\delta' \geq \gamma$. We denote by $\tilde{E}^t_{\beta, \gamma}(\tilde{M}, M)$ a space of functions $f : \tilde{M} \to M$ with $f_{i,j} : = \phi_i \circ f \circ \tilde{\phi}_j^{-1} (\tilde{\phi}_j(U_j) \cap \tilde{\phi}_j(f^{-1}(U_i)))$, $(f_{i,j} - \theta_{i,j}) \in E^t_{\beta, \gamma}(\tilde{\phi}_j(U_j) \cap \tilde{\phi}_j(f^{-1}(U_i)))$, $\phi_i(U_i)$ for each $i$ and $j$. When $At(M)$ is finite it is metrizable by a metric $(i) \rho_{\beta, \gamma}(f, \theta) := \sum |f_{i,j} - \theta_{i,j}|^t$ with $(ii) \lim_{t \to \infty} \rho_{\beta, \gamma}(f|_{\tilde{M}_t}, \theta) = 0$. For infinite countable $At(M)$ we denote by $\tilde{E}^t_{\beta, \gamma}(\tilde{M}, M)$ the strict inductive limit $\bigcup_{E \in \Sigma} \tilde{E}^t_{\beta, \gamma}(\tilde{M}, M)$, where $E \in \Sigma$, $\Sigma$ is the family of all finite subsets of $M$ directed by the inclusion $E < F$ if $E \subset F$, $\tilde{U}_E := \bigcup_{E \in E} \tilde{U}_j$, $(\tilde{U}_j, \tilde{\phi}_j)$ are charts of $At(\tilde{M})$, $\Pi_E : \tilde{E}^t_{\beta, \gamma}(\tilde{M}, M) \hookrightarrow \tilde{E}^t_{\beta, \gamma}(\tilde{M}, M)$ and $\Pi_E : \tilde{E}^t_{\beta, \gamma}(\tilde{M}, M)$ are uniformly continuous embeddings (isometrical for $0 < \gamma < \infty$). Evidently, $\tilde{E}^t_{\beta, \gamma}(\tilde{M}, M)$ is the space of functions $f$ of the class $\tilde{E}^t_{\beta, \gamma}(\tilde{M}, M)$ with supports $\text{supp}(f) := \overline{\{x \in \tilde{M} : f(x) \neq 0\}} \subset \tilde{U}_E \cap \tilde{U}_F$ is open for each $E \in \Sigma$.

Let $Hom(M)$ be a group of homeomorphisms of $M$ and $Diff^t_{\beta, \gamma}(M) := \{f \in Hom(M) : f \text{ and } f^{-1} \in \tilde{E}^t_{\beta, \gamma}(M,M)\}$ be a group of homeomorphisms (diffeomorphisms for $t \geq 1$) of class $\tilde{E}^t_{\beta, \gamma}$. When $At(M)$ is finite it is metrizable with the right-invariant metric

$$(iii) \ d(f, g) := \rho_{\beta, \gamma}(g^{-1}f, id),$$

where $\theta$ is the identity map for $\tilde{M} = M$, $\theta = id$ (in this case the index $\theta$ is omitted), $\beta \geq 0$ (see also [?] for finite-dimensional $M$, correctness of this definition is proved in theorem 3.1). Henceforth, we omit tilde in $\tilde{E}$.

2.4. Definition. A Riemannian metric $g$ for $M$ Hilbertian at infinity is called regular Hilbertian asymptotically, if there exist $\delta > 0$, $t' > 1$, $\beta' > 0$, $\infty > \gamma' \geq 0$ such that $\langle g - e \rangle x(\xi, \xi) \in E_{\beta', \gamma'}(M, \mathbb{R})$ by $x$ for each $\xi \in TM$, $\xi = (\xi_x : x \in M)$, $\|\xi_x\|_2 \leq 1$ for each $x \in M$, $\sup_{\xi \in TM, \|\xi\|_2 \leq 1} \|\langle g - e \rangle x(\xi, \xi)\|_{E_{\beta', \gamma'}(M, \mathbb{R})} \leq \delta$. For spaces $E_{\beta', \gamma'}(M, N)$ with $M = N$ or $N$ being a Banach space over $\mathbb{R}$ we assume that $\omega \geq \max(0, \beta)$ and $\beta' \geq \max(0, \beta)$, $t' > t + 1$, $\gamma' \geq \gamma$ in 2.2, 2.4.

2.5. Definition. Let $U$ be open in $\mathbb{R}^m$ and $V$ be open in $\mathbb{R}^n$ or $l_2$ neighbourhoods of $0$ with $m$ and $n \in \mathbb{N}$. We denote by $H^1_{\beta, \delta}(U, V)$ the following completion relative to a metric $d_{\beta, \delta}(f, g)$ of a family of infinitely differentiable functions $f, g : U \to V$ with $d_{\beta, \delta}(f, \theta) < \infty$, where $\theta : U \to V$ is a smooth mapping, $l$ is a nonnegative integer, $\beta \in \mathbb{R}$, $\infty > \delta \geq 0$, $\beta_{\beta, \delta}(f, g) = \langle \Sigma_{0 \leq|\alpha| \leq l} |x|^{-\beta+|\alpha|} D_x^\alpha(f(x) - g(x))\|_{L^2} \rangle^{1/2}$, $L^2 := L^2(U, F)$ (for $F := \mathbb{R}^m$ or $l_{2, \delta}$) is the standard Hilbert space of equivalence classes of measurable functions $h : U \to F$ for which exists $\|h\|_{L^2} := (\int_U |h(x)|^2 \mu_m(dx))^{1/2} < \infty$, $\mu_m$ is the Lebesgue measure on $\mathbb{R}^m$. Let $M$ and $N$ be manifolds fulfilling 2.2(i-vi) with finite atlases, modelled over $\mathbb{R}^m$ for $M$ and $\mathbb{R}^n$ or $l_2$ for $N$, $\theta : M \hookrightarrow N$ be a smooth mapping (for example, an embedding). Then $H^1_{\beta, \delta}(M, N)$ is the completion of a family of infinitely differentiable functions $g, f : M \to N$ with $\kappa_{\beta, \delta}(f, \theta) < \infty$, where $\kappa$ is a
The metric $\kappa_{\beta,\delta}(f,g) = (\sum_{i,j} [q_{\beta,\delta}(f_{i,j},g_{i,j})]^2)^{1/2}$ (see 2.3). The Hilbert spaces $H^{1}_{\beta,\delta}(U,F)$ and $H^{1}_{\beta,\delta}(TM)$ are called the weighted Sobolev spaces, where $H^{1}_{\beta,\delta}(TM) := \{f : M \to TM : f \in H^{1}_{\beta,\delta}(M,TM), \pi \circ f(x) = x \text{ for each } x \in M\}$ with $\theta(x) = (x,0) \in T_xM$ for each $x \in M$. For infinite atlases we use strict inductive limits as in 2.3. From this definition follows that for such $f$ and $g$ there exists $\lim_{R \to \infty} d_{\beta,\delta}(f(U_R),g(U_R)) = 0$. For $\beta = 0$ or $\gamma = 0$ we omit $\beta$ or $\gamma$ respectively in $Diff_{\beta,\gamma}(M)$ and in $H^{1}_{\beta,\delta}$. Then each topologically conjugated space $(H^{1}_{\beta}(TM))' := \{f : M \to TM : \forall x \in M, \theta(x) = (x,0) \in T_xM\}$ is also a Hilbert space with the standard norm.

2.6. Definition. Let $G$ be a topological group. A Radon measure $\mu$ on $Af(G,\mu)$ (or $\nu$ on $Af(M,\nu)$) is called left-quasi-invariant relative to a dense subgroup $G'$ of $G$, if $\mu(\phi,g) = \mu(g\phi,g)$ (or $\nu(\phi,g) = \nu(g\phi,g)$ respectively) for each $h \in G'$. Henceforth, we assume that a quasi-invariance factor $q_{\mu}(\phi,g) = \mu(\phi,g)/\mu(g,\phi)$ (or $q_{\nu}(\phi,g)$) is continuous by $(\phi,g) \in G' \times G$ (or $(G' \times M)$, $\mu : Af(G,\mu) \to [0,\infty)$, $\nu(V) > 0$ (or $\nu : Af(M,\nu) \to [0,\infty)$, $\nu(V) > 0$) for some (open) neighbourhood $V \subset G$ (or $M$) of the unit element $e \in G$ (or a point $x \in M$, $\mu(G) < \infty$ (or $\nu(M) < \infty$ and is $\sigma$-finite respectively), where $\mu(\phi,E) := \mu(\phi^{-1}E)$ for each $E \in Af(G,\mu)$, $Af(G,\mu)$ is the completion of $Bf(G)$ by $\mu$, $Bf(G)$ is the Borel $\sigma$-field on $G$. Let $(M,F)$ be a space $M$ of measures on $(G,Bf(G))$ (or $(M,Bf(M))$ with values in $\mathbb{R}$ and $G''$ be a dense subgroup in $G$ such that a topology $T$ on $M$ is compatible with $G''$, that is, $\mu \to \mu_h$ (or $\nu \to \nu_h$) is the homeomorphism of $(M,F)$ onto itself for each $h \in G''$. Let $T$ be the topology of convergence for each $E \in Bf(G)$ (or $\nu \in Bf(M)$) and $W$ a neighbourhood of the identity $e \in G$ such that $J$ is dense in $W$, where $J := \{h : h \in G'' \cap W =: W'' \text{, there exists } b \in (-1,1) \text{ and } g(b) = h \text{ with } [g(c) : c \in (-1,1) \subset W'']\}$. Let $\omega(c_1 + c_2) = \omega(c_1)\omega(c_2)$, $\omega(0) = e$ are one parameter subgroups, $c_1, c_2 \in \mathbb{R}$. We assume also that for each $f \in W''$ there are $g(b_1), ..., g(b_k) \in J$ such that $f = g(b_1)...g(b_k)$. A measure $\mu \in M$ (or $\nu \in M$ ) is called differentiable along $g(b)$ in a point $g(c)$ if $\mu(g(b)^{-1}E) - \mu(E) = (b-c)(\mu'(g(c);E) + \alpha(g(b);E))$ and there exists $\lim_{b \to c} \alpha(g(b);E) = 0$ and $\mu'(g(c);E) \in \mathbb{R}$ is continuous by $g(c)$ for each $E \in Bf(G)$, where $b$ and $c \in \mathbb{R}$, $\mu'(g(c);E)$ is called the derivative (by Lagrange) along $g(b)$ in $g(c)$ (analogously for $\nu$ on $M$). Let by induction $\lambda(\ast) = \mu^{(j-1)}(g(c_1),...,g(c_{j-1});\ast)$ and there exists $\lambda'(g(c_j);E)$, then it is denoted $\mu^{(j)}(g(c_1),...,g(c_j);E)$ and is called the $j$-th derivative (by Lagrange) of $\mu$ along $(g(b_1),...,g(b_j))$ in $(g(c_1),...,g(c_j))$, where $j \in \mathbb{N}$.

2.7. Note. For a manifold $N = \bigoplus\{M_j : j \in J\}$, $M_j = M$ for each $j$, $J \subset \mathbb{N}$, we have that $Diff^1_{\beta,\gamma}(N)$ is isomorphic to $S \otimes Diff^1_{\beta,\gamma}(M)$, where $S$ is a discrete symmetric group. Therefore, a quasi-invariant measure on $Diff^1_{\beta,\gamma}(M)$ provides a quasi-invariant measure on $Diff^1_{\beta,\gamma}(N)$.

Henceforward, we assume that $M$ and $M_k$ are connected for each $k > n$ and some fixed $n \in \mathbb{N}$. For a finite-dimensional manifold $M$ a space $E_{\beta,\gamma}(M,\mathbb{R})$ (or $Diff^1_{\beta,\gamma}(M)$)
is isomorphic with the usual weighted Hölder space $C^b_\beta(M, R)$ (or $\text{Diff}^b_\beta(M)$ correspondingly).

3 Structure of groups of diffeomorphisms.

3.1. Theorem. Let $G = \text{Diff}^b_\beta(M)$ be defined as in 2.3. Then it is a separable topological group. If $At(M)$ is finite, $G$ is metrizable by a left-invariant metric $d$.

Proof. Let at first $At(M)$ be finite. If $f$ and $g$ in $G$ then $f \circ g^{-1} \in G$ due to theorem 2.5 [?] and ch.5 in [?] about differentiation and difference quotients of composite functions and inverse functions, since $\phi_i \circ \phi_j^{-1} \in E^\infty_{\omega, S}$ for each $i$ and $j$. At first we have $d(f, id) > 0$ for $f \neq id$ in $G$, since there are $i$ and $j$ such that $f_{i,j} \neq id_{i,j}$. Then $d(hf, hg) = d(g^{-1}h^{-1}hf, id) = d(g^{-1}f, id) = d(f, g)$, hence $d$ is left-invariant, where $f, g \in G$. Therefore, $d(f^{-1}, id) = d(id, f)$, in view of 2.1 and 2.3(i,ii) we have that $d(id, f) = d(f, id)$, hence $d(f, g) = d(g, f)$.

It remains to verify, that the composition map $(f, g) \to f \circ g$ from $G \times G \to G$ and the inversion map $f \to f^{-1}$ are continuous relative to $d$. Let $W = \{f \in G : d^b_\beta(f, id) < 1/2\}$ and $f, g \in W$. We have $f_{i,j} \circ g_{j,l} - id_{i,l} = (f_{i,j} \circ g_{j,l} - f_{i,j}) + (f_{i,j} - id_{i,j})$ for corresponding domain as an intersection of domains of $f_{i,j} \circ g_{j,l}$ and $f_{i,j}$. Hence, using induction by $p = 1, 2, ..., |t| + 1$ and the Cauchy inequality we have that there are constants $\infty > C_1 > 0$, $\infty > C_2 > 0$ such that $d(f \circ g, id) \leq C_1 d(f, id) + d(g, id)$ and $d(f^{-1}, id) \leq C_2 d(f, id)$, since $\lim_{n \to \infty} d^b_n(\beta, \gamma)(f_{i,j}, id_{i,j}) + d^b_n(\beta, \gamma)(g_{j,l}, id_{j,l}) = 0$, $|t| + 1$ and $At(M)$ are finite, $r_{i,j} > 0$ and $g$ satisfies 2.4 [?].

Indeed, in normal local coordinates $x$ (omitting indexes $(i, j)$ for $f_{i,j}$), $M \ni x = (x^j : j \in \mathbb{N})$, $f = (f^j : C \to \mathbb{R} | j \in \mathbb{N})$, $C$ open in $l_2$, using the Cauchy inequality we get: $\sum_{i \in \mathbb{N}} (|f \circ g|^t - x^i |\zeta|^t)^2 \leq 2 \left(\sum_{i \in \mathbb{N}} |f \circ g|^t - g^i |\zeta|^t\right)^{1/2} \left(\sum_{i \in \mathbb{N}} |f \circ g|^t - g^i |\zeta|^t\right)^{1/2} + \sum_{i \in \mathbb{N}} |(f \circ g)^i - g^i |\zeta|^t|^2 + \sum_{i \in \mathbb{N}} |g^i - x^i |\zeta|^t|^2$ and $\sum_{i \in \mathbb{N}} |f \circ g|^t - g^i |\zeta|^t|^2 \leq (a + b + 2(\alpha_{i,j} b + ab^{1/2}) + 2a^{1/2}b^{1/2})$, where $a = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |f \circ g^i| - f^i j^i |\zeta|^t|^2$, $b = \sum_{i \in \mathbb{N}} |f \circ g^i| - f^i j^i |\zeta|^t|^2$, $\delta^i_j = 1$ for $i = l$ and $\delta^i_j = 0$ for each $l \neq i$, $f \circ g = f \circ g(x)$, $f, g \in G$.

Then we can proceed by induction for finite products of $D^b_\beta(f \circ g)^i$ and $D^b_\beta g^i$, because $D^b_\beta id(x) = 0$ for $|\alpha| > 1$. For $f = g^{-1}$ we can express recurrently $(D^b_\beta f^{-1})$ by $(D^b_\beta f)$ with $\xi^i \leq \alpha^i$ for each $i$, since $|\alpha| \leq t$. Analogously, for difference quotients, since $(1 + \zeta)^b = 1 + \sum_{m=0}^{\infty} b^m_{\zeta}$ for $0 < b < 1$ and $0 < |\zeta| < 1$, $\zeta \in \mathbb{R}$ and $(1 + \zeta b)^b = 1 + b \zeta + z(\zeta)$ with $z : \mathbb{R} \to \mathbb{R}$, $\lim_{\zeta \to 0} (z(\zeta)/\zeta^b) = 0$ [?]. For countable infinite $At(M)$ for each $f, g \in G$ there are $E(f), E(f^{-1}), E(g)$ and $E(g^{-1}) \in \Sigma$ such that $\text{supp}(f) \subset U^{E(f)}$, etc., consequently, $f(\text{supp}(f)) \cup g^{-1}(\text{supp}(g^{-1})) \subset U^{E}$ for some $F \in \Sigma$, whence $g^{-1} \circ f \in G$ and there is $E \in \Sigma$ with $\text{supp}(g^{-1} \circ f) \subset U^{E}$. If $(f_\gamma : \gamma \in \alpha)$ and $(g_\gamma : \gamma \in \alpha)$ are two nets converging in $G$ to $f$ and $g$ respectively, so for each neighbourhood $W \subset G$ there exist $E \in \Sigma$ and $\beta \in \alpha$ such that $g^{-1}_\gamma \circ f_\gamma \in W$, where $\alpha$ is a limit ordinal.
In view of the Stone-Weierstrass theorem and 2.1(i,ii) in each $E^\infty_{\gamma}(U,V)$ for open $U$ and $V$ in $l_2$ are dense cylindrical polynomial functions with rational coefficients, consequently, $G$ is separable, since $E^\infty_{\beta}(U,V)$ is dense in $E^\infty_{\gamma}(U,V)$. Due to conditions 2.2(i-vi) for each open submanifold $V \subset M$ with $V \supset M_k$ and $\epsilon > 0$ every $f \in \text{Diff}_{f_0}^\epsilon(M_k)$ has an extension $\tilde{f}$ onto $M$ such that $\tilde{f} \in \text{Diff}_{f_0}^\epsilon(M)$ with $\tilde{f}|(M \setminus M_k) \cap U \in \mathcal{F}(f, id) < \epsilon$.

3.2. Lemma. Let $M$ be a manifold defined in 2.2, 2.4 with submanifolds $M_k$ and $N_k$, $k = k(n)$, $n \in \mathbb{N}$. Then there exist connections $\kappa \nabla$ induced on $M_k$ by $\nabla$ are the Levi-Civita connections, where $\nabla$ is the Levi-Civita connection on $M$.

Proof. For each chart $(U_j, \phi_j)$ we have $\phi_j(U_j) \subset l_2$ and in $l_2$ for each sequence of subspaces $R^n \subset R^{n+1} \subset ... \subset l_2$ there are induced embeddings $\phi_j^{-1}(R^n) \cap U_j \hookrightarrow \phi_j^{-1}(R^{n+1}) \cap U_j \hookrightarrow U_j$. The Levi-Civita connection and the corresponding covariant differentiation $\nabla$ for the Hilbertian manifold $M$ induces the Levi-Civita connection $\nabla'$ for each submanifold $M'$ embedded into $M$, if $M'$ is a totally geodesic submanifold. That is, for each $x \in M'$ and $X \in T_x M'$ there exists $\epsilon > 0$ such that a geodesic $\tau = x t \subset M$ defined by the initial condition $(x, X)$ lies in $M'$ for each $t$ with $|t| < \epsilon$ ($\S$ in [?], $\S$VII.8 in [?]). Then using theorem 5 in $\S$4.2 [?] and geodesic completeness of $M$ we can choose such $M' = M_k$ with dimensions $\text{dim}(M_k) = k \in \mathbb{N}$ and $M_k(n) \hookrightarrow M_{k(n+1)} \hookrightarrow ... \hookrightarrow M$ with $U_k M_k$ dense in $M$. Each manifold $M_k$ was chosen Euclidean at infinity, since $M$ is Hilbertian at infinity. In view of §VII.3 in [?] and 5.2, 5.4 in [?] $k(n+1) \nabla$ on $M_{k(n+1)}$ induces $k(n) \nabla$ on $M_{k(n)}$. The latter coincides with that of induced by $\nabla$ on $M$. Here each $M_k$ is geodesically complete, but normal coordinates are defined in $M_k$ in general locally as in $M$ also, since may be $r_{inj}(x) < \infty$ for $x \in M$, so that $\text{At}(M)$ induces $\text{At}(M_k)$ for each $k = k(n)$, $n \in \mathbb{N}$.

3.3. Lemma. Let $M$ be a manifold fulfilling 2.2(i-vi), 2.4, $\text{At}(M)$ be finite and $T_x M$ be its tangent bundle. Then a metric $\rho^\varphi_{\gamma}$ in $E^\varphi_{\beta}(M, TM)$ defined in 2.3 is equivalent to the following norm

$$(i) \|f\|_{t, \beta, \gamma} := \sum_{j=1}^k \sup_{x \in U_j, y \in U_j} \sum_{m=0}^{\infty} |f|_{t, \beta, \gamma}^{2m} \|Y_j\|^2 \leq \|f\|_{t, \beta, \gamma}^{2m} \|Y_j\|^2$$

where $|f|_{t, \beta, \gamma}^{2m}(x) = \|\sigma(x)^{\beta + t} f(x)|U_j|_{t, \beta, \gamma}^{2m} / \|\sigma(x)^{\beta + t} f(x)|U_j\|^2$ and $\lambda = \max(1, \gamma)$.

Notation to lemma 3.3. There is assumed that $X_j$ are linearly independent (in each $x \in M_k$, $k = k(n)$) vector fields on $M_k$ corresponding to local normal coordinates such that $\|\lambda (x_1 - a_i)|U_j\| \leq i^{-2s}$, $s = \max(1, \gamma)$, $\alpha$ corresponds to $Y_1, ..., Y_l$, values of $Y_j$ are taken in each corresponding $x \in M_k$, here $\|Q\| := \sup_{a_j \in A, b_j \in A^*} \|Q(a_1, ..., a_p, b_1, ..., b_q)\| / (\|a_1\|...\|a_p\||\|b_1\|...\|b_q\|)$ for multilinear operator $Q : A^{\otimes p} \otimes (A^*)^{\otimes q} \to B$ and Banach spaces $A, A^*$ ($A^*$ denotes topologically conjugated space) that induces norms for $x \in M$ and $T_x M$.
instead of $A$, $\sigma(x) := (1 + d_M(x,x_0)^2)^{1/2}$, $d_M(x,y)$ is the Riemannian distance function in $M$ induced by $g$ and $\nabla$, $x_0$ is some fixed point in $M$, $\tau(x,y)$ is the tensor of parallel transport, $\rho_M(x)$ denotes the radius of injectivity of the exponential mapping for $TM$ and $x \in M$, $\sigma(\bar{x}) := \min(\sigma(x),\sigma(y))$, $\nabla$ corresponds to $M_a$ in 3.2.

**Proof.** For each $U \subset I_2$ and $f, g \in E^t_{\beta,\gamma}(U, TU)$ we have that $\|f - g\|_{t,\beta,\gamma} = \rho^t_{\beta,\gamma}(f, g)$. Here we use the normal Gaussian coordinates $(x^1, ..., x^n, \ldots)$ on $U$ as in 3.2. In view of lemma 1.5 [?], §7.6 [?], 2.2 and 2.4 the Christoffel symbols $\Gamma_x(V,W)$ are in $E^\infty_{\beta,\gamma}$, where $(x, V, W) \rightarrow \Gamma_x(V,W)$, $\beta^* \geq \max(\beta, 0)$, $V$ and $W \in E^t_{\beta,\gamma}(M, TM)$, $\pi \circ V_x = x$, $\pi \circ W_x = x$, $\pi : TM \rightarrow M$ is the canonical projection. Then $\partial f'(x'(x))/\partial x^j = (\partial f'(x'(x))/\partial x^p)(dx^p(x)/dx^j)$, where the sum is taken over $p \in \mathbb{N}$, $\partial x^p/\partial x^j$ are elements of the unitary matrices $T_{\phi_a, \phi_b} := D(\phi_b \circ (f)_{\phi_a}^{-1})$ for points in $U_a \cap U_b \neq 0$, $|x^j : j \in \mathbb{N}|$ and $|x^p : p \in \mathbb{N}|$ are the local coordinates of the charts $(U_a, \phi_a)$ and $(U_b, \phi_b)$ respectively [? , ?], hence $\sum_p(\partial f'(x'(x))/\partial x^p)^2 = \sum_p(\partial f'(x'))^2(\partial x^p)^2$. If $X_j = \partial/\partial x^j$ then the sum by $(Y_1, ..., Y_k) \subset (X_1, ..., X_n)$ in 3.3(i) corresponds to the sum by $(\alpha_1, ..., \alpha_n - 1)$ in 2.1(i), when $k = |\alpha| - 1$. From 2.4 we have that the norm on $E^t_{\beta,\gamma}(M, TM)$ is equivalent to $\rho^t_{\beta,\gamma}(f, g)$, since $A t(M)$ is finite and $\phi_j \circ \phi_i^{-1}$ for each $U_i \cap U_j \neq 0$ with $\omega \geq \max(\beta, 0)$, where $(\nabla x Y)|\psi(p) = (DY_\psi(p))(X_\psi(p) + \Gamma_\psi(p)(X_\psi(p), Y_\psi(p)))$ in local coordinates of a chart $(U, \psi)$ with $p \in U$.

Indeed, there are constants $C_1 > 0$ and $C_2 > 0$ such that for each $n$ and $U_i \cap U_j \neq 0$ we have $C_1 \sum_{n=0}^{\infty} \|(f - g)_{\beta,\gamma}(i,j), g_{\beta,\gamma}(i,j)\|^2 \leq C_2 \sum_{n=0}^{\infty} \|(f - g)_{\beta,\gamma}(i,j), g_{\beta,\gamma}(i,j)\|^2$ and $C_1 \sum_{n=0}^{\infty} \|(f - g)_{\beta,\gamma}(i,j), g_{\beta,\gamma}(i,j)\|^2 \leq C_2 \sum_{n=0}^{\infty} \|(f - g)_{\beta,\gamma}(i,j), g_{\beta,\gamma}(i,j)\|^2$ for each $g_{\beta,\gamma}(i,j) \in \mathbb{R}$, where $i, j = 1, ..., k < \infty$, because $\nabla V h \in E^t_{\beta,\gamma}$ for $h \in E^t_{\beta,\gamma}$ and due to properties of $\Gamma$. Then conditions $\lim_{R \rightarrow \infty} \rho^t_{\beta,\gamma}(f|M^c_R, 0) = 0$ and $\lim_{R \rightarrow \infty} \|f|M^c_R\|_{t,\beta,\gamma} = 0$ are equivalent, where $M^c_R := M \setminus M_R$.

3.4. **Theorem.** Let $M$ be a manifold fulfilling 2.2, 2.4 and $\text{Diff}^t_{\beta,\gamma}(M)$ be as in 2.3 with $t \geq 1$, $\beta \geq 0$, $\gamma \geq 0$. Then

(i) for each $E^t_{\beta,\gamma}(M, TM)$-vector field $V$ its flow $\eta_t$

is a one-parameter subgroup of $\text{Diff}^t_{\beta,\gamma}(M)$, the curve $t \rightarrow \eta_t$ is of class $C^1$, the mapping $\tilde{\text{Exp}} : T_e \text{Diff}^t_{\beta,\gamma}(M) \rightarrow \text{Diff}^t_{\beta,\gamma}(M), V \mapsto \eta_1$ is continuous and defined on a neighbourhood of the zero section in $T_e \text{Diff}^t_{\beta,\gamma}(M)$

(ii) $T_f \text{Diff}^t_{\beta,\gamma}(M) = \{V \in E^t_{\beta,\gamma}(M, TM) | \pi \circ V = f\}$;

(iii) $(V, W) = \int_M g_f(x)(V_x, W_x)\mu(dx)$

is a weak Riemannian structure on a manifold $\text{Diff}^t_{\beta,\gamma}(M)$, where $\mu$ is a measure induced on $M$ by $\phi_2$ and a Gaussian measure with zero mean value on $l_2$ produced by an injective self-adjoint operator $Q : l_2 \rightarrow l_2$ of trace class, $0 < \mu(M) < \infty$;

(iv) the Levi-Civita connection $\nabla$ induces the Levi-Civita connection
\( \nabla \) on \( \text{Diff}^f(M) \);

\( (v) \, \bar{E} : T\text{Diff}^f_{\beta,\gamma}(M) \to \text{Diff}^f_{\beta,\gamma}(M) \) is defined by

\( \bar{E}(V) = \exp_{\eta(x)} \circ V \eta \) on a neighbourhood of the zero section in \( T\eta \text{Diff}^f_{\beta,\gamma}(M) \) and is a \( E^\omega_{\delta} \) mapping by \( V \) onto a neighbourhood \( W_\eta = W_\text{id} \circ \eta \) of \( \eta \in \text{Diff}^f_{\beta,\gamma}(M) \); \( \bar{E} \) is the uniform isomorphism of uniform spaces \( V \) and \( W \).

**Proof.** Let at first \( A(M) \) be finite. In view of lemma 3.3 and [?] we have that

\[
T_f E^\beta_{\beta,\gamma}(M, N) = \{ g \in E^\beta_{\beta,\gamma}(M, TN) : \pi_N \circ g = f \},
\]

where \( N \) fulfills 2.2, 2.4, \( \pi_N : TN \to N \) is the canonical projection. Therefore, \( TE^\beta_{\beta,\gamma}(M, N) = E^\beta_{\beta,\gamma}(M, TN) = \bigcup T_f E^\beta_{\beta,\gamma}(M, N) \)

and the following mapping \( w_{\exp} : T_f E^\beta_{\beta,\gamma}(M, N) \to E^\beta_{\beta,\gamma}(M, N) \), \( w_{\exp}(g) = \exp \circ g \) gives charts for \( E^\beta_{\beta,\gamma}(M, N) \), since \( TN \) has a finite atlas of class \( E^\infty_\chi \) with \( \nu > \beta, \chi > \gamma \geq 0 \). In view of theorem 5 about differential equations on Banach manifolds in §4.2 [?] a vector field \( V \) of class \( E^\beta_{\beta,\gamma}(M) \) on \( M \) defines a flow \( \eta_t \) of class \( E^\beta_{\beta,\gamma}(M) \), that is \( d\eta_t /dt = V \circ \eta_t \) and \( \eta_0 = e \). Then lightly modifying proofs of theorem 3.1 and lemmas 3.2, 3.3 in [?] we get that \( \eta_t \) is a one-parameter subgroup of \( \text{Diff}^f_{\beta,\gamma}(M) \), the curve \( t \to \eta_t \) is of class \( C^1 \), the map \( S_{\exp} : T\text{Diff}^f_{\beta,\gamma}(M) \to \text{Diff}^f_{\beta,\gamma}(M) \) defined by \( V \to \eta_t \) is continuous.

The curves of the form \( t \to \tilde{E}(tV) \) are geodesics for \( V \in T\eta \text{Diff}^f_{\beta,\gamma}(M), d\tilde{E}(tV)/dt \) is the map \( m \to f \circ h \), \( f \to f \circ h \) is of class \( C^\infty \) on \( \text{Diff}^f_{\beta,\gamma}(M) \) for each \( h \in \text{Diff}^f_{\beta,\gamma}(M) \). Moreover, \( \text{Diff}^f_{\beta,\gamma}(M) \) acts on itself freely from the right, hence we have the following principal vector bundle \( \tilde{\pi} : T\text{Diff}^f_{\beta,\gamma}(M) \to \text{Diff}^f_{\beta,\gamma}(M) \) with the canonical projection \( \tilde{\pi} \).

Analogously to [?, ?, ?] we get the connection \( \nabla = \nabla \circ h \) on \( \text{Diff}^f_{\beta,\gamma}(M) \). Then \( \nabla_{\dot{Y}} \dot{Z} = \int_M \left< \nabla_{\dot{Y}} Xe, Ze \right>_{h(x)} + \left< Y_e, \nabla_{\dot{X}} Ze \right>_{h(x)} [\mu(dx)] = \int_M X_e g(Y_e, Ze) [\mu(dx)] = X(\dot{Y}, \dot{Z}) \), since \( X g(Y, Z) = g(\nabla_{\dot{Y}} Y, Z) + g(Y, \nabla_{\dot{X}} Z) \) (Satz 3.8 in [?]) and for each right-invariant vector field \( V \) on \( \text{Diff}^f_{\beta,\gamma}(M) \) there exists a vector
field $X$ on $M$ with $V_h = X \circ h$ for each $h \in Diff^s_{\beta,\gamma}(M)$, where $\dot{X} := X \circ h$ (see also 3.5 in I.\[?]). If $\nabla$ is torsion-free then $\nabla$ is also torsion-free. From this it follows that the existence of $\tilde{E}$ and $Diff^s_{\beta,\gamma}(M)$ is the Banach manifold of class $E^\infty_{\omega,\delta}$, since $exp$ and $M$ are of class $E^\infty_{\omega,\delta}$, $\alpha_h(f) = f \circ h, f \mapsto f \circ h$ is a $C^\infty$ map with the derivative $\alpha_h : E^s_{\beta,\gamma}(M',TN) \rightarrow E^s_{\beta,\gamma}(M,TN)$ whilst $h \in E^s_{\beta,\gamma}(M,M')$, $\tilde{E}_h(V) := exp_{h(x)}(V(h(x)))$, $\tilde{V}_h = V \circ h, V \in \Xi(M), \tilde{V} \in \Xi(Diff^s_{\beta,\gamma}(M))$.

The case of infinite $At(M)$ may be treated analogously to the proof of theorem 3.2 in II.\[?].

4 Quasi-invariant measures on a group of diffeomorphisms.

At first we give few preliminary definitions and results. Then we formulate the main theorem 4.5.

4.1. Definition. Let $U$ and $V$ be open in $\mathbb{R}^n$, suppose $\theta : U \rightarrow V$ is a smooth mapping, $\infty > \delta > 0$. We define $E^s_{\beta,\gamma}(U,V)$ as a completion of $Q$ relative to the family of metrics given below $[\chi_l,r,s,\delta : l,r,s,\delta \in \mathbb{N}], Q := \{f : f \in E^s_{\beta,\gamma}(U,V), \text{ there exists } t \in \mathbb{N} \text{ such that } supp(f) \subset U \cap \mathbb{R}^n, \chi_l,r,s,\delta(f,\theta) < \infty \text{ for each } l,r,s\}$, where

\[(i) \quad d_{l,r,s,\delta}(f,g) := \sup_{n \in \mathbb{N}} \sup_{x \in U} \rho_{s,n,\delta}(f,g)(n!)^l < \infty \]

and $\lim_{R \rightarrow \infty} d_{l,r,s,\delta}(f|_{U_R},g|_{U_R}) = 0, f \in \rho_{s,n,\delta}$ are taken corresponding to $U \cap \mathbb{R}^n$, that is $f|_{U \cap \mathbb{R}^n} : U \cap \mathbb{R}^n \rightarrow f(U) \subset V$ (see 2.1 and 2.3), $\rho_{s,n,\delta}(f_i,j,id_{i,j}) = \rho_{s,n,\delta}(f_i,j(id_{i,j}(U_i,j) \cap M),id_{i,j}(U_i,j) \cap M)$. $\chi_l,r,s,\delta(f_i,j,\theta)$ are taken corresponding to $\chi_l,r,s,\delta(f_i,j)$. $\rho_{s,n,\delta}$ depends on parameters $(x^j : j > n)$, $\rho_{s,n,\delta}$ is the metric by $x^1,...,x^n$ in $E^s_{\beta,\gamma}(U \cap \mathbb{R}^n, V)$ for $f$ as functions by $(x^1,...,x^n)$, $\rho_{s,n,\delta}$ depends on parameters $(x^j : j > n)$, we omit $\theta$ for $\theta = 0$.

Let $M$ fulfills 2.2, 2.4, $At(M)$ be finite and $(\phi_j \circ \phi_i^{-1} - id_{i,j}) \in E^s_{\gamma}(U_i,j,l_2)$ for each $U_i \cap U_j \neq \emptyset$, a metric $g$ (see 2.4 ) is of class $E^s_{\gamma}$, where $U_i,j$ are (open in $\mathbb{R}^n$) domains of $\phi_j \circ \phi_i^{-1}, l'(n) \geq l(n) + 2, \gamma'(n) \geq \gamma(n)$ for each $n, \infty > \chi > \delta$. Then we can define $\tilde{E}_{\gamma}(\tilde{M},M)$ as in 2.3 and $G' := Diff_{\beta,\gamma}(M) := \{f \in Diff_{\gamma}(M), (f)_{i,j} - id_{i,j} \}$ and $(f_i,j - id_{i,j}) \in E^s_{\gamma}(U_i,j,l_2)$ for each charts $\{U_i,\phi_i\}$ and $\{U_j,\phi_j\}$ with $U_i \cap U_j \neq \emptyset$ with the topology given by the following family of left-invariant metrics $\chi_{l,r,s,\delta}(f,g) := \chi_{l,r,s,\delta}(g^{-1}f, id)$,

\[(i) \quad \chi_{l,r,s,\delta}(f, id) := \sum_{i,j=1}^{k} d_{l,r,s,\delta}(f_{i,j}, g_{i,j}), \]

$g_{i,j}(U_{i,j}) = 0 \in l_2, f_{i,j} \in l_2, \phi_i(U_i) \subset l_2, U_{i,j} = U_{i,j}(x^{n+1},x^{n+2},...) \subset l_2$ is the domain of $f_{i,j}$ by $x^1,...,x^n$ for chosen $(x^j : j > n)$, $\rho_{s,n,\delta}$ are dependent on parameters $(x^j : j > n)$,
$U_{i,j} \subset \mathbb{R}^n \rightarrow l_2$, when $(x^j : j > n)$ are fixed and $U_{i,j}$ is the region in $\mathbb{R}^n$ by $(x^1, ..., x^n)$.

For countable infinite $At(M)$ spaces $E^{(l)}_{(r),\delta}$ and $D^{(l)}_{(r),\delta}$ are defined with the help of the strict inductive limits as in §2. Particularly, for a finite-dimensional manifold $M$ the group $D^{(l)}_{(r),\delta}(M^n)$ is isomorphic to $Diff_{\beta l}(M^n)$ with $l = l(n)$, $\gamma = \gamma(n)$, where $n = \dim_{\mathbb{R}}(M^n)$.

4.2. Lemma. Let $M'$ and $M$ be as in 4.1, then $G'$ is the topological group dense in $Diff_{(\beta l)}(M)$.

Proof. Let at first $At(M)$ be finite. The minimal algebraic group $gr(Q)$ generated by $Q$ is dense in $G'$ and $Diff_{(\beta l)}(M)$ due to the Stone-Weierstrass theorem, since $\cup_k M_k$ is dense in $M$ and there is $l \in \mathbb{N}$ such that $(n!)^l \geq n^{-\delta}$ for every $|\alpha| \leq l$ and $n$. It remains to verify that $G'$ is the topological group. Let $\chi_{3(l)},r,s,\delta(f, id) < 1/2$ and $\chi_{3(l),r,s,\delta}(g, id) < 1/2$ then $\chi_{3(l),r,s,\delta}(f \circ g^{-1}, id) \leq C_{r,s,l}(\chi_{3(l)},r,s,\delta(f, id) + \chi_{3(l),r,s,\delta}(g, id))$, where $C_{r,s,l}$ is a constant depending on $r,s,l$ and independent of $f$ and $g$, since for the Bell polynomials there are inequalities $Y_n(1, ..., 1) < n! e^n$ for each $n$ and $Y_n(F_1, ..., F_(n+1)) < (2n)! e^n$ for $F_{p} := F_{p} = \prod_{j=1}^{n+p}(n+1)$ (see ch.5 in [?], theorem 2.5 in [?]). Then the case of infinite $At(M)$ may be considered analogously to the proof of theorem 3.1.

4.3. Definition. Let $\{l\} := \{l(n) : n \in \mathbb{N}\} \subset \mathbb{Z}$ and $\{\gamma\} := \{\gamma(n) : n \in \mathbb{N}\} \subset \mathbb{R}$ be two sequences with $\{M_k : k = k(n), n \in \mathbb{N}\}$ be as in 3.2 and 4.1 at first with finite $At(M)$. Now let $H^{(l)}_{(r),\delta}(M,TM)$ denotes the completion relative to the norm defined below $\|\zeta\|_{(l),\gamma,\delta}$ of a linear span $sp_{\mathbb{R}}K$ over $\mathbb{R}$ of $K := \{\zeta \in E^\infty_{((\gamma,\delta))}(M,TM) :$ there exists $1 \leq j \leq k$ such that $supp(\zeta) \subset U_j$ and $\zeta\vert_{M_k \cap TM} \in H_{(r)}^{(l)}(M_k,TM)$ for each $n \in \mathbb{N}, \|\zeta\|_{(l),\gamma,\delta} < \infty, \lim_{R \rightarrow \infty} \|\zeta\|_{(l),\gamma,\delta} = 0\},$ where

\[(i) \quad \|\zeta\|_{(l),\gamma,\delta} = \left[ \sum_{n=1}^{\infty} \omega_n \sup_{x \in M} \|\zeta\|_{(n)} \right],\]

$(x^i : i \in \mathbb{N})$ are the local coordinates in $M$, $(x^1, ..., x^k)$ correspond to that of $M_k$, $\| \ast \|_{H^{(l)}_{(r),\delta}(M_k,TM)}$ are taken by $(x^1, ..., x^k)$ and are functions by $(x^{n+1}, x^{n+2}, ...)$.

$H^{(l)}_{(r),\delta}(M_k,TM) := \{\zeta : M_k \rightarrow TM \pi(\zeta(x)) = x \text{ for each } x \in M_k, \zeta \in H_{(r)}^{(l)}(M_k,TM)\}$ (see 2.5), $\|\zeta\|_{k,(n)} = (\sum_{i,j,n \leq k} q_{i,j,(n)}(\zeta_i,j,n), \xi_i,j), \nabla \zeta_i,j) \right)^{1/2}$; $q_{i,j,(n)} = \sum_{\alpha \neq 0} \|\nabla^{\alpha} \zeta\|_{(\alpha)} < x_{(n) \gamma,\delta}^{\alpha} \begin{cases} \frac{1}{\sqrt{2}} \text{ for } n > 1 \end{cases}$ analogously to 2.5 for $l > 0$ and $\gamma \in \mathbb{R}$, but with $\alpha^n \neq 0$, $\|\zeta\|_{(l),\delta} := \|\zeta\|_{H^{(l)}_{(r),\delta}(M_k,TM)} \text{ for } l \leq 0$ we take $\sup\|\zeta\|_{\tau} = 1 < x_{(n) \gamma,\delta}^{\alpha} \left( D^2_{\tau,j,k} \zeta_i,j - \xi_i,j \right) \right) ||_{(u_{i,j,n},\tau,\delta)} \text{ instead of } \| x_{(n) \gamma,\delta}^{\alpha} \right) \begin{cases} \text{ for } \tau \in H_{(r)}^{(l)}(M_k,TM), \text{ for } x > x_{(n) \gamma,\delta}^{\alpha} \end{cases}$
4.4. Lemma. Let \( M \) fulfil 2.2, 2.4, \( \text{At}(M) \) be finite \( E^\theta_{1}(TM) \) as in 2.3, \( H^\theta_{(1)}(TM) \) be as in 4.3 with \( l(n) = \lceil t \rceil + \lfloor n/2 \rfloor + 3 + \text{sign}(t) \), \( \gamma(n) = \beta + \lfloor n/2 \rfloor + 7/2 + \text{sign}(t) \).

Then there exist constants \( C > 0 \) and \( C_n > 0 \) such that \( \| \zeta \|_{E^\theta_{1}(TM)} \leq C \| \xi \|_{H^\theta_{(1)}(TM)} \) for each \( \zeta \in H^\theta_{(1)}(TM) \), \( \omega_n = k(n)^2C_n \), \( \| \xi \|_{c^\theta_{(1)}(TM)} \leq C_n \| H^\theta_{(1)}(TM) \), for \( k = k(n) \),

\[
l'(k) = \lceil t \rceil + 2 + \text{sign}(t), \gamma'(k) = \beta + 5/2 + \text{sign}(t), \zeta \in H^\theta_{(1)}(TMk), \text{ where sign}(\epsilon) = 1 \text{ for } \epsilon > 0, \text{sign}(0) = 0, \{ t \} = t - \lceil t \rceil \geq 0.
\]

Proof. In view of theorems in [?] and \( f_{(Rn)} \) there are embeddings \( H^\theta_{(n)}(TM) \rightarrow H^\theta_{(n)}(TMn) \), since \( 2(\lfloor n/2 \rfloor + 1) \geq n + 1 \). Moreover, there exist constants \( C_n > 0 \) for each \( k = k(n), n \in \mathbb{N} \) such that \( \| \xi \|_{c^\theta_{(1)}(TM)} \leq C_n \| H^\theta_{(n)}(TMk) \) for each \( \zeta \in H^\theta_{(1)}(TMk) \). Then \( D^\alpha f(y^n, \ldots, y^n, x^n, x^{n+1}, \ldots) = \sum_{n=0}^{\infty} (D^\alpha f(y^n, \ldots, y^n, x^n, x^{n+1}, \ldots) - D^\alpha f(y^n, \ldots, y^n, x^n, x^{n+1}, \ldots)) f \in H^\theta_{(1)}(TM) \) in local coordinates, where \( f(y^n, \ldots, y^n, x^n, x^{n+1}, \ldots) = f(x)^{\alpha_1, \ldots, x^n, x^{n+1}, \ldots} \) if \( n = 0, \alpha = (\alpha_1, \ldots, \alpha_m) \), \( m \in \mathbb{N}, \alpha^1 \in \mathbb{N}_0 \). Therefore, for \( x^n < x^h \) we have: \( D^\alpha f(y^n, \ldots, y^n, x^n, x^{n+1}, \ldots) )_{l_i, \delta} \leq \sum_{i=1}^{s} \int f_{\beta_j}(U_j \cap M_k) \quad \exists z = (y^n, \ldots, y^n, x^n, x^{n+1}, \ldots) \leq \sum_{n=0}^{\infty} (D^\alpha f(y^n, \ldots, y^n, x^n, x^{n+1}, \ldots) )_{l_i, \delta} \leq \sum_{i=1}^{s} \int f_{\beta_j}(U_j \cap M_k) \quad \exists z = (y^n, \ldots, y^n, x^n, x^{n+1}, \ldots) \)

4.5. Theorem. Let \( M \) be a Hilbert manifold and \( g \) be a Riccimannian metric fulfilling 2.2, 2.4 and 4.1, \( l \in \mathbb{N} \) or \( l = \infty \). Then for each \( \text{Diff} f_{\beta, \omega}(M) = G \) with \( 0 \leq t < \infty \) and \( \infty > \beta \geq 0 \), \( \infty > \delta \geq 0 \), there exists a probability measure \( \mu \) on \( \text{Diff}(G) \) and a dense subgroup \( G' \subset G \) such that \( \mu \) is quasi-invariant relative to \( G' \) and \( l \) times differentiable relative to \( G'' \).

The proof of this theorem is divided into several parts.

4.6. Definition. Let \( (M, g) \) fulfil 2.2, 2.4 and 4.1. \( 1 \leq q \leq t, 0 \leq \gamma \leq \beta \). For \( f \in G \) we can define \( f_{\gamma, \beta}(M) = G \) with \( 0 \leq t < \infty \) and \( \infty > \beta \geq 0 \), \( \infty > \delta \geq 0 \) there exists a probability measure \( \mu \) on \( \text{Diff}(G) \) and a dense subgroup \( G' \subset G \) such that \( \mu \) is quasi-invariant relative to \( G' \) and \( l \) times differentiable relative to \( G'' \).

The proof of this theorem is divided into several parts.
(see 3.9(iv) in [?], $T^r_s(l_2)$ is the tensor space of type $(r, s)$ over $l_2$, $T^r_s(M)$ is the tensor bundle of type $(r, s)$ over $M$, $T^r_s(M)$ is the tensor bundle of type $(r, s)$ over $M$, $T^r_s(M)$ corresponds to $(r, s) = (1, 0)$ [?], $(df)_\xi := f_* \xi := D\xi f$, $(D\xi)\langle \xi( x) \rangle := D\xi(\xi(x))$, $\nabla_{\partial^r}(X_1, ..., X_s) = (\nabla^r)(X_1, ..., X_s; \xi)$, $X_j(x) \in T^r_{xM}$, $X_j \in \Xi(M)$, $j = 1, ..., s \in \mathbb{N}$, $r \in \mathbb{N}$, $(df)^{-1}\langle \xi(x) \rangle := (df|^{r-1}(\xi(x)))^{-1}(x) \in T^r_{f^{-1}(\xi(x))\big \vert M}$; $df$ and $\nabla^m df$ are well defined for $f \in G$ analogously to §4 [?]. Indeed, the differential $f_* = df$ is a section of $T^*M \otimes f^*TM$ (with the induced connection in $f^*TM$).

4.7. Lemma. In the notation of 4.6 $D\xi \xi( x) \in \bigwedge^{r-1}_{s+1, \delta}(M, T^r_s(M))$ and $D\xi$, $(df)^{-1}$ are continuous mappings of $\bigwedge^{r-1}_{s+1, \delta}(M, T^r_s(M))$ into $\bigwedge^{r-1}_{s+1, \delta}(M, T^r_s(M))$.

Proof. Let $\{M_k : k = k(n), n \in \mathbb{N}\}$ be a sequence of submanifolds as in 3.2 with atlases $At(M_k) = [(U_j, k, \phi_j) : j] = At(M)$ for each $j, k$. If $h \in T^r_s(l_2)$ then $\|h\|$ is given analoguously to $\|Q\|$ in 3.2. Then $\xi^\nabla$ in 3.3 corresponds to $f^\nabla$ for $M_k$ in 3.2. From definitions in §2 and 4.6, lemmas 3.2 and 3.3, theorems 3.1 and 3.4 it follows that $D\xi$ and $(df)^{-1}$ are continuous, since $f_* \xi \in \bigwedge^1_{s+1, \delta}(M, TM)$ and $(df)^{-1}$ corresponds to $f^*_{\xi^{-1}}$, $t \geq 1$.

4.8. Lemma. Let $D\xi$ be as in 4.6, $f$ and $\phi \in Diff^\beta_{\delta}(M)$ as in 4.5. Then

\[
\sum_{\sigma \in S_l} \left( \sum_{\omega(l)} D\xi_{\sigma(1)} ... D\xi_{\sigma(l)}(\phi \circ f) = \sum_{\omega(l)} \left[ \prod_{i,j} f_i \right]_{\omega(l)} \right)
\]

\[
\times \left( \prod_{i,j} f_i \right)_{\omega(l)} \sum_{\omega(l)} S \left( D\xi_{\sigma(1)} ... D\xi_{\sigma(l)} f, ..., D\xi_{\sigma(l)} f, D\xi_{\sigma(l+1)} f \right)
\]

\[
\times \left( \prod_{i,j} f_i \right)_{\omega(l)} \sum_{\omega(l)} S \left( D\xi_{\sigma(1)} ... D\xi_{\sigma(l)} f, ..., D\xi_{\sigma(l)} f, D\xi_{\sigma(l+1)} f \right)
\]

\[
\nabla_{\phi, f, \xi}^{\nabla, f, \xi} \nabla_{\phi, f, \xi}^{\nabla, f, \xi} \xi_1 f \text{ for } l > 1, D\xi \phi \circ f = \phi_0, f \xi, S_1 \text{ is the symmetric group of } \{1, ..., l\}
\]

\[
\text{elements of which are considered as bijective mappings } \sigma : \{1, ..., l\} \rightarrow \{1, ..., l\}, \ S \text{ is the symmetrizer of } (D^{(1+ ... +i_m)}(a, ..., z)) \text{ by all arguments } (a, ..., z) = (D\xi_{\sigma(1)} ... D\xi_{\sigma(l)} f, ..., D\xi_{\sigma(l)} f), \ S g(a_1, ..., a_p) := \sum_{\sigma \in S_l} g(a_{\sigma(1)}, ..., a_{\sigma(p)}) \text{ for a function } g \text{ of arguments } a_1, ..., a_p; \ S_{\omega(l)} \text{ denotes the summation by all partitions } \omega(l) \text{ of } l, \text{ that is by all representations of } l \text{ as } l_1 l_2 ... l_m, \ l_1 \geq l_2 \geq ... \geq l_m > 0.
\]

Proof. For $f$ and $\phi \in Diff^\beta_{\delta}(M)$ and $[\ell] \geq l$ in view of theorem 2.5 in [?], §5.1-5.3 in [?] and improving lemma 1 in [?] we have the equality (4.1) due to lemma 4.3.

4.9. Definition. Let $(M, g)$ and $D\xi$ be as in 4.6, let us denote with the help of 4.8 the following expression

\[
\left( D_{f, \xi_1} ... \xi_l \phi(\xi) := \sum_{\omega(l)} \prod_{i,j} f_i \right)_{\omega(l)} \sum_{\omega(l)} S \left( D\xi_{\sigma(1)} ... D\xi_{\sigma(l)} f, ..., D\xi_{\sigma(l)} f, D\xi_{\sigma(l+1)} f \right), \ \text{where } \xi \in \bigwedge_{s+1, \delta}(M, T^r_s(M)), [\ell] \geq l, \beta \geq \gamma \geq 0.
\]

4.10. Lemma. Let $(M, g)$ and $\{M_k : k = k(n), n \in \mathbb{N}\}$ be as in 2.2, 2.4 and 4.1 with each atlas $At(M_k)$ inherited from $At(M)$. Then there exists the locally finite partition of unity $\{\psi_i : i \in J\}$, $J \subset \mathbb{N}$, for $M$ such that
(i) \( V_i \subseteq \text{supp}(\psi_i) \subseteq U_{p(i)}, V_i \) are open, \( \phi_{p(i)}(V_i) \) are locally convex, \( p = p(i) \in \{i, ..., s\} \), \( \bigcup_{i \in J} V_i = M \);
(ii) vector fields \( [\xi_{l,i} : i \in \mathbb{N}] \) are in \( \Xi(M) \) of class \( E^{(l')}_{[\gamma']}, \text{supp}(\xi_{l,i}) \subseteq \text{supp}(\psi_l) \), \( \xi_{l,i} \in \Xi(M_i) \) for each \( i \);
(iii) \( [\xi_{l,i}(x) : i \in \mathbb{N}] \) is a linear basis in \( T_x M \) for each \( x \in V_l \);
(iv) for each \( l \in J \) there exists \( x \in V_l \) with \( \xi_{l,i}(x) = e_i \) for every \( i \), where \( e_i \) is the standard basis in \( l^2 \);
(v) \( 1/2 \leq \inf_{x \in V_l} \|\xi_{l,i}\|_{E^{(l')}_{[\gamma']}((TV_l))} \leq \inf_{x \in V_l} \|\xi_{l,i}\|_{E^{(l')}_{[\gamma']}((TV_l))} \leq 2 \) and \( \sup_{i \neq j, l \in J, x \in V_l} |\langle \xi_{l,i}(x), \xi_{l,j}(x) \rangle|_{l^2} < 1/2 \) for some \( t' > 1, \beta' \geq \gamma'(k(1)) \).

Proof. The manifold \((M, g)\) is Riemannian and modelled on \( l^2 \), hence it possesses the partition of unity \( \{\epsilon_l : l \in J\}, J \subseteq \mathbb{N} \) of class \( E^{(l')}_{[\gamma']} \), fulfilling (i) due to §2.3 in [?], that is, \( \sum_{l \in J} \epsilon_l(x) = 1 \) for each \( x \in M \), \( \epsilon_l(x) \geq 0 \) for each \( l \) and \( x \), \( \text{supp}(\epsilon_l) = \{x \in M : \epsilon_l(x) \neq 0\} \subseteq U_{p(l)}, cl(B) \) denotes the closure of \( B \subseteq M \).
Let \( \{M_k : k = k(n), n \in \mathbb{N}\} \) be as in 3.2 then \( \exp \) for \( M \) induces \( \exp \) for \( M_k \) as restrictions on corresponding neighbourhoods of the zero sections in \( T M_k \). Therefore, the Gaussian coordinates in \( M \) induce corresponding coordinates in \( M_k \), since each \( M_k(n) \) has tubular neighbourhoods in \( M_k(n+1) \) (for \( j > 0 \)) and in \( M \) (§4.4-4.6 [?]).

4.11. Definition. Let \( \{\xi_{l,i} : l, i\} \) be the same as in 4.10 \( f \in Diff^{(p)}_{\beta, \delta}(M), q > \deg(A_{n;m}(n)) \). Let us define operators:

\[
(4.4) A_{n;m}(n)(f(x)) = \sum_{l \in J} \psi_l(x) A_{n;m}(n) f(\xi_{l,1}, ..., \xi_{l,n}),
\]

where \( A_{n;m}(n), f(\xi_{l,1}, ..., \xi_{l,n}) := \sum_{p=1}^{n} F_{s,n;m}(n) (g_{s,n}^{(1)}(j(1)), ..., g_{s,n}^{(a)}(j(a)) ; D_{\xi_{l,1}}^{(2)}, ..., D_{\xi_{l,s}}^{(2)(s-1)} ; D_{\xi_{l,s+1}}^{(2)(s+1)} ; ..., D_{\xi_{l,n}}^{(2)(n)} \sum_{p=0}^{2(n)-1} \alpha(p, 2(1)) D_{\xi_{l,n}}^{(p)} ([df]^{-1} \circ D_{\xi_{l,n}}^{(2)(p)} f) \) with \( s := \min_{i \geq 2(m(n))} j, \deg(A_{n;m}(n)) \) is a degree of \( A_{n;m}(n) \) as the differential operator, \( n \) and \( m(n) \in \mathbb{N}, \alpha(p, j) \in \mathbb{Q}, a = i(1) + ... + i(n) = 2(m(n))n, g_{s,n}^{(a)} \) are components of \( g_{s,n}^{(a)} \) on \( M_n \), the Riemannian metric \( g_{s,n} \) on \( M_n \) is induced by \( g_s \) on \( M \) for each \( x \in M_n \), \( (g_{s,n})^{ij} = g_{s,n}(\partial/\partial x^i, \partial/\partial x^j) \), \( F_{s,n,m}(n) \) are operators of polynomial types by \( g_{s,n}^{(a)} \) and \( D_{\xi_{l,p}} \).
4.12. Lemma. Let the operator $A_{n;m}(f)$ and $f$ be the same as in 4.11, $\mathcal{M}(f)$ be finite. Then there exists $F_{s,n,m}(n)$ and $\alpha(p,j)$ such that $F_{s,n,m}(n)$ is continuously Fréchet differentiable by $V$ mapping from $Y \otimes (E_{\beta,\delta}(\mathcal{M}(f)))^{\otimes 4(m(n))}$ into $E_{\beta,\delta}(\mathcal{M}(f)))^{\otimes 4(m(n))}$ for $q \geq 4m(n)$, $\nabla V A_{n;m}(n) = \Delta_{n;m}(n)$, $\nabla V A_{n;m}(n) = \Delta_{n;m}(n)$, and $\alpha(p,j) = \Delta_{n;m}(n) \in E_{\beta,\delta}(\mathcal{M}(f)))^{\otimes 4(m(n))}$ for $q > 4m(n)$, $\phi = K_{2m(n)} + Q_{n}: E_{\beta,\delta}(\mathcal{M}(f)))^{\otimes 4(m(n))}$.

Proof. For the submanifold $M_n$ in $M$ with the covariant differentiation $\nabla$ the torsion tensor $T_{ij}^k = 0$, $n \nabla x g_{ij} = n \nabla x (g_{ij,n,i,j} + 0$, where $X \in X(M_n)$. In view of proposition III.7.6 [?] the curvature tensor field is given by the equation $R_{ijkl} = \partial \partial_{jk}^k - \partial \partial_{jl}^l + \partial \partial_{ij}^i + \partial \partial_{jl}^l - \partial \partial_{ik}^i + \partial \partial_{jl}^l - \partial \partial_{ik}^i$, where $g = g_{ij,n}$ (see corollary IV.2.4 [?] and 4.7), or $R(X,Y,Z)_{\phi(p)} = D\Gamma_{\phi(p)}(X,\phi(p)) - D\Gamma_{\phi(p)}^{\otimes 4(m(n))}(X,\phi(p)) + \Gamma_{\phi(p)}(X,\phi(p)) - \Gamma_{\phi(p)}(X,\phi(p))$ in local coordinates in infinite-dimensional case ($\S 8.3$ [?]). Consequently, the Riemannian connection $\Gamma$ in $M$ and $\mathcal{R}$ are in the class $E_{\beta,\delta}(\mathcal{M}(f)))^{\otimes 4(m(n))}$, $\phi \in \text{Diff}^\beta,\delta_q(M)$, supp$(\phi) := \text{cl}(x \in M : \phi(x) \neq 0) \subset \phi^{-1}(\phi(U_j) \cap \mathbb{R}^n)$ for some $j \in \{1,...,s\}$, $W$ is some open neighbourhood of id in $\text{Diff}^\beta,\delta_q(M)$, $\phi \in W$, $f \in W$, $\mathcal{R}(Y) = W$, $Y$ is an open neighbourhood of $0$ in $\text{Diff}^\beta,\delta_q(M)$ (see 3.4).

There are constant coefficients $\alpha(j,u)$ fulfilling the following system of linear algebraic equations $\sum_{u=0}^d \binom{d}{u} \alpha(j,u) = 0$ with $d = 0, ..., w - 1, w = 1, ..., \min(2m(n) - 1, u) = p$ and $\sum_{j=1}^q \alpha(j,u) = 1$, since this system is equivalent to $\sum_{j=1}^p \binom{d}{u} \alpha(j,u) = 1$ for $k = 0, 1, ..., p$, where $u > p$, det$\{\binom{a-k}{j-k}\}_{j,k} \neq 0, \binom{a}{n} = 0$ for $a < d$ or $d < 0$, $\binom{a}{d} = 0$ for $d = 0, ..., a$ are the binomial coefficients.

Using the following facts: (i) the equality $[\nabla^p,\nabla^q] = \sum_{a=0}^{p-1} \nabla^a [\nabla^i,\nabla^j] \nabla^{p-a-1}$ for $p = 2,3,..., v^0 := I$ for infinitely differentiable vector fields; and then (ii) for the corresponding to $\nabla$ pseudo-differential operators with additional terms belonging to $S^{-\infty}$ with the well-known rules for their compositions [?]; (iii) the coefficients $\alpha(j,u)$ as above; (iv) smoothness of $\Gamma$ and $R$; (v) the expression of the Beltrami-Laplace operator in normal coordinates for the Levi-Civita connection in $M_k$: $\Delta_k = (g_{ij,k})^{i,j}(\mathbb{R})$ (see note 14 in v.2 [?]); (vi) lemmas 3.2, 4.3, 4.4 and 4.6 - we can find polynomials $F_{s,n,m(n)}$ by $D_{\xi,l}$ with coefficients depending on $x$ as functions in $E_{\infty,\lambda}(M_k, \mathbb{R})$ such that to fulfil demands of 4.12, since $B_{p,i}(x)$ are polynomials of $D_{\xi}^j f$ with $j = 1,..., i-2m(n)$ and $(D_{\xi}^j \phi)(f(x))$ with
1 \leq j \leq i. The differentiability by \( \hat{V} \) follows from the existence of \( E_{(\gamma)}^{(\nu)} \subset E_{\nu}^{\infty} \) mapping of some neighbourhood \( Y \) of 0 in the Banach space \( TDiff f_{\beta,\delta}^{\alpha}(M) \) onto a neighbourhood \( W \) of \( id \) in \( Diff f_{\beta,\delta}^{\alpha}(M) \).

Indeed, there is a neighbourhood \( W \) of \( id \in Diff f_{\beta,\delta}^{\alpha}(M) \) such that \( W^2 \subset U \) and it is given with the help of \( \hat{E} \) in 3.4(v) (analogously for the class of the smoothness of \( M \) considered here). Hence the differentiability by \( f \in W \) can be reduced to the differentiability of \( A_{n,m(n)} \) by \( \hat{X} \in Y \).

**4.13. Definition.** Let \( \psi \in G' \) (see 4.1) and \( A_{n,m(n)} \) be chosen as in 4.11 together with \( \{\xi_{\gamma} \} \) and constants \( B_n > 0 \). We denote\( A(\psi) := \sum_{n=1}^{\infty} B_n A_{k(n);m(k(n))}(\psi)e'_n \in \bigoplus_{n=1}^{\infty}(H_{(\gamma)}^{(\nu)}(TM) \otimes e'_n) \) and give below conditions when \( A \) is well defined in a neighbourhood of \( id \), where \( \{e'_n : n \in \mathbb{N}\} \) are linearly independent vectors.

**4.14. Lemma.** Let \( A \) and \( \psi \) be as in 4.13 and \( At(M) \) be finite. Then there are the Banach space \( \hat{H} \), neighbourhoods of \( W \ni id \) in \( W \subset U \subset Diff f_{\beta,\delta}^{\alpha}(M) \) with the topology induced from \( H_{(\gamma)}^{(\nu)}(TM) \), \( \hat{E}(W_0) =: U \), \( W_0 \) is some open neighbourhood of 0 in \( H_{(\gamma)}^{(\nu)}(TM) \), \( W \) is open in \( U \), \( V \ni 0 \) in \( \hat{H} \) and \( \{\xi_{\gamma} \} \) such that \( A : W \to V \) is a uniform isomorphism, where \( inf - \lim_{n \to \infty} m(n)/n = c > 1 \) and \( N \ni 2m(n) > n \) for each \( n \).

**Proof.** In view of the results in [?, ?, ?, ?, ?] and lemma 4.12 \( A'_{n,m(n)} : H_{(\gamma)}^{(\nu)}(M_n TM) \to H_{(\gamma)}^{(\nu)}(M_n TM) \) are the linear uniform isomorphisms for each \( n \in \mathbb{N} \), where \( A'_{n,m(n)} := \nabla_{\hat{V}} A_{n,m(n)}(\hat{E}(V)) \big|_{V=0} \), for \( 4m(n)n > l(n) \) are considered their continuous extensions as pseudo-differential operators defined modulo terms in the class \( S^1_{\nu,0} \). Let \( B_n \) be such that \( \|B_n A_{k(n);m(k(n))}\| = 1 \) for each \( n \). Since \( TG' \subset H_{(\gamma)}^{(\nu)}(M, TM) \subset E_{\nu,\delta}(M, TM) \) we can consider the following restriction \( \hat{E}|H_{(\gamma)}^{(\nu)}(M, TM) \). Then there exists continuous \( \nabla_{\hat{V}} A(\hat{E}(V)) = \sum_{n=1}^{\infty} e'_n \times B_n \nabla_{\hat{V}} A_{k(n);m(k(n))}(\hat{E}(V)) \) in some neighbourhood of 0 in \( H_{(\gamma)}^{(\nu)}(M, TM) \) such that \( A' = \sum_{n} e'_n B_n A_{k(n);m(k(n))} \).

Now let \( H := \{\xi = \{e'_n B_n A_{k(n);m(k(n))} \xi : n \in \mathbb{N}\} \in \mathbb{N} \} \) \( \xi \in H_{(\gamma)}^{(\nu)}(TM) \) be a linear space with a norm \( \|\xi\|_H := \sum_{n} \omega_n \|B_n A_{k(n);m(k(n))}\|_k(n) \) where \( \|\*\|_k(n) \) are defined as in 4.3 \( \|\*\|_k(n) \) with \( \nu(n) = l(n) - 4m(n)n \) and \( \gamma(n) = \gamma(n) + 4m(n)n \) instead of \( l(n) \) and \( \gamma(n) \) respectively for each \( n \). There are \( \{\nu_j(n) : n \} \) and \( \{\gamma_j(n) : n \} \) such that \( A_{j,m(n)} H_{(\gamma)}^{(\nu)}(TM) \subset H_{(\gamma_j)}^{(\nu_j)}(TM) \). We consider \( e'_n \) as linearly independent vectors in \( \bigoplus_{(\gamma_j)}^{(\nu_j)}(TM) \), \( e'_j \in H_{(\gamma_j)}^{(\nu_j)}(TM) \).

Therefore, a set \( K := \{\xi \in E_{\nu,\delta}(TM) : \) there are \( 1 \leq j \leq s \) such that \( supp(\xi) \subset U_j \) and \( \xi|_{U_j} A_{\xi} M_j \in H_{(\gamma_j)}^{(\nu_j)}(M_n TM) \) for each \( n = k(p), p \in \mathbb{N} \), \( \|\*\|_H < \infty, \lim_{R \to \infty} \|\*|_{M_R}\|_H = 0 \) is dense both in \( H \) and \( H_{(\gamma)}^{(\nu)}(TM) \), where \( s \) is a number of charts in \( At(M) \). From the aforementioned isomorphisms and the Banach theorem about the inverse operator it follows that \( A' : H_{(\gamma)}^{(\nu)}(TM) \to H \) is the linear uniform isomorphism and \( \hat{H} = H \) is complete relative to \( \|\*\|_H \). Hence, \( A(\hat{E}(V)) \) is continuously differentiable operator in some neighbourhood \( W_0 \) of 0 in \( H_{(\gamma)}^{(\nu)}(TM) \) by \( \hat{V} \in W_0 \), since \( T_e G' \subset H_{(\gamma)}^{(\nu)}(TM) \subset T_e Diff f_{\beta,\delta}^{\alpha}(M) \) (see 3.4 and 4.12).

**4.15. Lemma.** Let the conditions of lemma 4.14 be satisfied. Then there exists a
neighbourhood $P$ of id in $G'$ with $2m(n) > n$ for each $n$, $\inf n \to \lim \frac{m(n)}{n} = c > 1$, such that $S_\phi(\hat{V}) : P \times V \to Y$ is uniformly continuous (by $(\phi, \hat{V}) \in P \times W_0$) differentiable by $\hat{V} \in V$ mapping, $Y$ is a Banach space with an embedding operator of trace class $J : H \to Y$, where $S_\phi(\hat{V}) := A[\phi(A^{-1}(\hat{V}))] - \hat{V}$.

Proof. Let $Y$ be a Banach space of the same type as $H$ in 4.14, but with $l''(n) = l(n) - 4m(n)n + 2m(n)$ and $\gamma''(n) = \gamma(n) + 4m(n)n - 2m(n)$ instead of $l'(n)$ and $\gamma'(n)$ for each $n$. Hence, $Y \supset H$ and the natural embedding $J$ is of trace class (or nuclear, see §III.7 in [?]). Indeed, for the finite atlas $At(M)$ and each chart $(U_i, \phi_i)$ we can consider linearly independent cylinder functions $x^m e_n < x >^s / m! = f(x)$, where $\{e_n : n \in \mathbb{N}\} \subset l_2$ is the standard orthonormal base, $x^m := x_1^{m_1} \cdots x_s^{m_s}$, $m! = m_1! \cdots m_s!$, $< x >^s = (1 + \sum_{i=1}^s (x_i^2)^{1/2}$, $s \in \mathbb{N}$, $e(s) = e_R$. The linear span over $\mathbb{R}$ of such $f(x)$ is dense in $H$ and $Y$. Then $D^a f(x) = e_n \sum \binom{a}{b} (D^a x^m/m!) (D^a x^n < x >^s / m!) - (\delta + 1) / 4! = c_s$ exists for each $l$, $c_s < \infty$, $\lim_{n \to \infty} q^n / \sqrt{n} = 0$ for each $\infty > q > 0$, $\sum_{s=1}^\infty \sum_{m|\geq 2m(s), m} [m! / s^s] < \infty$, since $\sum_{s=1}^\infty [2m(s)/s]^{-s} < \infty$ and $n! = n^a e^{-n} e^{\theta(a)(2\pi n)^{1/2}}$, $|\theta(n)| < 1/(12n)$ due to the Stirling formula, here $m = (m_1, \ldots, m_s)$, $|m| := m_1 + \ldots + m_s$, $\epsilon_i \in \mathbb{Z}$.

Proof of theorem 4.5. Let at first $At(M)$ be finite, $t \geq 1$, $H = H$ and $Y$ be as in lemmas 4.14 and 4.15, then they are separable. For $G = Diff^{t,\delta}(M)$ let $G' = Diff^{t,\delta}(\nu)$ with $\infty > \zeta > 2(\delta + 1)$ (see definition 4.1). In view of theorem I.4.4 [?] there exists a separable Hilbert space $Z$ over $\mathbb{R}$ such that $\theta : Z \hookrightarrow H$, $Z$ is dense in $H$, $\theta$ is the inclusion mapping and $J \circ \theta : Z \hookrightarrow Y$ is of trace class. From lemmas 4.2 and 4.4 it follows that $G'$ acts uniformly continuous from the left on $W \subset U$. There is a neighbourhood $P$ of id in $G'$ such that $PW \subset U$. Then conditions of theorem 26.2 in [?] are satisfied for the operators on $Z$ induced by $I + S_\phi$ for each $\phi \in P$ and a countably additive Gaussian measure $\nu$ on $Z$ with a correlation operator $B$ and a zero mean value induces a measure $\mu$ on $Bf(W)$, $\mu(Q) := \nu(A(Q))$ for each $Q \in Bf(W)$. The space $GW = G_0$ is in the uniformity induced from $H^{(t)}(\nu), \delta(M, TM)$, $G_0$ is separable, Lindelöf and paracompact, consequently, there exists some locally finite open covering $\{g_i W(i) : i \in \mathbb{N}\}$ of $G_0$ with $g_i \in G'$ and $W(i) \subset W$, $W(0) := W$, so $\mu(Q) := \sum_{i=0}^\infty 2^{-j} \mu((g^{-1}_i Q) \cap W(i))$ is countably additive and quasi-invariant relative to $G'$ on $Bf(G_0)$.

Since $G' \subset G_0$ is dense in $Diff^{t,\delta}(M)$ the measure $\mu$ induces the measure $\mu$ such that $\mu(Q) = \tilde{\mu}(\tilde{Q})$, where $\tilde{Q} := \{x \in G_0 : (h_1(x), \ldots, h_s(x)) \in H, Q := \{x \in Diff^{t,\delta}(M) : (h_1(x), \ldots, h_s(x)) \in R, R \in Bf(R^s), h_i \in K := \{h : Diff^{t,\delta}(M) \to R, h$ are continuous\} (R is with the standard uniformity). Indeed, the minimal $\sigma$-fields over $G_0$ and $Diff^{t,\delta}(M)$ generated by such $\tilde{Q}$ and $Q$ coincide with $Bf(G_0)$ and $Bf(Diff^{t,\delta}(M))$ respectively. If $Q_1 \cap Q_2 = \emptyset$ for such $Q_1$, then $\tilde{Q}_1 \cap \tilde{Q}_2 = \emptyset$, hence $\mu$ is additive and has $\sigma$-additive extension on $Bf(Diff^{t,\delta}(M))$, since the uniformity in $G_0$ is stronger than in $Diff^{t,\delta}(M)$. 18
\[
\frac{\partial (AL_n(\tau ; *)h)A^{-1}}{\partial \tau} = AV \circ \eta(\tau ; *)hA^{-1} \quad \text{for} \quad h \in G' \quad \text{and} \quad \eta(\tau ; *) \in J',
\]
where \( V \) is corresponding to \( \eta \) vector field, \( J' = Exp(Q_0) \) (see 3.4(1)), \( Q_0 \subset F_0 \cap TG' \), \( F_0 \) is a neighbourhood of 0 in \( TDiff_{f,n}^b(M) \) as in theorem 3.4(v), \( L_\phi f := \phi \circ f \) in \( Diff_{f,n}^b(M) =: G \).

Practically we have constructed just above \( G' \) dense in \( G \) to fulfil definition 2.6 and then have denoted it also \( G' \).

Indeed, without loss of generality we may suppose that \( G' \supset G'' \), where \( G'' \) is the minimal group generated by \( J^1 \) and some countable family of \( \{ g_n : n \in \mathbb{N} \} \subset C \text{Diff}_{f,n}^b(M) \) as in theorem 3.4(v), \( L_\phi f := \phi \circ f \) in \( Diff_{f,n}^b(M) =: G \). Now let \( 0 < t(1) < 1 \leq t \) and \( \{ z(i) : i = 1, 2, \ldots \} \) be dense in \( M \). This is possible, since \( M \) is separable. We may define in accordance with lemma 4.14 the following subsets of \( W \) and \( W(1) \subset Diff_{f,n}^b(M) =: G(1), W(1) \cap G =: W, W(k, t(1), c; f) := \{ g \in W(1) : \rho(k; g, f) \leq c \}, W(k, t, c; f) := \{ g \in W : \rho(k; g, f) \leq c \}, \) where \( \infty > c > 0, k \in \mathbb{N}, f \in W, \) the mappings \( \rho(k, k'; g, f) := \sum a_{b}(\| \tilde{\sigma}(x) \|^{1/\beta})^{\beta} \| g - \phi^{-1} \|^{\beta} \) for each \( k < k' \) are continuous on \( G(1) \). Then we put \( W(k, t(1), c; f) := W(k, t, c; f) \) for each \( t(1) < 1, \tilde{\sigma}(x) := \min(\sigma(x), \sigma(y)) \) for a pair \((x, y), h_{ab} = \phi \circ h \circ \phi_{b}^{-1} \) as in \( \text{§2}, \) so

Then we put \( \mu(W(k, t(1), c; f)) := \mu(W(k, t, c; f)) \) for each \( c > 0, f \in W, k \in \mathbb{N}, \) whence \( \mu_1 \) is finitely additive, since from \( E(1) \cap L(1) = \emptyset \) in \( G(1) \) it follows \( E \cap M = 0 \) in \( G \), where \( E(1) \) and \( L(1) \) are in \( V(1), E \) and \( L \) are corresponding sets in \( V := \{ W(k, t, c; f) : c > 0, k \in \mathbb{N}, f \in W \} \). From the definition of \( \mu \) follows that
\[
\mu(\bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} W(k,t,1/k; f_n)) = 0, \text{ consequently, } \mu_1 \text{ is countably additive on } Bf(W(1)).
\]

Each \( \rho(k; g, f) \) is left-invariant: \( \rho(k; hg, hf) = \rho(k; g, f) \), hence \( hW(k, t, c; f) = W(k, t, c; hf) \) and \( hB'(f, c) = B'(hf, c) \) for each \( h, f \in G \), where \( B'(f, c) := \bigcap_{n=1}^{\infty} W(k, t, c; f) \), \( c > 0 \).

Therefore, \( \lim_{n \to 0} \mu(hB'(f, c))/\mu(B'(f, c)) = \lim_{n \to 0} \mu_1(hB_p(G(1), f, c))/\mu_1(B_p(G(1), f, c)) \)
\( = q_2(h^{-1}, f) = \mu_2(hdf)/\mu_1(df) =: q_2(h^{-1}, f) \) for each \( h \in G' \) and \( f \in G \), since \( \bigcap_{n>0} B'(f, c) = \{ f \} \). Taking \( \{ f_n \} \subset G \) with \( \lim_{n \to \infty} f_n = f \in G(1) \) we have the existence of \( \lim_{n \to 0} \mu(hW(k, t, c; f_n))/\mu(W(k, t, c; f_n)) \)
\( = \mu_1(hW)/\mu_1(W) =: q_2(h^{-1}, f) \) and analogously for \( h^k \) and \( h^{k'} \), consequently, \( q^2 \) are the quasi-invariant measures relative to \( G' \) on \( Bf(G(1)) \) with analogous properties of the quasi-invariance factor for \( G(1) \) instead of \( G \), but with the same \( G' \) and continuity instead of uniform continuity on corresponding neighbourhoods. Indeed, let \( \phi(\tau) = \mu(\eta(\tau, *)E) \) for \( E \in Bf(G) \), so \( \phi(0) = 0 \) if \( \mu(E) = 0 \). Using the Jordan-Hahn decomposition \( \mu = \mu^+ - \mu^- \) we obtain a \( \mu \)-integrable function \( f(\eta(\xi, *)g) \) by \( g \) for each \( \xi \in (-1,1) \) (that is, the logarithmic derivative of \( \mu \).
along \( \eta(\tau; \ast) \) such that \( \eta' \eta(\xi; \ast) E = \int_{E} f(\eta(\xi; \ast); g) d\mu(\xi) \) (see the case of linear spaces in §IV.2.2 in \cite{[3]}). Therefore, for each \( (\eta(b; \ast), h, f) \in J' \times G' \times G(1) \) and \( \epsilon > 0 \) there is a neighbourhood \( L \times U' \times U(1) \subset G' \times G' \times G(1) \) for each \( \tilde{f} \in U(1) \) there are sequences \( \{\tilde{f}_{n}\} \subset U \) converging to \( \tilde{f} \) in \( U(1) \), \( \{f_{n}\} \subset U \) converging to \( f \) in \( U(1) \cap G = U \), \( k(0), n(0) \in \mathbb{N}, \delta > 0 \) such that

\[
|\mu(\eta' \eta(\xi; \ast) W(k, t, 1/k; \tilde{f}_{n}) - \mu(\eta' \eta(\xi; \ast) W(k, t, 1/k; \tilde{f}))| / (b - c) - |\mu(\eta' \eta(\xi; \ast) W(k, t, 1/k; f_{n}) - \mu(\eta' \eta(\xi; \ast) W(k, t, 1/k; f))| / (b - c) | < \epsilon \quad \text{for each } k > k(0)
\]

\( n > n(0), 0 < |b - c| < \delta, (\eta'(b; \ast), \tilde{h}, \tilde{f}) \in (L \cap J') \times U' \times U(1) \).

Now let \( At(M) \) be infinite. There is a sequence \( E_{1} \subset \ldots \subset E_{n} \subset \ldots \subset E_{n+1} \subset \ldots, E_{n} \in \Sigma \) for each \( n \in \mathbb{N} \) such that \( \bigcup_{n} E_{n} = \mathbb{N} \) and \( U_{E_{n}} \) are open submanifolds fulfilling the same conditions as \( M \). Shrinking slightly \( U_{j} \) if necessary we may consider that there are natural embeddings \( G_{n} := D{i}ff_{\beta, \delta}^{1}(U_{E_{n}}) \subset D{i}ff_{\beta, \delta}^{1}(U_{E_{n+1}}) \subset D{i}ff_{\beta, \delta}^{1}(M) =: G \) such that \( G = \bigcup_{n} G_{n} \), since \( G \) is defined with the help of the strict inductive limit and for each \( \eta \in D{i}ff_{\beta, \delta}^{1}(M) \) there is \( E(\eta) \in \Sigma \) with \( \text{supp}(\eta) \subset U \). The groups \( G \) and \( G_{n} \) are complete, hence \( G \setminus G_{n} \) and \( G_{n+1} \setminus G_{n} \) are open in \( G \) and \( G \) respectively \cite{[3]}. Then we choose left-quasi-invariant (\( l \) times differentiable) measures \( \mu_{n} \) on \( G_{n} \) relative to \( G_{n}^{l} \subset G_{n+1}^{l} \) and consider \( G' = \text{str}-\text{ind}-\lim_{n} G_{n} \). This is possible, since \( G_{n} \) are the Polish spaces and hence Radonian \cite{[3]}.

4.16. Notes. 1. In the definition of \( D{i}ff_{\beta, \delta}^{1}(M) \) were imposed conditions \( \beta > 0 \) and

\[
\lim_{R \to \infty} \rho_{\beta}(f|M_{R}), id = 0,
\]

so \( D{i}ff_{\beta, \delta}^{1}(M) \) are separable and the constructed above measures \( \mu \) are dependent on all the natural coordinates of charts of atlas induced by \( \tilde{E} \) and \( T D{i}ff_{\beta, \delta}^{1}(M) \). Indeed, there is a sequence \( \{R^{n} : n \in \mathbb{N}\} \) of the Euclidean spaces embedded both in \( Z \) and \( T D{i}ff_{\beta, \delta}^{1}(M) \) such that their union is dense in the latter two spaces. On the other hand without the condition 2.3(ii) the corresponding diffeomorphisms group \( D{i}ff^{1}(l_{2}) \) contains the general linear group \( GL(l_{2}) \) over \( l_{2} \). The latter contains all permutations of the standard orthonormal base \( \{e_{j} : j \in \mathbb{N}\} \) in \( l_{2} \), hence \( GL(l_{2}) \) and \( D{i}ff^{1}(l_{2}) \) are not separable and not locally compact. Therefore, \( D{i}ff^{1}(l_{2}) \) permits only cylinder measures quasi-invariant relative to a non-dense subgroup that can't depend on all coordinates, since this is the case for unseparable Hilbert spaces \cite{[3]}.

2. The conditions 2.2, 2.4 and 4.1 imposed on \( M \) in the particular case of \( \text{dim}_{R} M = n \in \mathbb{N} \) are fulfilled for a \( C^{\infty} \)-manifold Euclidean at infinity with \( g \) regularly Euclidean.
These conditions can be weakened with the use of pseudo-differential operators on manifolds with roughly Euclidean ends [?].

4.17. Theorem. Let \( \mu \) be a quasi-invariant relative to \( G' \) measure on \( Bf(G) \) with \( G := Diff_{\beta,\delta}^t(M) \) as in theorem 4.5. Assume also that \( H := L^2(G, \mu, C) \) is the standard Hilbert space of equivalence classes of square-integrable (by \( \mu \)) functions \( f : G \rightarrow C \). Then there exists a strongly continuous injective homomorphism \( T : G' \rightarrow U(H) \), where \( U(H) \) is the unitary group on \( H \) in a topology induced from a Banach space \( L(H \rightarrow H) \) of continuous linear operators supplied with the operator norm.

Proof. Let \( f \) and \( h \) be in \( H \), their scalar product is given by \( (f, h) := \int_G \overline{h(g)} f(g) \mu(dg) \), where \( f \) and \( h : G \rightarrow C \), \( \overline{h} \) denotes complex conjugated \( h \).

There exists the regular representation \( T : G' \rightarrow U(H) \) defined by the following formula:

\[
T(z)(f) := \left[ q(z^{-1}, g) \right]^{1/2} f(zg),
\]

where \( q(z, g) = \mu_z(dg)/\mu(dg) \), \( \mu_z(S) = \mu(zS) \) for each \( S \in Bf(G) \), \( z \in G' \). For each fixed \( z \) the quasi-invariance factor \( q(z, g) \) is continuous by \( g \), hence \( T(z)f \) is measurable, if \( f \) is measurable (relative to \( \mu_{G(G, \mu)} \) and \( Bf(C) \)). Therefore, \( (T(z^{-1})(f), T(z^{-1})(g)) = \int_G \overline{g(zg)} f(zg) q(z^{-1}, g) \mu(dg) = (f, h) \), consequently, \( T \) is unitary. From \( \mu_{z^2}(dg)/\mu(dg) = q(z, g) = q(z, (z')^{-1}g)q(z', g) = [\mu_{z^2}(dg)/\mu_{z^2}(dg)][\mu_{z^2}(dg)/\mu(dg)] \) it follows that \( T(z')T(z) = T(z'z) \) and \( T(id) = I, T(z^{-1}) = T^{-1}(z) \).

For each \( v > 0 \) and a continuous function \( f : G \rightarrow C \) with \( \|f\|_H = 1 \) there is an open neighbourhood \( V \) of \( id \) in \( G' \) (in the topology of \( G' \)), such that \( |q(z, g) - 1| < v \) for each \( z \in V \) and each \( g \in F \) for some open \( F \) in \( G, id \in F \) with \( \mu_z(G \setminus F) < v \) for each \( z \in V \), where \( \mu_z(dg) := |f(g)|\mu(dg) \), \( \mu_z(S) := \mu_z(zS) \) for each \( S \in Bf(G) \) (see the proof of theorem 4.1). Indeed, this can be done analogously for the corresponding Banach space from which \( \mu \) was induced, \( f \in \{f_1, ..., f_j\} = \{f\} \), \( j \in \mathbb{N} \).

In \( H \) continuous functions \( f(g) \) are dense, hence \( \int_G |f(g) - f(zg)(q(z, g))^{1/2}|^2 \mu(dg) < 4v \) for each finite family \( \{f\} \) with \( \|f\|_H = 1 \) and \( z \in V' = V \cap V'' \), where \( V'' \) is an open neighbourhood of \( id \) in \( G' \) such that \( \|f(g) - f(zg)\|_H < v \) for each \( z \in V'', 0 < v < 1 \), consequently \( T \) is strongly continuous (that is, \( T \) is continuous relative to the strong topology on \( U(H) \) induced from \( L(H \rightarrow H) \), see its definition in [?]). Moreover, \( T \) is injective, since for each \( g \neq id \) there is \( f \in C(0, G \rightarrow C) \cap H \), such that \( f(id) = 0, f(g) = 1 \), and \( \|f\|_H > 0 \), so \( T(f) \neq I \). In general \( T \) is not continuous relative to the norm topology on \( U(H) \), since for each \( z \neq id \in G' \) and each \( 1/2 > v > 0 \) there is \( f \in H \) with \( \|f\|_H = 1 \), such that \( \|f - T(z)f\|_H > v \), when \( J := \text{supp}(f) \) is sufficiently small with \( J \cap J = \emptyset \).

5 Measurability of representations of a group of diffeomorphisms.

5.1. Theorem. Let \( G = Diff_{\beta,\delta}^t(M) \) be as in theorem 4.5 with a quasi-invariant measure
\( \mu \). Then \( G \) has the family \( E \) of non-measurable characters and weakly nonmeasurable irreducible unitary representations with the cardinality \( \text{card}(E) = 2^c \), where \( c := \text{card}(\mathbb{R}) \).

**Proof.** In view of theorems 3.4(1) and 4.5 there is some symmetric open neighbourhood \( W = W^{-1} \ni \text{id} \) such that \( \mu(W) > 0 \) and the following family \( J := \{ g \in W : g \text{ is with infinite order } g^b \in W \text{ for each } b \in [-1,1] \subset \mathbb{R} \} \) is such that \( \text{card}(J \cap U) = c \) for each \( U \subset W \) open in \( W \). Then from \( [?] \) it follows that in each neighbourhood \( W \) of \( \text{id} \) in \( G \) a family \( J_W \subset J \) of algebraically independent elements is dense such that \( \text{card}(J_W \cap U) = c \).

Then due to \( \text{card}(\cup[n!a^n : n \in \mathbb{N}]) = \text{card}(a) \) for each \( \text{card}(a) \geq \aleph_0 \) \( [?] \) and the Kuratowski-Zorn lemma \( [?] \) there are algebraic automorphisms \( f \) of \( G \) satisfying the following conditions: (i) \( \text{card}(f(V') \cap V) = c \) for each \( V' \) and \( V \) open in \( Y \); (ii) \( \text{card}(f(V') \cap V) = c \) for each \( V' \) and \( V \) open in \( W \) (or in \( W \cap G' \)), where \( Y \subset U \), \( \mu(U \setminus Y) < c \times \min(1, m(U)) \), \( \mu(U) > 0 \), \( U \subset W \), \( S \subset W \), \( \mu(W \setminus S) < c \times \min(1, m(W))/6 \); (iii) \( S \), \( U \) and \( Y \) are symmetric, \( S \) and \( Y \) are compact, intersections of each one-parameter subgroup with \( S \) and \( Y \) correspond to subsets \( \{ b \} \) that contain open subsets in \( (−1,1) \subset \mathbb{R} \); (iv) the function \( q(g,x) = \mu(g(dx)/\mu(dx)) \) is uniformly continuous by \( g \in Y \cap G' \), \( x \in S \) and \( |q(g,x) − 1| \leq c \times \min(1, m(W))/6 \) for such \( (g,x) \), \( 0 < c < \min(1, m(Y))/25 \), where \( G' \) is dense in \( G \), \( U \) is fixed and open.

Indeed, there exists a family \( P = \{(L, f)\} \) of subgroups \( L \subset G \) together with their automorphisms \( f \) satisfying (i,ii) for \( S \), \( U \) and \( Y \) also fulfilling (iii − iv), but with cardinals \( y \) instead of \( c \) for \( c \geq y \geq \aleph_0 := \text{card}(\mathbb{N}) \). Also let each \( (L, f) \in P \) fulfills: (v) if \( g \in G \setminus L \) and there is not any \( h \in L \) and \( n \in \mathbb{N} \) with \( h^{1/n} = g \), then \( f \) has the extension on \( \text{gr}(L \cup \{ g \}) \), here \( \text{gr}(L) = L \) for \( (L, f) \in P \), where \( \text{gr}(A) \) is the minimal subgroup of \( G \) containing \( A \) such that if \( g \in \text{gr}(A) \) and \( g^{1/n} \in G \) for some \( n \in \mathbb{N} \) then \( g^{1/n} \in \text{gr}(A) \) with \( A \subset G \); (vi) \( (L, f) < (L', f') \) if \( L \neq L' \), \( L \subset L' \) and the restriction \( f'|_{L} = f \). Also let \( (L', f')R(L, f) \) if (vi) is fulfilled, where \( R \) is the partial order in \( P \). The family \( P \) is non-void, since there are \( (L, f) \) with \( y = \aleph_0 \), that may be constructed by induction. Then for each totally ordered \( P' \subset P \), (that is, all elements of \( P' \) are comparable by relations \( >, < \) or \( = \)) there is \( \text{sup}_c(P') = \text{inf}_R(P') = (\bigcup_{(L,f) \in P'}(L, f)) \in P \), hence \( P \) contains well-ordered subsets. Therefore, by the Kuratowski-Zorn lemma there are well-ordered \( P' \) with \( \text{inf}_R(P') = (G, f) \in P \) and the family of all such \( f \) has cardinality \( 2^c \), since \( \text{card}(J_W) = c \).

Let \( \mu^* \) be the outer measure for \( \mu \) (see IX.1.9 in [?]). Let us suppose, that for each open disjoint subsets \( \hat{R} \) and \( \hat{P} \) in \( W \) there is the equality \( (vii) \mu^*(q(\hat{R}) \cap \hat{P}) = 0 \) (where \( q = f \) or \( q = f^{-1} \) for \( \mu^*(q(W) \cap W) \geq \mu(W)/2 \), hence \( (viii) \mu^*(Z \cap q(Z)) = \mu^*(q(Z)) \) for each open \( Z \) in \( W \). Then we choose \( Z^{-1} = Z \subset Z^2 \subset \hat{R} \subset W \), \( \mu(\hat{R}) > 5c > 0 \), \( \hat{R} \cap \hat{R} = \emptyset \) for some \( g \in W \cap Y \cap G' \), there is \( b \in B \subset Z \cap Y \), \( \text{card}(B) = c \), \( q(b) = g \), \( b \in G' \), where \( c < 1 \).

From this it follows that \( \mu^*(q(BZ)) = \mu^*(q(B)q(Z)) \geq (1-c)\mu^*(q(Z)) \geq (1-c)(\mu(Z) - c) \) in contradiction with (vii), since \( q(b)Z \subset \hat{P} \) for \( \hat{P} = g\hat{R} \). Therefore, \( \mu^*(q(\hat{R}) \cap \hat{R}) > 0 \).
and \( \mu^*(q(\hat{R}) \cap \hat{P}) > 0 \) for some open disjoint \( \hat{R} \) and \( \hat{P} \) in \( W \).

Let \( T: G \rightarrow U(H) \) be some non-trivial weakly measurable irreducible unitary representation or a character, where \( U(H) \) is the unitary group on some Hilbert space \( H \). In view of the N.Lusin theorem \[?]\ and \( hc(W) = 8_0 \) (\( hc \) is the hereditary Suslin number \[?]\), there exists nontrivial function \( z(g) = (T(f(g))y, y) \) that is not measurable relative to the complete \( \sigma \)-field corresponding to \( \mu \) on \( Bf(G) \) and the Borel \( \sigma \)-field on \( C \), where \( y \) is in the Hilbert space \( H \) and \( U(H) \) is the unitary group with the topology induced by the operator norm (see also \[?\], \[?\], \[?\], \[?\], \[?\]).

6 Irreducible unitary representations of a group of diffeomorphisms of a Hilbert manifold.

6.1. Theorem. Let \( M \) be a Hilbert manifold fulfilling 2.2 and 2.4, \( G = \text{Diff}_{\beta, \gamma}^t(M) \) be a group of diffeomorphisms with \( t \geq 1, \beta \geq \omega \) and \( \gamma > 2(1+\delta) \). Then (for each \( 1 < l < 00 \)) there exists a quasi-invariant (and \( l \) times differentiable) measure \( \nu \) on \( M \) relative to \( G \).

Proof. The exponential mapping \( \exp \) is defined on a neighbourhood of the zero section of the tangent bundle \( TM \) and \( \exp \) is of class \( E^s \) due to 2.2. For each \( x \in M \) we have \( T_xM \cong l_2 \). Suppose \( F \) is a nuclear (of trace class) operator on \( l_2 \) such that \( Fe_i = F_i e_i \), where \( i^b < F_i < i^c \) for each \( i \), \( \{e_i : i\} \) is the standard base in \( l_2 \), \( 1 - \gamma + 2\delta < b < c < -1 \). Then there exists a \( \sigma \)-additive Gaussian measure \( \lambda \) on \( l_2 \) with zero mean and a correlation operator equal \( F \). Therefore, \( \exp_x \) induces a \( \sigma \)-additive measure \( \nu \) on \( W \times x \), where \( W = \exp_x(W), 0 < V \in \exp_x(W) \). Then there is a countable family \( \{g_j : j \in \mathbb{N}\} \subset G, g_1 = e, W_1 = W \) and open \( W_j \subset W \) such that \( \{g_jW_j : j\} \) is a locally finite covering of \( M \) with \( W_1 = W, g_1 = id \). For \( C \in Bf(M) \) let \( \nu(C) := \sum_{j\in\mathbb{N}} \nu((g_j^{-1}C) \cap W_j)^{-2^{-j}} \) (without multipliers \( 2^{-j} \) the measure \( \nu \) will be \( \sigma \)-finite, but not necessarily finite).

The following mapping \( Y_g := (\exp o g o \exp^{-1}_{\gamma}) \) on \( TM \) for each \( g \in G \) satisfies conditions of theorems 1,2 in §26 \[?\]. Indeed, \( (\partial g^t/\partial x^i)_{i,j\in\mathbb{N}} \) in local natural coordinates \( (x^i) \) is in the class \( E^{\gamma t+1}_{s+1,\gamma} \) (see 2.4). In view of these theorems and \[?]\ the measure \( \nu \) is quasi-invariant and \( l \) times differentiable, since \( (Y_g)' = I \) is of trace class on the Hilbert space \( l_2 \) and \( dg^t/dt = V o g^t \) (see the proof of theorems 3.4 and 4.5 above ), where \( g^t = \eta_t, Qx = \sum_j x^j e_j, x = \sum_j x_j e_j \in l_2, x^j \in \mathbb{R} \).

6.2. Definition.1. Let \( M \) satisfies conditions in 2.2 and 2.4. For a given atlas \( \mathcal{A}(M) \) we consider its refinement \( \mathcal{A}'(M) = \{(U^t_j, \psi_j) : j \in \mathbb{N}\} \) of the same class \( E^{s}_{\gamma t} \), such that \( \{U^t_j\} \) is a locally finite covering of \( M \), for each \( U^t_j \) there is \( i(j) \) with \( U_{i(j)} > U^t_j \), \( \exp^{-1}_x \) is injective on \( U^t_j \) for some \( x \in U^t_j \), \( \exp^{-1}_x(U^t_j) \) is bounded in \( T_xM \cong l_2 \). Henceforward, \( M \) will be supplied by such \( \mathcal{A}'(M) \) and \( \text{Diff}_{\beta, \gamma}^t(M) \) will be given relative to such atlas.
2. Let $\mu$ be a non-negative measure on $M$ relative to $G = \text{Diff}^\beta(M)$ such that $\mu(M) = \infty$, $\mu$ is $\sigma$-finite and $\mu(U'_j) < \infty$ for each $j$. Then $\mu$ is considered on $Af(M, \mu)$. We consider $X = \prod_{i \in \mathbb{N}} M_i$, where $M_i = M$ for each $i$. Take $E_i \in Af(M_i, \mu)$, put $E = \prod_{i \in \mathbb{N}} E_i$, which is called a unital product subset of $X$ if it satisfies the following conditions:

- (UPS1) $\sum_{i \in \mathbb{N}} |\mu(E_i) - 1| < \infty$ and $\mu(E_i) > 0$ for each $i$;
- (UPS2) $E_i$ are mutually disjoint.

6.3. Note. In view of 6.2 the above definitions 1.1, 1.2 and lemmas 1.1, 1.2 [?] are valuable for the case considered here $(G, M, \mu)$ for infinite-dimensional $M$. Henceforward, we denote by $G$ the connected component of $id \in \text{Diff}^\beta(M)$ from 6.2.2. Further, the construction of irreducible unitary representations follows schemes of [?] for finite-dimensional $M$ and II [?] for non-Archimedean Banach manifolds, so proofs are given briefly with emphasis on features of the case of a Hilbert manifold $M$.

6.4. Let $E$ be cofinal with $E'$ $(ERE'')$ if and only if

$$
(CF) \sum_{i \in \mathbb{N}} \mu(E_i \Delta E''_i) < \infty,
$$

$E$ be strongly cofinal with $E'$ $(E\cong E')$ if and only if

$$
(SCF) \text{there is } n \in \mathbb{N} \text{ such that } \mu(E_i \Delta E''_i) = 0 \text{ for each } i > n,
$$

where $E_i \Delta E''_i = (E_i \setminus E''_i) \cup (E''_i \setminus E_i)$, $\Sigma(E) := \{E' : E' \cong E''\}$.

Put $\nu_E(E') = \prod_{i \in \mathbb{N}} \mu(E''_i)$ for each $E' \in \Sigma(E)$. In view of the Kolmogorov’s theorem [?] $\nu_E$ has the $\sigma$-additive extension onto the minimal $\sigma$-algebra $M(E)$ generated by $\Sigma(E)$.

The symmetric group of $\mathbb{N}$ is denoted by $\Sigma_{\infty}$, its subgroup of finite permutations of $\mathbb{N}$ is denoted by $\Sigma_\infty$. For $g \in G$ there is $gx = (gx_i : i \in \mathbb{N})$, where $x = (x_i : i \in \mathbb{N}) \in X$, for $\sigma \in \Sigma_\infty$ let $x\sigma = (x'_i : i \in \mathbb{N})$, $x'_i = x_{\sigma(i)}$ for each $i$. Quite analogously to lemma 5.5 II [?] or 1.3 [?] we have 6.5 due to $\text{supp}(g) \subset U^E(g)$ for some $E(g) \in \Sigma$ and $\mu(U^E(g)) < \infty$, where $U^E = \bigcup_{j \in E} U_j$, $(U_j, \psi_j)$ are charts of $At'(M)$.

6.5. Lemma. Let $E$ be a unital product subset of $X$. Then (i) $(gE)RE$ for each $g \in G$, (ii) $\Sigma(E)$ is invariant under $G$ and $\Sigma_\infty$.

6.6. In view of 2.6, 6.2.1 and the proof of 6.1 we may choose $\mu$ such that for each $g \in G$ there is its neighbourhood $W_g$ and there are constants $0 < C_1 < C_2 < \infty$ such that

$$
(i) \ C_1 \leq q_\mu(f, z) \leq C_2
$$

for each $x \in m$ and $f \in W_g$ with $\text{supp}(f) \subset U^E(g)$. Indeed, for each $U_j$ there exists $y \in U_j$ such that $\exp_{y}^{-1}U_j$ is bounded in $T_y M$. Hence for each fixed $R, \infty > R > 0$, for operators $Y_f = U$ of non-linear transformations the term $|\det((Y_f)'(x))|^{-1} \exp\{\sum_{i=1}^{\infty} 2(x -
\[ Y_f^{-1}(x), c_1)(x, c_1) - (x - Y_f^{-1}(x), c_1)^2 / F_1 \] is bounded (see \( f \) after (i)) for each \( x \in \mathbb{I}_2 \) with \( \|x\| < R \). For \( z \in M \setminus U^{E(g)} \) we have \( q_p(f, z) = 1 \). Therefore, we suppose further that \( \mu \) satisfies (i).

If \( S \in \alpha f(M, \mu) \) and \( \mu(S) < \infty \) we may consider measures \( \mu_k = \mu \) on \( E_k \setminus S \) and \( \nu_k = 0 \) on \( S \), suppose \( L_n = \prod_{i=1}^n M_i, \mu_{L_n} = \bigotimes_{i=1}^n \mu_i, P_n : X \to L_n \) are projections, \( \rho_k(x) = \nu_k(dx) / \mu(dx) \). Then \( \rho_k(x) = 0 \) for each \( x \in S \). Using the analog of lemma 16.1 [?] for our case we obtain the analog of lemmas 1.4, 1.6, 1.7 and theorem 1.5 [?], since \( M \) has a countable open base \( \{\bar{U}_j : j \in \mathbb{N} \) there is \( E \in \Sigma \) such that \( \bar{U}_j \subset U^E \).

6.7. The manifold \( M \) is Polish, hence \( M \) is Radonian [?] and for each unital product subset \( E \) for each \( i \) there is a compact \( \hat{E}_i \subset M \) such that \( \mu(\hat{E}_i \Delta \hat{E}_i) < 2^{-i} \) and \( \hat{E}_i \subset U^{h(i)} \) for corresponding \( h(i) \in \Sigma \). Since each open covering of \( \hat{E}_i \) has a finite subcovering we may choose \( E_i' \in \alpha \ell(M, \mu) \) with finite number of connected components. As in §1.8 [?] we can construct \( E_i'' \cap \hat{E}_i \) such that \( E_i'' \cap \hat{E}_i \) are mutually disjoint.

6.8. Proposition. Each unital product subset \( E \) is cofinal with \( E_0 \) satisfying the following conditions:

\[
\text{(UP3)} \text{ the closure } \text{cl}(E_i^0) \text{ and } \text{cl}(\bigcup_{j \neq i} E_j^0) \text{ are mutually disjoint and } E_i^0 \text{ is open for each } i \text{ and } \inf \inf_{x \in E_i^0, y \in \bigcup_{j \neq i} E_j^0} d_M(x, y) > 0, \quad E_i^0 \subset U^{h(i)}, \quad h(i) \in \Sigma; \quad \text{(UP4)} E_i^0 \text{ and } E_{i,k}^0 \text{ are connected}.
\]

and simply connected, there is \( n \in \mathbb{N} \) such that for each \( k > n \) and \( i \in \mathbb{N} \) there exists \( g \in G \) with \( g(E_i^0, k) = B_{i,k} \) being an open ball in a coordinate neighbourhood of \( M_k \) with \( g | (M \setminus M_k) = \text{id} \) and \( \inf_{x \in \partial B_{i,k}, y \in E_i^0} d_M(x, y) > 0 \), \( g(E_i^0, k) = B_{i,k} \), where \( B := \text{cl}(B) \), \( E_{i,k}^0 := E_i^0 \cap M_k \). For \( i \neq j \), \( E_i^0 \) and \( E_j^0 \) can be connected by an open path \( P_{i,j} \) such that \( P_{i,j} \cap \text{cl}(\bigcup_{k \neq i,j} E_k^0) = \emptyset \).

Proof. In view of 2.7 and \( M_k \) are connected for each \( k > n \) and some fixed \( n \in \mathbb{N} \). Then using 3.1, locally finite coverings of \( M \) and \( M_k \) [?] and shrinking slightly \( E_i^0 \) such that \( \partial E_i^0 \) are of class \( C^\infty_{\alpha, \delta} \) analogously to steps 1-4 [?] and using properties of \( \mu \) we prove this proposition. Indeed, \( \mu \) is approximable from beneath by the class of compact subsets [?].

6.9. Henceforth, \( \Pi : \Sigma_\infty \to U(V(\Pi)) \) denotes a unitary representation on a Hilbert space \( V(\Pi) \) over \( \mathbb{C} \), \( H(\Sigma) \) denotes a Hilbert space that is the completion of \( \bigcup_{E \in \Sigma(\Sigma)} H_{|E|} \), with the scalar product \( \langle \phi_1, \phi_2 \rangle := \sum_{\sigma \in \Sigma_\infty} \int_{E_1 \cap E_2} \phi_1(x), \Pi(\sigma)^{-1} \phi_2(x^{-1}) >_{V(\Pi)} \nu_E(dx) \), where \( H_{|E|} := L^2(E'; \mathcal{M}(E); \nu_E|E'|; V(\Pi)) \), \( \Pi_E \) is a Hilbert space of functions on \( E' \) with values in \( V(\Pi) \), \( \Sigma := (\Pi; \mu, E) \). \( E' \cap E \), \( E \) is a unital product subset of \( X \). Then we define a representation

\[
(i) \quad T_\Sigma(g) \phi(x) := \rho_E(g^{-1}|x|)^{1/2} \phi(g^{-1}x),
\]
where \( \rho_E(g^{-1}|x) := (\nu_E)_g(dx)/\nu_E(dx), (\nu_E)_g(C) := \nu_E(g^{-1}C), \rho_E(g|x) = \Pi_{i\in \mathbb{N}} \rho_M(g; x_i), \rho_M(g; x_i) := q_x(g^{-1}; x_i) \) (see §2 [?] and 5.9 [?]).

6.10. Proposition. The formula 6.9(i) determines a strongly continuous unitary representation of \( G \) (given by 6.2 and 6.3) on the Hilbert space \( H(\Sigma) \).

Proof. The space \( H'(\Sigma) \) is isomorphic with the completion \( H'(\Sigma) \) of \( \bigcup_{E \in \Sigma} H_{E|E}^{\|} \) with the scalar product \( f, g > : f_E < f_1(x), f_2(x) >_{\nu_E} \nu_E(dx) \), where \( f_i \in H_{E|E(i)}^{\|}, E^{(i)} \in \Sigma(E), F \in M(E) \) for \( \sigma \in \Sigma_\infty \) are disjoint and \( \text{supp}(f_1(x)f_2(x)) \subset \bigcup_{\sigma \in \Sigma_\infty} F_\sigma \). Here \( H_{E|E}^{\|} \) is a space of functions \( f = Q_\Sigma \phi \), where \( \phi \in H_{E|E}^{\|} \) and (i) \( Q_\Sigma \phi := \sum_{\sigma \in \Sigma} (R_\sigma \Pi_\sigma) \phi, (Q_\Sigma \phi)(x) = \Pi_\sigma^{-1} \phi(x); \) (ii) \( R_\sigma \phi(x) := \phi(x) \); (iii) \( \Pi_\sigma \phi(x) = \Pi_\sigma(\phi(x)), \| f \|^2 = f_E \| f(x) \|^2_{\nu_E} < \infty \), since \( E' \sigma \) for \( \sigma \in \Sigma_\infty \) are disjoint for different \( \sigma \). Therefore, as in 2.1 [11] we get \( < T(\Sigma) g_1 f_1, f_2 > = < v_1, v_2 >_{\nu_E} \times \prod_{i \in \mathbb{N}} \int_{[g^{(i)}]} \rho_M(g^{-1}; x_i) dx_i \) for \( f_j = Q_\Sigma \phi_j, \phi_j = \chi_{B^{(i)}} \circ v_j \), where \( \chi_C \) is the characteristic function of \( C \) (see also 6.6(i)).

Let us fix \( J \in \Sigma \) and take \( U(J) = \bigcup_{i \in I} U_i \subset M \). As in the proof of theorem 5.6 (see 6.6(i)) we can find a neighbourhood \( W \ni id \) in \( G \) and \( 0 < c_1 < c_2 < \infty \) such that \( c_1 \leq \rho_M(g^{-1}; y) \leq c_2 \) for each \( y \in U^J \) and \( \rho_M(g^{-1}; y) = 1 \) for each \( y \notin U^J \) for each \( g \in W \) with \( \text{supp}(g) \subset U^J \). Hence for each \( \epsilon > 0 \) there exists \( W \ni id \) such that \( |< T(\Sigma) g_1 f_1, f_2 > - < v_1, v_2 >| < \epsilon \), consequently, due to the Banach-Steinhaus theorem (11.6.1 [22], [26]) there exists a neighbourhood \( V \ni id \) such that \( \| (T(\Sigma) g_1 - I) f_1 \| < \epsilon \) and \( T(\Sigma) \) is strongly continuous.

It is interesting to note that 6.10 may be proved from the inequality: \( \| T(\Sigma) g_1 - f_1 \|_{H'(\Sigma)} \leq |v|^2 f_E \| f_1(x) - f_1(g^{-1}x) \rho_E(g^{-1}x) dx_i \|^2 \nu_E(dx) \). Then we consider restrictions \( g \mid M_k \) and properties of \( (Y_g)^v \) (for \( g \) on \( M \mid M_k \)) such that \( \text{card}(i : \text{supp}(g) \cap F_{i,k}) < \infty \) for each \( k \in \mathbb{N} \). In view of theorems 26.1.2 [?] for each sequence \( g_n \) with \( \text{lim} \, g_n = e \) and for each \( \epsilon > 0 \) there is \( m \) such that \( f_E \| f_1(x) - f_1(g_n^{-1}x) \rho_E(g_n^{-1}x) dx_i \|^2 \nu_E(dx) < \epsilon \) for all \( n > m \), since there is \( E \in \Sigma \) with \( \text{supp}(g_n) \subset U^E \) for every \( n > m \).

6.11. Let \( E_1, \ldots, E_r \) be mutually disjoint open subsets of \( M, H_1 := \bigotimes_{i=1}^r L^2(E_i), L^2(E_i) := L^2(E_i, \mu|E_1), G_1 := \prod_{i=1}^r G|E_i, G|E_i := \{ g \in G : \text{supp}(g) \subset E_i \} \), denote by \( G(E_i) \) the connected component of \( id \in \text{Diff}_{\beta,\gamma}^\Lambda (E_i) \), also let \( E_{i,j} : j \in J_i \) be the connected components of \( E_i \). Then \( G|E_i \) is \( G(E_{i,j}) \), since for each continuous mapping \( F : [0, 1] \rightarrow G \) we have by continuity that (i) \( F(\epsilon)(E_{i,j}) \subset E_{i,j} \) for each \( \epsilon \in [0, 1] \subset \mathbb{R} \) and each \( j \in J_i \). Indeed, suppose \( J \) is the connected subset of \( [0, 1] \) such that \( 0 \in J \) and for each \( \epsilon \in J \) is satisfied (i). If \( v = \text{sup}(J) < 1 \) then by continuity there is \( w > v \) for which \( [0, w] \) have the same properties as \( J \). Hence the maximal such \( J \) coincides with \( [0, 1] \).

We define and consider \( G(E') := \prod_{i \in \mathbb{N}} G(E_i) := \{ g = (g_i : i : g_i \in G(E_i), \text{supp}(g_i) \subset U^E(g_i), \{ \text{supp}(E_i) \} \in \Sigma \} \). Therefore, \( \prod_{J \in J} G(E_{i,j}) = G|E_i \). Then quite analogously to lemma 3 [?] and lemma 5.12 [?] we get that the following representation \( L_1 \) of \( G_1 \) is irreducible: \( (L_1(g)f)(y) = \prod_{i=1}^r \rho_M(g_i^{-1}, y_i)^{1/2} f(g^{-1}y) \) for \( f \in H_1, g = (g_i : i) \in G_1 \).
and \( y = (y_i : i) \in \prod_{i=1}^\infty E_i \), since \( G|_{E_i} \) is dense in \( G_i := G \cap \prod_{j \in J_i} G(E_{i,j}) \) and \( L_1 \) is strongly continuous, \( G|_{E_i} \subset \prod_{j \in J_i} G(E_{i,j}) \). Indeed, in view of proposition 6.8 \( G|_{E_i} \) is connected, since \( G \) is connected.

Then \( L_1 \) on \( G|_{E_i} \) is decomposable into irreducible components, since \( L_1 \) of \( G(E_{i,j}) \) on \( L^2(E_{i,j}) \) is irreducible. In view of strong continuity of \( L_1 \) on the dense subgroup \( G|_{E_i} \) it follows that its strongly continuous extension on \( G_i \) is also unitary. Then the rest of §3.1 may be transferred onto the case considered here.

Let \( L \in L^2(E_\sigma(\xi)) \) for \( \xi \in \Sigma(E) \wedge E_\sigma(\xi) \) and \( \sigma \in \Sigma_\infty \), and such that \( g(E_{\sigma(i),k}) = E_{\sigma(i),k} \) for each \( i \in \mathbb{N} \) and \( g|_{M \setminus M_k} = id \), where \( E' = \prod_{i \in \mathbb{N}} E_{\sigma(i),k} \) (\( E' \subset M \)) satisfies (UP3 - 4) and \( E' \in \Sigma(E) \). \( E_{\sigma(i),k} = E_i \cap M_k \). In view of the foliated structure in \( M \) this group is dense in (ii) \( \{ g \in G : supp(g) \subset \bigcup_{i \in \mathbb{N}} E_i \} \).

6.14. Lemma. Let \( E' \in \Sigma(E) \) satisfy (UP3 - 4). Then for any \( \sigma \in \Sigma_\infty \) there is \( n \) such that for each \( k > n \) there exists \( g \in G((E')) \) with \( g(E_{\sigma(i),k}) = E_{\sigma(i),k} \) for each \( i \), moreover, \( g|_{M \setminus M_k} = id \) if \( \sigma(i) = i \).

Proof is quite analogous to that of lemma 3.4, since each \( M_k \) is locally compact and connected, also due to properties of \( \mu \) induced as the image of the Gaussian \( \sigma \)-additive measure. On the other hand, the latter is fully characterised by its weak distribution and is with the Radonian property (see lemma 2 and theorem 1 in §2). 6.15. Let \( E' \) be as in 6.12, \( H^1\Pi_{E'} = L^2(E', M(E)|E', \nu_E|E') \). For each \( g \in G((E')) \) there are \( \sigma \in \Sigma_\infty \) and \( k = k(n), n \in \mathbb{N} \) such that \( g(E_{\sigma(i),k}) = E_{\sigma(i),k} \) for each \( i \in \mathbb{N} \) and \( g(M \setminus M_k) = id \). Suppose \( f = Q\Pi H_{\Pi_{E'}} \). If \( (\alpha) \) \( \phi \) depends only on \( \{ x = (x_i : i)|x_i \in E_{i,k} \} \) then \( T_{\sum(g)}(f)(x) = \rho_E(g^{-1}|x)^{1/2}\Pi(\sigma)\phi(g^{-1}x)\sigma \). If \( (\beta) \) \( \phi \) depends only on \( \{ x = (x_i : i)|x_i \in E_{i,k} \} \) then \( T_{\sum(g)}(f)(x) = f(x) \). Then if \( \phi(x) = \phi_1(x) \times \phi_2(x) \), where \( \phi_2(x) \) is of type \( (\alpha) \) or \( (\beta) \) and \( \phi_1 : E' \rightarrow \mathbb{C} \) is also of type analogous to \( (\alpha) \) or \( (\beta) \) then \( T_{\sum(g)}(f) \in H^1\Pi_{E'} \). Let \( G_k((E')) = \{ g \in G((E')) : g|(M \setminus M_k) = id \} \), then \( \bigcup_k G_k((E')) \) is dense in \( G((E')) \). Denote \( H_k := \{ \phi \in H^1\Pi_{E'} : \phi(x) \) is constant on \( M \setminus M_k \} \). Then we obtain analogously to lemma 4.2 the following lemma.

6.16. Lemma. Let \( F = \prod_{i \in \mathbb{N}} F_i \) satisfy (UP3 - 4). Then there exists \( F' \in \Sigma(F) \)
satisfying \((UP3 - 4)\) and
\[ (UPS5) \quad M \setminus \text{cl}\left( \bigcup_{i \in \mathbb{N}} F_i \right) \text{ is connected for every } N > 0. \]

**Proof.** Consider \(F_{i,k} = F_i \cap M_k\) and measures \(\mu_k\) on \(M_k\) induced by \(\mu\) on \(M\) and the projection \(P_k : l_2 \to \mathbb{R}^k\) and choose \(F'\) such that \(|\mu_{k(\alpha+1)}(F'_{k(\alpha+1)}) - \mu_k(F'_{i,k(n)} \Delta F_{i,k(n)})| < 3^{-i-2(k(n)+1)}\mu(F_i)\) for each \(k = k(n)\) and \(i, n \in \mathbb{N}\). Then use theorem 3.1 [?].

**6.17. Theorem.** The unitary representation \(T_\Sigma\) of \(G\) on \(H(\Sigma)\) is irreducible.

**Proof.** Considering the sequences \(\{M_k : k\}, \{G_k((E')) : k\}\) and \(\{H_k : k\}\), using 6.2-6.16 and strong continuity of \(T_\Sigma\) we get from the proof of theorem 4.1 [?] that \(T_\Sigma\) is irreducible. Indeed, we may consider \(\Delta := \{E' : E' \cong E^0\}\), \(E'\) satisfies \(UP3 - 4\) instead of \(\Delta\) in §4.3 [?].

**6.18. Theorem.** Suppose \(T_{\Sigma_i}\) are unitary representations of \(G\) with parameters \(\Sigma_i = (\Pi_i; \mu, E')\). Then, \((T_{\Sigma_i}, H(\Sigma_i)), i = 1, 2\) are mutually equivalent if and only if there exists \(a \in \tilde{\Sigma}_\infty\) such that \(\Pi_1 = a \Pi_2\) and \(E_1 \in \Sigma(E_2 a^{-1})\), where \((a \Pi)(\sigma) := \Pi(\alpha^{-1} \sigma a)\).

**Proof.** In view of 6.8 and 6.9 we may assume without loss of generality that \(E^i\) satisfies \((UP3 - 4, UPS5)\) for \(i = 1\) and \(2\). Then we consider \(G^{(1)} := G((E^{(1)})) \cap G((E^{(2)})) \subset G\) and \(G^{(2)} := \Pi_k \in \mathbb{N} \Pi(G_k),\) where \(C_k\) are all connected components of \(E^{(1)}_{i,j} = E^{(2)}_{j,i}\) (with \(E^{(2)}\) here instead of \(F^{(2)}\) in [?]). Instead of equations (5.7) we have corresponding expressions as intersections with \(M_k\) in both sides for some \(k = k(n), n \in \mathbb{N}\). Using the sequences \(\{M_k\}, \{G_k((E'))\}\) and strong continuity of \(T_{\Sigma_i}\) we get the statement of theorem 6.18 analogously to §5 [?].

**6.19. Note.** The construction presented above of irreducible unitary representations is valid as well for each dense subgroup \(G'\) of \(Diff_{\tilde{\beta}}(M)\) such that the corresponding non-negative measure \(\lambda\) on \(M\) is left-quasi-invariant relative to \(G'\) and satisfies 6.2 and 6.6.

**References**


