APPROXIMATE SOLUTION OF A NONLINEAR $m$-ACCRETIVE OPERATOR EQUATION

C.E. Chidume  Habtu Zegeye
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

Let $E$ be real Banach space which is both uniformly convex and uniformly smooth. Let $T : D(T) \subseteq E \longrightarrow E$ be bounded $m$-accretive operator, where the domain of $T$, $D(T)$, is a proper subset of $E$. For a given $f \in E$, an iterative method is constructed which converges strongly to the unique solution of the equation $x + Tx = f$. A related result deals with operator equations of the dissipative type. Our Theorems generalize important known results and our method is of independent interest.

MIRAMARE – TRIESTE
September 1996

---

$^1$Permanent address: Department of Mathematics, Bahir Dar Teachers College, P.O. Box 79, Bahir Dar, Ethiopia.
1 Introduction

Let $E$ be a real Banach space. An operator $T$ with domain $D(T)$ in $E$ to $E$ is called accretive [2] if for each $x, y \in D(T)$ and all $t \geq 0$, the following inequality is satisfied

$$\|x - y\| \leq \|x - y + t(Tx - Ty)\|$$

(1)

The operator $T$ is said to be $m$-accretive if $T$ is accretive and $(I + \lambda T)(D(T)) = E$ for all $\lambda > 0$ where $I$ denotes the identity operator on $E$. $T$ is called dissipative (respectively $m$-dissipative) if $(-T)$ is accretive (respectively $m$-accretive). The notion of accretive operators was introduced in 1967 by Browder [2] and Kato [22]. The main interest in this class of mappings arises from the fact that many physically significant problems can be modelled in terms of an initial value problem of the form

$$\frac{dx}{dt} + Tu = 0$$

$$x(0) = x_0$$

(2)

where $T$ is either accretive or $m$-accretive or dissipative in an appropriate Banach space. Typical examples of how such equations arise are found in models involving either the heat or wave or the Schrödinger equation (See for example [31]). In [2], Browder proved that if $T$ is locally Lipschitzian and accretive then $T$ is $m$-accretive. In particular, for any given $f \in E$, the equation

$$x + Tx = f$$

(3)

has a solution. In [25], Martin extended this result to the continuous accretive operators. Methods of approximating a solution of equation (3) when it is known to exist have been studied by various authors (see, for example, [4-6, 8-9, 11-14, 31]). Two well known iterative schemes for successive approximation of a solution of (3) when the domain of $T$ is assumed to be the whole of $E$ or when $T$ is self-mapping of a nonempty convex subset of $E$ are the Mann Iteration Process (see e.g., [24] or [19]) and the Ishikawa Iteration Process (see e.g., [16] or [28]). These iteration schemes have successfully been employed by various authors to approximate solutions of nonlinear operator equations in Banach spaces (see, for example, [3-4, 7-10, 18-21, 23, 24, 26, 28]). In several applications, however, the operator $T$ of equation (3) is, in general, not defined on the whole of $E$. The domain of $T$, $D(T)$, is generally a proper subset of $E$. In such a situation neither the Mann nor the Ishikawa iteration scheme is well defined. In this connection and in the case that $K$ is a proper closed convex subset of a Hilbert space $H$ and $T : K \rightarrow H$ is a map, Chidume
[6] introduced the following iteration process $x_0 \in K$,

$$\begin{cases} 
  x_{n+1} = P_K p_n \\
  p_n = (1 - c_n)x_n + c_n(f - Tx_n) 
\end{cases}$$

where $P_K : H \rightarrow K$ is the nearest point mapping, and proved that $\{p_n\}$ converges strongly to the unique solution of equation (3) if $T$ is Lipschitz and accretive (monotone). We observe immediately that (4) reduces to the Mann-type iteration process if $T$ is a self-map. A key tool in the proof of Chidume [6] is the fact that in Hilbert spaces, the nearest point map is nonexpansive (i.e., $\|P_Kx - P_Ky\| \leq \|x - y\|$ for all $x, y \in H$). Unfortunately, this fact also characterizes Hilbert spaces so that it is not available in general Banach spaces. In this connection, we state the following result for m-accretive maps.

**LEMMA 1.** [27] Let $E$ be a Banach space which is both uniformly convex and uniformly smooth (defined below). Let $T : D(T) \subseteq E \rightarrow E$ be m-accretive and let $J_r = (I + rT)^{-1}$. Then for each $x \in E$, the strong limit $\lim_{r \to 0} J_r(x)$ exists. Denote this strong limit by $Qx$. Then $Q : E \rightarrow \text{cl}(D(T))$ is a nonexpansive retraction of $E$ onto $\text{cl}(D(T))$. (Here $\text{cl}(D(T))$ denotes the closure of $D(T)$.)

It is known that under the hypothesis of Lemma 1, $\text{cl}(D(T))$ is convex (see e.g., [1]). Utilizing Lemma 1, Chidume and Osilike [13] extended the iteration process (4) to Banach spaces much more general than Hilbert Spaces: If $K$ is a closed convex subset of a uniformly convex and uniformly smooth Banach space $E$ and $T$ is an m-accretive map from $K$ into $E$, the sequence $\{x_n\}_{n=1}^\infty$ in $K$ is defined by

$$\begin{cases} 
  x_0 \in K \\
  x_{n+1} = Qp_n \\
  p_n = (1 - \alpha_n)x_n + \alpha_n(f - TQy_n) \\
  y_n = (1 - \beta_n)x_n + \beta_n(f - Tx_n), n \geq 0 
\end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

We observe immediately that if $T$ is a self-map of $K$, then $Q$ becomes the identity operator and (5), reduces to the well-known Ishikawa iteration process, and reduces to the Mann scheme, if in addition, $\beta_n = 0$ for each $n$.

The main results of Chidume and Osilike [13] are the following theorems.

**THEOREM CO1** ([13], Theorem 3, p. 230). Let $E$ be a real Banach space which is both uniformly convex and q-uniformly smooth. Let $T : D(T) \subseteq E \rightarrow E$ be a Lipschitz
m-accretive operator with a closed domain $D(T)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying: (i) $0 \leq \alpha_n, \beta_n < 1$, $n \geq 0$; (ii) $\lim_{n \to \infty} \alpha_n = 0$, $\lim_{n \to \infty} \beta_n = 0$; (iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then for any $x_0 \in D(T)$, the sequence $\{p_n\}$ in $E$ generated from $x_0$ by
\[
p_n = (1 - \alpha_n)x_n + \alpha_n(f - TQy_n), \quad n \geq 0 \\
y_n = (1 - \beta_n)x_n + \beta_n(f - Tx_n), \quad n \geq 0 \\
x_n = Qp_{n-1}, \quad n \geq 1
\]
converges strongly to the unique solution of $x + Tx = f$.

**THEOREM CO2.** ([13], Theorem 5, p. 234). Let $E$ be a real Banach space which is both uniformly convex and uniformly smooth. Let $T : D(T) \subseteq E \to E$ be an m-accretive operator with closed domain $D(T)$, and bounded range $R(T)$. Let $\{c_n\}$ in $(0,1)$ be a real sequence satisfying: (i) $\lim_{n \to \infty} c_n = 0$; (ii) $\sum_{n=0}^{\infty} c_n(1 - c_n) = \infty$; (iii) $\sum_{n=0}^{\infty} c_n b(c_n) < \infty$. Then for any $x_0 \in D(T)$, the sequence $\{p_n\}$ in $E$ generated from $x_0$ by
\[
p_n = (1 - c_n)x_n + c_n(f - TQy_n), \quad n \geq 0 \\
x_n = Qp_{n-1}, \quad n \geq 1
\]
converges strongly to the unique solution of the equation $x + Tx = f$.

The iteration method (5) has also recently been studied by Xie Ping Ding [17] who proved the following Theorem:

**THEOREM XPD.** [17] Suppose $E$ is a real Banach space which is both uniformly convex and uniformly smooth, $T : D(T) \subseteq E \to E$ is an m-accretive operator with closed domain $D(T)$ and bounded range $R(T)$. Suppose $\{u_n\}$, $\{v_n\}$ are two sequences in $E$ and $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying: (i) $\sum_{n=0}^{\infty} ||u_n|| < \infty$, $\lim_{n \to \infty} ||u_n|| = 0$; (ii) $\lim_{n \to \infty} \beta_n = 0$; (iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (iv) $\lim_{n \to \infty} \alpha_n b(\alpha_n) = 0$. Then for any $x_0 \in D(T)$ and $f \in E$, the sequence $\{p_n\}$ and $\{x_n\}$ generated from $x_0$ by
\[
x_{n+1} = Qp_n \\
p_n = (1 - \alpha_n)x_n + \alpha_n(f - TQy_n) + u_n \\
y_n = (1 - \beta_n)x_n + \beta_n(f - Tx_n) + v_n, \quad n \geq 0
\]
converge strongly to the unique solution of $x + Tx = f$.

In Theorem CO2 and XPD, ‘$b$’ is a function that depends on the geometry of $E$ (see, e.g., [13, 17]).
It is our purpose in this paper to prove Theorems which significantly generalize and improve the results of Chidume and Osilike [13], Xie Ping Ding [17] and a host of other authors. In particular, the condition that \( T \) has bounded range (imposed in Theorem CO2 and in Theorem XPD) will be replaced by a weaker condition; the dependence of the iteration parameter, \( \alpha_n \), on the geometry of the underlying Banach space (condition (iii) in Theorem CO2 and condition (iv) in Theorem XPD) which restricts applications will not be needed in our Theorems. Furthermore, the requirement that \( T \) be Lipschitz imposed in Theorem CO1 will not be needed. In fact, no continuity assumption whatsoever will be imposed on our main Theorems. Finally our method of proof is of independent interest.

2 Preliminaries

In the sequel we shall need the following preliminaries and results. Let \( E \) be a Banach space. We shall denote by \( J \) the normalized duality mapping from \( E \) to \( 2^E^* \) given by

\[
Jx = \{ j \in E^* : \langle x, j \rangle = ||j||^2 = ||x||^2 \}
\]

where \( \langle ., . \rangle \) denotes the generalized duality pairing. If \( E^* \) is uniformly convex, then \( J \) is single-valued and is uniformly continuous on bounded sets.

As a consequence of a result of Kato [22], a mapping \( T \) with domain \( D(T) \) and range \( R(T) \) in \( E \) is accretive (see e.g., [22]) if for each \( x,y \in D(T) \) there exists \( j \in J(x - y) \) such that \( \langle Tx - Ty, j \rangle \geq 0 \). For \( p > 1 \), following [29], we shall associate the generalized duality map \( J_p \) from \( E \) to \( 2^{E^*} \) defined by

\[
J_p(x) = \{ j \in E^* : \langle x, j \rangle = ||x||^p, ||j|| = ||x||^{p-1} \}
\]

Observe that \( J_2 \) is the usual normalized duality map, \( J \) on \( E \). It is well known (e.g., [29]) that

\[
J_p(x) = ||x||^{p-2}J(x), \quad x \neq 0
\]

A Banach space \( E \) is called smooth if, for every \( x \in E \) with \( ||x|| = 1 \), there exists unique \( j \in E^* \) such that \( ||j|| = j(x) = 1 \) (see, e.g., [15]). The modulus of smoothness of \( E \) is the function

\[
\rho_E : [0, \infty) \rightarrow [0, \infty),
\]

defined by

\[
\rho_E(\tau) = \sup \left\{ \frac{1}{2} (||x + y|| + ||x - y||) - 1 : x, y \in E, ||x|| = 1, ||y|| = \tau \right\}.
\]
The Banach space E is called uniformly smooth (e.g., [30]) if
\[ \lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0, \]
and for \( q > 1 \), E is said to be \( q \)-uniformly smooth if there exists a constant \( c > 0 \) such that
\[ \rho_E(\tau) \leq c\tau^q, \quad \tau \in [0, \infty). \]
It is well known (see, e.g., [29]) that
\[ \ell_p \text{ uniformly smooth, if } 1 < p < 2 \]
\[ \ell_p \text{ uniformly smooth, if } p > 2. \]

The Banach space E is called uniformly convex if given any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for all \( x, y \in E \) with \( \|x\| < 1, \|y\| < 1 \) and \( \|x - y\| \geq \varepsilon \) we have
\[ \left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta. \]

It is well known that \( L_p \) spaces \( (1 < p < \infty) \) are uniformly convex.

In [30], the following result which will be needed in the sequel is proved.

**Lemma 2.** [30] Let E be a real uniformly smooth Banach space. Then
\[ \|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + D\max\{\|x\| + \|y\|, \frac{c}{2}\} \rho_E(\|y\|) \]
for every \( x, y \in E \) where \( D \) and \( c \) are positive constants.

**Lemma 3.** [32] Let E be a Banach space and \( T : D(T) \subseteq E \to E \) be an \( m \)-accretive operator. Then, for any given \( f \in E \), the equation \( x + Tx = f \) has a unique solution.

### 3 Main Results

Let E be a real Banach space which is uniformly smooth and uniformly convex. Let 
\( T : D(T) \subseteq E \to E \) be a bounded \( m \)-accretive operator with closed domain \( D(T) \). Define
\( S : D(T) \to E \) by \( Sx = x + Tx - f \). Let \( Q \) be the operator defined in Lemma 1. For arbitrary \( x_0 \in D(T) \) and \( f \in E \), define the sequences \( \{x_n\}, \{y_n\} \) and \( \{p_n\} \) by
\[ \begin{align*}
  x_{n+1} &= Qp_n \\
p_n &= x_n - \alpha_n SQy_n - \alpha_n \beta_n Sx_n + u_n \\
y_n &= x_n - \beta_n Sx_n + v_n, \quad n \geq 0
\end{align*} \tag{6} \]
where \( \{u_n\} \) and \( \{v_n\} \) are two sequences in \( E \) and \( \{\alpha_n\} \) and \( \{\beta_n\} \) two real sequences satisfying the following conditions:

\[
(i) \quad 0 < \alpha_n \leq T(x_{n_0}) := \min\{\beta, \frac{\|Sx_{n_0}\|}{2K(x_{n_0})}\}, \ n > 0
\]

\[
(ii) \quad 0 \leq \beta_n \leq T_1(x_{n_0}) := \min\{\beta, T(x_{n_0}), \frac{\delta}{3K(x_{n_0})}\}, \ n \geq 0
\]

\[
(iii) \quad \sum_{n=0}^{\infty} \alpha_n = \infty
\]

\[
(iv) \quad \alpha_n \to 0, \beta_n \to 0 \quad \text{as} \quad n \to \infty.
\]

\[
(v) \quad \|v_n\| \leq \min\{\|Sx_{n_0}\|, \frac{\delta}{2}\}, \|v_n\| \to 0 \quad \text{as} \quad n \to \infty
\]

\[
\|u_n\| \leq \min\{K(x_{n_0})\alpha_n, \frac{\max\{7\|Sx_{n_0}\|, \frac{\delta}{2}\}}{4\|Sx_{n_0}\|} - \rho_E(3\alpha_nK(x_{n_0}))\}
\]

where \( K(x_{n_0}) := \sup\{\|Sx\| : \|x - x_{n_0}\| \leq 7\|Sx_{n_0}\|\} \), \( n_0 \) is the smallest positive integer such that \( Sx_{n_0} \neq 0 \), and

\[
\beta := \max\{\beta > 0 : \beta^{-1}\rho_E(3\beta K(x_{n_0})) \leq \frac{\|Q_{y_{n_0+1}} - x^*\|^2}{3\max\{7\|Sx_{n_0}\|, \frac{\delta}{2}\}}\}
\]

and \( \delta \) is some positive fixed constant such that \( \|x - y\| < \delta \) implies \( \|Jx - Jy\| < \frac{\|Q_{y_{n_0+1}} - x^*\|^2}{2K(x_{n_0})} \) for all \( x, y \in B(0, 7\|Sx_{n_0}\|). \) With these notations we prove the following Theorem.

**THEOREM 4.** Let \( E \) be a real Banach space which is uniformly smooth and uniformly convex. Let \( T : D(T) \subseteq E \to E \) be a bounded \( m \)-accretive operator with closed domain \( D(T) \). Then for any \( x_0 \in D(T) \) and \( f \in E \), each of the sequences \( \{p_n\}, \{x_n\} \) and \( \{y_n\} \) generated from \( x_0 \) by (6) converges strongly to the unique solution of the equation \( x + Tx = f \).

**Proof.** Since \( T \) is \( m \)-accretive, by Lemma 3 the equation \( x + Tx = f \) for any \( f \in E \) has a unique solution \( x^* \in D(T) \). Observe that \( x^* \) is the unique solution of the equation \( Sx = 0 \) iff it is the unique solution of \( x + Tx = f \). By Lemma 1 the sequences \( \{x_n\}, \{p_n\} \) and \( \{y_n\} \) are well defined.

Moreover, \( S \) has the following property:

\[
\langle Sx - Sy, J(x - y) \rangle \geq \|x - y\|^2 \quad \text{for} \quad x, y \in D(T) = D(S)
\]

so that \( \|x - x^*\| \leq \|Sx\| \). In particular, we have

\[
\|x_n - x^*\| \leq \|Sx_n\| \quad \text{for any} \quad n \geq 0
\]
Clearly, \( ||S_{x_{n_0}}|| \leq K(x_{n_0}) \) and \( K(x_{n_0}) > 0 \). Since \( E \) is uniformly smooth we have \( \lim_{\tau \to 0} \frac{p_E(x)}{\tau} = 0 \). Thus we can choose \( \beta \) such that \( \beta^{-1}p_E(3\beta K(x_{n_0})) \leq \frac{||Qy_{n_0+1} - x^*||^2}{3\text{Dmax}(|S_{x_{n_0}}|, \frac{1}{2})} \). Moreover, \( J \) is uniformly continuous on the open ball \( B(0, 7||S_{x_{n_0}}||) \). Hence for \( \epsilon = \frac{||Qy_{n_0+1} - x^*||^2}{2K(x_{n_0})} \) there exists a \( \delta > 0 \) such that for \( x, y \in B(0, 7||S_{x_{n_0}}||) \), with \( ||x - y|| < \delta \) we have \( ||Jx - Jy|| < \frac{||Qy_{n_0+1} - x^*||^2}{2K(x_{n_0})} \).

Now we show that \( \{x_n\}, \{y_n\} \) and \( \{p_n\} \) are bounded and converge strongly to \( x^* \). We know that by (8), \( ||x_{n_0} - x^*|| \leq ||S_{x_{n_0}}|| \). By (6) and property of the map \( Q \),

\[
||Qy_{n_0} - x^*|| \leq ||y_{n_0} - x^*|| \leq ||x_{n_0} - x^*|| + \beta_{n_0}||S_{x_{n_0}}|| + ||v_{n_0}|| \\
\leq ||S_{x_{n_0}}|| + ||S_{x_{n_0}}|| + ||S_{x_{n_0}}|| = 3||S_{x_{n_0}}||
\]

(by (ii), then (i) and also (iv))

Therefore

\[
||Qy_{n_0} - x_{n_0}|| \leq ||y_{n_0} - x^*|| + ||x^* - x_{n_0}|| \leq 3||S_{x_{n_0}}|| + ||S_{x_{n_0}}|| = 4||S_{x_{n_0}}||
\]

and hence \( ||SQy_{n_0}|| \leq K(x_{n_0}) \).

Consequently,

\[
||x_{n_0+1} - x^*|| \leq ||x_{n_0} - x^*|| + \alpha_{n_0}||SQ_{y_{n_0}}|| + \alpha_{n_0}||S_{x_{n_0}}|| + ||u_{n_0}|| \\
\leq ||S_{x_{n_0}}|| + ||S_{x_{n_0}}|| + ||S_{x_{n_0}}|| + ||S_{x_{n_0}}|| = 4||S_{x_{n_0}}||
\]

(by (i), (ii), (v), \( ||S_{x_{n_0}}|| \leq K(x_{n_0}) \) and \( ||SQ_{y_{n_0}}|| \leq K(x_{n_0}) \))

and

\[
||x_{n_0+1} - x_{n_0}|| \leq ||x_{n_0+1} - x^*|| + ||x_{n_0} - x^*|| \leq 4||S_{x_{n_0}}|| + ||S_{x_{n_0}}|| = 5||S_{x_{n_0}}||
\]

So, \( ||S_{x_{n_0+1}}|| \leq K(x_{n_0}) \).

Thus we get, by (6),

\[
||Qy_{n_0+1} - x^*|| \leq ||y_{n_0+1} - x^*|| \\
\leq ||x_{n_0+1} - x^*|| + \beta_{n_0+1}||S_{x_{n_0+1}}|| + ||v_{n_0+1}|| \\
\leq 4||S_{x_{n_0}}|| + ||S_{x_{n_0}}|| + ||S_{x_{n_0}}|| = 6||S_{x_{n_0}}||
\]

and

\[
||Qy_{n_0+1} - x_{n_0}|| \leq ||y_{n_0+1} - x^*|| + ||x^* - x_{n_0}|| \\
\leq 6||S_{x_{n_0}}|| + ||S_{x_{n_0}}|| = 7||S_{x_{n_0}}||
\]
Hence \( ||SQy_{n_0+1}|| \leq K(x_{n_0}) \).

Using (6), (7), and the inequality of Lemma 2, we obtain the following estimates

\[
||x_{n_0+2} - x^*||^2 = ||Qp_{n_0+1} - x^*||^2 \leq ||p_{n_0+1} - x^*||^2 \\
= ||(x_{n_0+1} - x^*) - \alpha_{n_0+1}(SQy_{n_0+1} - Sx^*) - \alpha_{n_0+1}\beta_{n_0+1}(Sx_{n_0+1} - Sx^*) + u_{n_0+1}||^2 \\
\leq ||x_{n_0+1} - x^*||^2 - 2\alpha_{n_0+1}||SQy_{n_0+1} - Sx^*||J(x_{n_0+1} - x^*) \\
+ 2\langle u_{n_0+1}, J(x_{n_0+1} - x^*) \rangle \\
+ D_{max}\{||x_{n_0+1} - x^*|| + \alpha_{n_0+1}||SQy_{n_0+1}|| + \alpha_{n_0+1}\beta_{n_0+1}||Sx_{n_0+1}|| + ||u_{n_0+1}||, \frac{c}{2}\} \rho_{E}(\alpha_{n_0+1}||SQy_{n_0+1}|| + \beta_{n_0+1}||Sx_{n_0+1}|| + K(x_{n_0})), (by \ property \ (v) ) \\
\leq ||x_{n_0+1} - x^*||^2 - 2\alpha_{n_0+1}||SQy_{n_0+1} - Sx^*||J(x_{n_0+1} - x^*) - J(Qy_{n_0+1} - x^*) \\
- 2\alpha_{n_0+1}||SQy_{n_0+1} - Sx^*||J(Qy_{n_0+1} - x^*) \\
+ 2\langle u_{n_0+1}, J(x_{n_0+1} - x^*) \rangle + D_{max}\{7||Sx_{n_0}||, \frac{c}{2}\} \rho_{E}(3\alpha_{n_0+1}K(x_{n_0}))) \\
\leq ||x_{n_0+1} - x^*||^2 + 2\alpha_{n_0+1}||SQy_{n_0+1}||\cdot||J(x_{n_0+1} - x^*) - J(Qy_{n_0+1} - x^*)|| \\
- 2\alpha_{n_0+1}||Qy_{n_0+1} - x^*||^2 + 2||u_{n_0+1}||\cdot||x_{n_0+1} - x^*|| \\
+ D_{max}\{7||Sx_{n_0}||, \frac{c}{2}\} \rho_{E}(3\alpha_{n_0+1}K(x_{n_0}))) \\
\leq ||x_{n_0+1} - x^*||^2 + 2\alpha_{n_0+1}||SQy_{n_0+1}||\cdot||J(x_{n_0+1} - x^*) - J(Qy_{n_0+1} - x^*)|| \\
- 2\alpha_{n_0+1}||Qy_{n_0+1} - x^*||^2 + 3D_{max}\{7||Sx_{n_0}||, \frac{c}{2}\} \rho_{E}(3\alpha_{n_0+1}K(x_{n_0}))) (by \ (v)) \quad (9)
\]

Clearly, by (ii), (v) and \( ||Sx_{n_0+1}|| \leq K(x_{n_0}) \) we obtain that

\[
||x_{n_0+1} - Qy_{n_0+1}|| \leq ||x_{n_0+1} - y_{n_0+1}|| \leq \beta_{n_0+1}||Sx_{n_0+1}|| + ||v_{n_0+1}|| < \frac{\delta}{3} + \frac{\delta}{2} < \delta
\]

and \( (x_{n_0+1} - x^*), (y_{n_0+1} - x^*) \in B(0, 7||Sx_{n_0}||) \) so that by the uniform continuity of \( J, \)
\[ ||SQy_{n_0+1}||\cdot||J(x_{n_0+1} - x^*) - J(Qy_{n_0+1} - x^*)|| \leq K(x_{n_0}) \frac{||Qy_{n_0+1} - x^*||^2}{2K(x_{n_0})} \]

Substituting this in (9) and using the fact that \( \frac{\rho_{E}(z)}{z} \) is nondecreasing, we have that

\[
||x_{n_0+2} - x^*||^2 \\
\leq ||x_{n_0+1} - x^*||^2 - \alpha_{n_0+1}||Qy_{n_0+1} - x^*||^2 + 3D_{max}\{7||Sx_{n_0}||, \frac{c}{2}\} \rho_{E}(3\alpha_{n_0+1}K(x_{n_0}))) \\
\leq ||x_{n_0+1} - x^*||^2 - \alpha_{n_0+1}\left\{||Qy_{n_0+1} - x^*||^2 - 3D_{max}\{7||Sx_{n_0}||, \frac{c}{2}\} \rho_{E}(3\beta K(x_{n_0}))) \right\} \\
\leq ||x_{n_0+1} - x^*||^2 - \alpha_{n_0+1}\left\{||Qy_{n_0+1} - x^*||^2 - 3D_{max}\{7||Sx_{n_0}||, \frac{c}{2}\} \cdot \frac{||Qy_{n_0+1} - x^*||^2}{3D_{max}\{7||Sx_{n_0}||, \frac{c}{2}\}} \right\}
\]

Hence \( ||x_{n_0+2} - x^*||^2 \leq ||x_{n_0+1} - x^*||^2 \). That is \( ||x_{n_0+2} - x^*|| \leq ||x_{n_0+1} - x^*|| \)

Similarly we can prove that
\[ |x_{n+1} - x^*| \leq |x_n - x^*| \leq \ldots \leq |x_{n_0+1} - x^*| \text{ for all } n \geq n_0+1. \] Therefore \( \lim_{n \to \infty} |x_n - x^*| \) exists and consequently \( \{Sx_n\} \) is bounded. Let \( \lim_{n \to \infty} |x_n - x^*| = r. \) By (6) we also have

\[
\begin{align*}
\lim_{n \to \infty} |x_n - x^*| &= \lim_{n \to \infty} (|x_n - x^*| - \beta_n |Sx_n| - |v_n|) \\
&\leq \inf_{n \to \infty} |y_n - x^*| \leq \sup_{n \to \infty} |y_n - x^*| \\
&\leq \lim_{n \to \infty} (|x_n - x^*| + \beta_n |Sx_n| + |v_n|) = r
\end{align*}
\]

So, \( \lim_{n \to \infty} |y_n - x^*| = r. \) Moreover, since

\[
|y_n - x_n| = \beta_n |Sx_n| + |v_n| \to 0 \text{ as } n \to \infty \quad \text{and}
\]

\[
\begin{align*}
\lim_{n \to \infty} |x_n - x^*| &\leq \lim_{n \to \infty} (|x_n - Qy_n| + |Qy_n - x^*|) \\
&\leq \lim_{n \to \infty} (|x_n - y_n| + |Qy_n - x^*|) = r.
\end{align*}
\]

we get

\[
\lim_{n \to \infty} |Qy_n - x^*| = r.
\]

Therefore we have that \( \{Qy_n\} \) bounded and hence \( \{SQy_n\} \) bounded. Consequently since

\[
|x_{n+1} - x^*| \leq |p_n - x^*| \leq |x_n - x^*| + \alpha_n |SQy_n| + \beta_n |Sx_n| + |u_n|
\]

we obtain that \( \{|p_n - x^*|\} \) is bounded and converges to \( r. \)

Now we want to show that \( r = 0. \) If this is not the case, we may assume \( r > 0. \) Since \( |Qy_n - x^*| \to r \text{ as } n \to \infty, \) we can choose a positive integer \( N_1 \) such that

\[
|Qy_n - x^*| > \frac{r}{2} \text{ for all } n \geq N_1.
\]

Then using (6), (7), Lemma 2 and the above relations for \( n \geq n_0+1 \) we obtain the following estimates:

\[
\begin{align*}
|x_{n+1} - x^*|^2 &\leq |p_n - x^*|^2 = |(x_n - x^*) - \alpha_n (SQy_n - Sx^*) - \alpha_n \beta_n (Sx_n - x^*) + u_n|^2 \\
&\leq |x_n - x^*|^2 - 2\alpha_n \langle SQy_n - Sx^*, J(x_n - x^*) \rangle + 2\alpha_n \langle J(x_n - x^*) \rangle \\
&\quad + Dmax\{||x_n - x^*|| + \alpha_n ||SQy_n|| + \alpha_n \beta_n ||Sx_n|| + ||u_n||, \frac{c}{2}\} \\
&\quad + \rho_E(\alpha_n ||SQy_n|| + \beta_n ||Sx_n|| + K(x_{n_0}))) \\
&\leq |x_n - x^*|^2 - 2\alpha_n \langle SQy_n - Sx^*, J(x_n - x^*) - J(Qy_n - x^*) \rangle \\
&\quad - 2\alpha_n \langle SQy_n - Sx^*, J(Qy_n - x^*) \rangle \\
&\quad + 2||u_n|| ||x_n - x^*|| + Dmax\{||Sx_{n_0}||, \frac{c}{2}\} \rho_E(\alpha_n 3K(x_{n_0}))
\end{align*}
\]
\[
\begin{align*}
&\leq ||x_n - x^*||^2 + 2\alpha_n(||SQy_n|| + ||J(x_n - x^*) - J(Qy_n - x^*)||) \\
&\quad - 2\alpha_n||Qy_n - x^*||^2 + 3D \max\left\{7||Sx_{n_0}||, \frac{c}{2} \right\} \rho_E(\alpha_n 3K(x_{n_0})) \\
&\leq ||x_n - x^*||^2 - \alpha_n \left\{ \frac{1}{2} r^2 - (2\alpha_n + 3D \max\left\{7||Sx_{n_0}||, \frac{c}{2} \right\} \rho_E(\alpha_n 3K(x_{n_0})) \right\} \\
&\text{for } n \geq N_1 \text{ where } \alpha_n = ||SQy_n|| + ||J(x_n - x^*) - J(Qy_n - x^*)||. \quad (10)
\end{align*}
\]

for \( n \geq N_1 \) where \( \alpha_n = ||SQy_n|| + ||J(x_n - x^*) - J(Qy_n - x^*)||. \) Since
\( ||Qy_n - x_n|| \leq \beta_n ||Sx_n|| + ||u_n|| \to 0 \) as \( n \to \infty \) and \( J \) is uniformly continuous on bounded subset of \( E \) we see that \( \alpha_n \leq ||K(x_{n_0})|| + ||J(x_n - x^*) - J(Qy_n - x^*)|| \to 0 \) as \( n \to \infty. \) Moreover, since \( \alpha_n \to 0 \) and so \( \frac{\rho_E(\alpha_n)}{\alpha_n} \to 0, \) we choose a positive integer \( N_2 \) such that \( 2\alpha_n + 3D \max\left\{7||Sx_{n_0}||, \frac{c}{2} \right\} \frac{\rho_E(\alpha_n 3K(x_{n_0}))}{\alpha_n} < \frac{1}{4} r^2 \) for all \( n \geq N_2. \) Then substituting this in (10) yields
\[
||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 - \alpha_n \left( \frac{r^2}{4} \right) \text{ for all } n \geq N_3 = \max\{N_1, N_2, n_{0+1}\}.
\]

This gives \( \frac{1}{4} \alpha_n r^2 \leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 \) for all \( n \geq N_3. \)

Therefore \( \frac{1}{4} r^2 \sum_{n=N_3}^{\infty} \alpha_n \leq ||x_{N_3} - x^*||^2 \) which contradicts \( \sum_{n=0}^{\infty} \alpha_n = \infty. \)

This contradiction shows that \( r = 0. \) Hence \( \{x_n\}, \{y_n\} \) and \( \{p_n\} \) strongly converge to \( x^*. \)

The proof of Theorem 4 is complete. \( \square \)

**COROLLARY 5.** Let \( E \) and \( T \) be as in Theorem 4. Let \( \{\alpha_n\}_{n=0}^{\infty} \) be a real sequence and \( \{u_n\}_{n=0}^{\infty} \) a sequence in \( E \) satisfying the following conditions:

(i) \( 0 < \alpha_n \leq T(x_{n_0}) = \min\{\beta, \frac{||Sx_{n_0}||}{2K(x_{n_0})}\} \)

(ii) \( \sum_{n=0}^{\infty} \alpha_n = \infty, \alpha_n \to 0 \) as \( n \to \infty \)

(iii) \( ||u_n|| \leq \min\{K(x_{n_0})\alpha_n, \frac{D \max\left\{7||Sx_{n_0}||, \frac{c}{2} \right\} \rho_E(\beta \alpha_n K(x_{n_0}))}{4||Sx_{n_0}||}\} \)

where \( K(x_{n_0}) \) and \( \beta \) are as in Theorem 4. Then, for any \( x_0 \in D(T) \) the sequences \( \{x_n\} \) and \( \{p_n\} \) generated from \( x_0 \) by

\[
\begin{align*}
x_{n+1} &= Qp_n \\
p_n &= x_n - \alpha_n Sx_n + u_n, n \geq 0.
\end{align*}
\]

converge strongly to the unique solution of the equation \( x + Tx = f \) for any \( f \in E. \)

**Proof.** This follows from Theorem 4 with \( \beta_n = 0 \) and \( v_n = 0 \) for every \( n. \)
REMARK 6. It is clear that if an operator has a bounded range then it is bounded. Hence Theorem 4 is a significant generalization of Theorem CO2 in the following sense:

(i) The bounded range condition in Theorem CO2 is weakened to the boundedness of T,

(ii) the dependence of the iteration parameter $c_n$ on the geometry of the Banach space (condition (iii) in Theorem CO2 ) E is eliminated in our Theorem:

(iii) The iteration method of Theorem 4 above includes error terms $u_n$, $v_n$. In fact, the iteration method of Theorem CO2 is a special case of that of Theorem 4 in which $\beta_n = 0$, $u_n = 0$, and $v_n = 0$ for each $n$.

Theorem 4 also extends and generalizes Theorem XPD in the sense of (i) and (ii). Furthermore, since $x + Tx = f$ has a solution in $D(T)$, the Lipschitz continuity condition on T implies the boundedness of T and, since every q-uniformly smooth Banach space is uniformly smooth, Theorem 4 also generalizes Theorem CO1 to a more general class of operators and to the more general uniformly convex and uniformly smooth Banach spaces. Finally, our method of proof which is different from that used in [13] and [17] is of independent interest.

In Theorem 4, if $D(T) = E$, the map Q becomes the identity map on E will not be necessary and E need not be uniformly convex. In particular, we have the following Theorem.

THEOREM 7. Let E be a uniformly smooth Banach space, $T : E \longrightarrow E$ be a bounded m-accretive operator. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in E and $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences as in Theorem 4, where the map Q is replaced by I, the identity map on E. Then for any $x_0 \in E$ and $f \in E$, the sequences $\{x_n\}$ and $\{y_n\}$ generated from $x_0$ by

\[
\begin{align*}
\Delta \{ \quad & x_{n+1} = x_n - \alpha_n S y_n - \alpha_n \beta_n S x_n + u_n \\
& y_n = x_n - \beta_n S x_n + v_n, n \geq 0 \\
\end{align*}
\]

converge strongly to the unique solution of the equation $x + Tx = f$.

Proof. Obvious.

COROLLARY 8. Let E be a uniformly smooth Banach space, $T : E \longrightarrow E$ be a bounded continuous accretive operator. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in E and $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences as in Theorem 4, where the map Q is replaced by I, the identity map on E. Then for any $x_0 \in E$ and $f \in E$, the sequences $\{x_n\}$ and $\{y_n\}$ generated from
Let $x_0$ by
\[
(\Delta) \quad \begin{cases} 
  x_{n+1} = x_n - \alpha_n S y_n - \alpha_n \beta_n S x_n + u_n \\
  y_n = x_n - \beta_n S x_n + v_n, & n \geq 0 
\end{cases}
\]
converge strongly to the unique solution of the equation $x + Tx = f$.

**Proof.** By Martin [25] $T$ is m-accretive, thus the equation $x + Tx = f$ has a unique solution $x^* \in E$. The rest of the argument follows as in the proof of Theorem 4.

We now turn our attention to convergence Theorems in approximating a solution of the equation $x - \lambda Tx = f$, where $T : D(T) \subseteq E \rightarrow E$ is m-dissipative and $\lambda$ is a positive real constant. In particular, we prove the following Theorem.

**THEOREM 9.** Let $E$ be a real Banach space which is both uniformly convex and uniformly smooth. Let $T : D(T) \subseteq E \rightarrow E$ be a bounded m-dissipative operator with closed domain $D(T)$. Let \( \{u_n\}, \{v_n\} \) be two sequences in $E$ and \( \{\alpha_n\}, \{\beta_n\} \) be two real sequences satisfying (i)-(iv) of Theorem 4 with $S$ replaced by $S'$. Then for any $f \in E, x_0 \in D(T)$ and $\lambda > 0$, the iterative sequences $\{x_n\}, \{y_n\}$ and $\{p_n\}$ generated from $x_0$ by
\[
\begin{align*}
  x_{n+1} &= Q p_n \\
  p_n &= x_n - \alpha_n S' Q y_n - \alpha_n \beta_n S' x_n + u_n \\
  y_n &= x_n - \beta S' x_n + v_n, & n \geq 0 
\end{align*}
\]
converge strongly to the unique solution of the equation $x - (\lambda T)x = f$ where $S'x = (I - \lambda T)x - f$.

**Proof.** The existence of the unique solution follows from the m-dissipativity of $T$. The rest of the argument now follows as in the proof of Theorem 4 since $(-\lambda T)$ is accretive.

**COROLLARY 10.** Let $T, E$ and $D(T)$ be as in Theorem 9. Let $\{u_n\}$ be a sequence in $E$ and $\{\alpha_n\}$ be a sequence satisfying the following conditions:

\[
\begin{align*}
  (i) & \quad 0 < \alpha_n \leq T(x_{n_0}) = \min \{ \beta, \frac{||S'x_{n_0}||}{2K(x_{n_0})} \} \\
  (ii) & \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \alpha_n \rightarrow 0 \text{ as } n \rightarrow \infty \\
  (iii) & \quad ||u_n|| \leq \min \{ K(x_{n_0}) \alpha_n, \frac{D \max \{ T ||S'x_{n_0}||, \frac{c}{2} \} \beta E(3\alpha_n K(x_{n_0}))}{4 ||S'x_{n_0}||} \} 
\end{align*}
\]
where \( K(x_{n_0}) \) and \( \beta \) are as in Theorem 9. Then for any \( f \in E, \ x_0 \in D(T) \) and \( \lambda > 0 \) then the Mann type iterative sequences \( \{x_n\} \) and \( \{p_n\} \) generated from \( x_0 \) by

\[
x_{n+1} = Qp_n \quad p_n = x_n - \alpha_nS'x_n + u_n, \quad n \geq 0
\]

converge strongly to the unique solution of the equation \( x - \lambda Tx = f \).

**Proof.** Obvious from Theorem 9 with \( v_n = 0 \) and \( \beta_n = 0 \) for every \( n \).

We can use the same method of Theorem 9 and Corollary 10 to restate the results of \( m \)-accretive operators in terms of \( m \)-dissipative operators for the equation \( x - \lambda Tx = f \).

For example, Theorem 7 can be stated for dissipative operators as follows.

**THEOREM 11.** Let \( E \) be a uniformly smooth real Banach space and \( T : E \rightarrow E \) be a bounded continuous dissipative operator. Let \( \{u_n\}, \{v_n\}, \{\alpha_n\}, \{\beta_n\} \) be as in Theorem 7 with \( S \) replaced by \( S' \). Then given any \( x_0 \in E \) and \( f \in E \) the sequence \( \{x_n\} \) generated from \( x_0 \) by

\[
x_{n+1} = x_n - \alpha_nS'y_n - \alpha_n\beta_nS'x_n + u_n \\
y_n = x_n - \beta_nS'x_n + v_n, \quad n \geq 0
\]

converges strongly to the unique solution of the equation \( x - \lambda Tx = f \).

**Proof.** \( T \) is bounded, continuous and dissipative imply that \((-\lambda T)\) is bounded, continuous and accretive. Hence the result follows from Theorem 7.

**REMARK 12.** Theorem 9, Corollary 10 and Theorem 11 are, in an obvious manner, significant generalizations of their corresponding results in [13].
References


