NEW GENERALIZATIONS OF FARKAS’ THEOREM
AND THEIR APPLICATIONS IN OPTIMAL CONTROL

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ABSTRACT

This paper deals with some new generalizations of Farkas’ theorem for a class of set-valued mappings with arbitrary convex cones in infinite-dimensional Banach spaces. A modified Farkas’ theorem with no closedness assumption is given. The generalized Gale alternative theorem in nonlinear programming is derived as an easy consequence. The results are applied to constrained controllability theory in Banach spaces as well as to some multiobjective optimization problem.

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1. INTRODUCTION

It is well known that Farkas’ theorem plays an important role in mathematical programming and optimization problems. The classical Farkas theorem [6] on linear systems gives a necessary and sufficient condition for solvability of the linear system

\[ \sum_{j=1}^{n} a_{ij} x_j = b_j, \quad x_j \geq 0, \quad 1 \leq i \leq m. \]

Over the years, this theorem has been generalized to systems involving polyhedral or arbitrary cones [1,7], to sublinear, generalized convex and then to convex processes [3–5], etc. In all those works, necessary and sufficient conditions for the solvability of the system with convex cones are derived by using a Hahn- Banach separation theorem.

It should be noticed that when extending Farkas’ theorem for a linear abstract system of the form

\[ Ax = b, \quad x \in S, \quad (1) \]

where \( S \) is a convex cone, a crucial closedness condition, namely \( A(S) \) is closed, is assumed to be hold (see, e.g., [3, 5]). Without this closedness condition only the weaker solvability can be derived from the usual Farkas theorem [2, 9], where the weaker solvability means that there exists a sequence \( x_n \in S \), such that \( Ax_n \to b \) as \( n \to \infty \). In this paper we will give a set-valued generalization of the Farkas theorem: instead of linear system (1) we consider an inclusion of the form:

\[ F(K) \subseteq T(S), \quad (2) \]

where \( F, T \) are convex set-valued functions; \( K, S \) are arbitrary convex cones in infinite-dimensional Banach spaces. The conditions for the solvability of system (2) (in the sense that for every \( x \in K \) there is \( y \in S \) such that \( F(x) \subseteq T(y) \)) are obtained in terms of the dual pairs involving the set-valued adjoint functions \( F^*, T^* \) and the positive dual cones \( K^*, S^* \). We provide a modification of Farkas’ theorem without the closedness assumption for set-valued convex systems in Banach spaces. The generalizations allow us to obtain useful applications in controllability problems for linear discrete-time systems with constrained controls [10–12] as well as in some multiobjective optimization problem.
A generalized Gale alternative theorem applied in nonlinear programming \cite{9, 15} is derived as easy consequences.

2. NOTATIONS AND PRELIMINARIES

We begin by adopting the notations, definitions and some preliminary results that will be used throughout the paper.

Let \(X, Y, Z\) be infinite-dimensional Banach spaces. Their topological dual spaces are denoted by \(X^*, Y^*\) and \(Z^*\), respectively. The value of functional \(x^* \in X^*\) at \(x \in X\) denotes by \(\langle x^*, x \rangle\). Let \(M \subseteq X\) be a nonempty set containing zero. The positive polar cone of \(M\) at zero is defined by

\[
M^* = \{x^* \in X^* : \langle x^*, x \rangle \geq 0, \forall x \in M\}.
\]

In the sequel, by \(\text{cl} M, \text{int} M\) and \(\text{sp} M\) we denote the closure, the interior and the linear hull of \(M\), respectively; \(\mathbb{R}^+\) denotes the set of all nonnegative real numbers.

We need the following important properties of the positive polar cone for later use.

**Proposition 2.1.** \cite{14} Let \(M, N\) be convex sets containing zero in \(X\). Then

(i) \(M^{**} = \text{cl} M\), if \(M\) is a convex cone.

(ii) If \(M \subseteq N\) then \(N^* \subseteq M^*\). The converse assertion holds if \(M, N\) are convex cone and \(N\) is closed.

Let \(f : X \rightarrow \mathbb{R}\) be a single-valued function, \(f(.)\) is convex if

\[
f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2),
\]

for all \(x_1, x_2 \in X, \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1\). \(f(x)\) is concave if \(-f(x)\) is a convex function. For a convex function \(f(.)\) we define its subdifferential \(\partial f(x)\) at \(x \in X\) by

\[
\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x \rangle, \forall y\}.
\]

Let \(F : X \rightarrow Y\) be a set-valued function. The graph, the domain and the inverse image of \(F\) are respectively defined by

\[
\text{gr} F = \{(x, y) \in (X \times Y) : y \in F(x)\},
\]
\[
\text{dom } F = \{ x \in X : \quad F(x) \neq \emptyset \},
\]
\[
F^{-1}(x) = \{ x \in X : \quad y \in F(x) \}.
\]

A set-valued function \( F(x) \) is said to be convex, closed iff its graph is a convex and closed set. If \( \text{dom } F = X \) we shall say that \( F \) is a strict function. In addition if \( \text{gr } F \) is a cone, then \( F \) is called a convex process.

In the sequel we denote by \( L(X,Y) \) the space of all linear and bounded operators, by \( \mathcal{L}(X,Y) \) the set of all convex processes mapping \( X \) into \( Y \), by \( \text{Ker } A \) the kernel of \( A \), by \( \text{Im } A \) the image of \( A \) and finally, by \( I \) the identity operator.

Associate with a set-valued function \( F \in \mathcal{L}(X,Y) \) we define its adjoint set-valued function \( F^* : Y^* \to X^* \) by

\[
x^* \in F^*(y^*) \iff \langle y^*, y \rangle \geq \langle x^*, x \rangle, \quad \forall (x,y) \in \text{gr } F.
\]

or equivalently,

\[
\text{gr } F^* = \{(y^*, x^*) : \quad (-x^*, y^*) \in (\text{gr } F)^* \}.
\]

It is clear that \( F^* \in \mathcal{L}(Y^*, X^*) \). For instance, if \( A \in L(X,Y) \) and \( K \) is a convex closed cone in \( X \), the set-valued function \( F(x) = Ax + K \) is a convex process and its adjoint is defined as

\[
F^*(y^*) = \begin{cases} 
A^* y^* & \text{if } y^* \in K^*, \\
\emptyset & \text{if not }.
\end{cases}
\]

When \( K = 0 \) we have \( F^* = A^* \), and the notion of adjoint is an extension of transposition of linear operators. As is known, the problem of finding the adjoint of set-valued functions, in general, is not easy (see Appendix). For later use we need the following result, the proof of which can be found in the author’s paper [13].

**Proposition 2.2.** [13] Let \( F \in \mathcal{L}(X,Y) \) and let \( M \subseteq X \) be a convex set containing zero. Then

\[
F(M)^* = F^*^{-1}(M^*).
\]

**Proposition 2.3.** Let \( F \in \mathcal{L}(X,Y), T \in \mathcal{L}(Z,Y) \) and let \( S \subseteq Z, K \subseteq X \) be convex cones. Then the following conditions are equivalent:

(i) \( T(S)^* \subseteq F(K)^* \).
(ii) $T^*(y^*) \cap S^* \neq \emptyset \implies F^*(y^*) \cap K^* \neq \emptyset$.

**Proof.** (i) $\implies$ (ii): Let $s^* \in T^*(y^*) \cap S^*$. Then $s^* \in S^*$ and $y^* \in T^{*-1}(s^*) \subseteq T^{*-1}(S^*)$.

Taking into account (i) and Proposition 2.2, we have

$$y^* \in T^{*-1}(S^*) = T(S)^* \subseteq F(K)^* = F^*(K^*).$$

Therefore, there is $k^* \in K^*$, such that

$$y^* \in F^{*-1}(k^*),$$

which implies $y^* \in F^*(K^*)$ and hence $k^* \in F^*(K^*) \cap K^*$.

The implication (ii) $\implies$ (i) is proved similarly.

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**3. GENERALIZATIONS OF FARKAS’ THEOREM**

We start with the following so-called range cone inclusion theorem for set-valued functions which forms the generalization of Farkas’ theorem.

**Theorem 3.1.** (Range cone inclusion theorem). Let $F \in \mathcal{L}(X, Y)$, $T \in \mathcal{L}(Z, Y)$ and let $S \subseteq Z, K \subseteq X$ be two convex cones. Assume that $T(S)$ is closed. Then the following conditions are equivalent.

(i) $F(K) \subseteq T(S)$.

(ii) $T^*(y^*) \cap S^* \neq \emptyset \implies F^*(y^*) \cap K^* \neq \emptyset$.

**Proof.** The proof is immediately followed by Propositions 2.1 and 2.3.

We are going to present a set-valued generalization of the Farkas theorem. Let $A \in \mathcal{L}(X, Y), T \in \mathcal{L}(Z, Y)$. Let $K \subseteq X, S \subseteq Z$ be two convex cones. We consider the following solvability problem

$$\text{for every } x \in K, \text{ there exists } z \in S : \ Ax \in T(z). \quad (SP)$$

**Theorem 3.2.** (Generalized Farkas’ theorem). Let $A \in \mathcal{L}(X, Y), T \in \mathcal{L}(Z, Y)$ and let $K \subseteq X, S \subseteq Z$ be convex cones. Assume that $T(S)$ is closed. Then problem (SP) is
solvable if and only if

\[ T^*(y^*) \cap S^* \neq \emptyset \implies A^*y^* \in K^*. \]

Proof. It suffices to consider the special case of Theorem 3.1 when \( F = A \).

As a special case of Theorem 3.2, we consider the solvability problem for a linear system of the form

\[ b \in T(z), \quad z \in S, \quad (LP) \]

where \( T \in \mathcal{L}(Z, Y) \), then taking \( X = R, K = R^+ \), the linear single-valued map \( A : R \to Y \) is given by \( A\lambda = \lambda b, \lambda \in R \), where \( b \in Y \), the cone \( A(K) \) is the ray \( \{\lambda b : \lambda \geq 0\} \), \( \langle A^*y^*, \lambda \rangle = \lambda \langle y^*, b \rangle \), the condition \( A^*y^* \in K^* \) is equivalent to \( \langle y^*, b \rangle \geq 0 \), and hence we have

**Corollary 3.1.** Let \( T \in \mathcal{L}(Z, Y) \) and let \( S \subseteq Z \) be a convex cone. Then a necessary (and sufficient if \( T(S) \) is closed) condition for the solvability of the problem (LP) is

\[ T^*(y^*) \cap S^* \neq \emptyset \implies \langle y^*, b \rangle \geq 0. \]

We have obtained the generalized Farkas theorem with the closedness assumption on \( T(S) \). We now want to provide a modified Farkas theorem without this closedness assumption. There was a result of [8], which states Farkas’ theorem without the closedness condition for a linear system over the cone of positive semidefinite matrices. Here we shall give a modified Farkas’ theorem extended to a more general class of systems in infinite-dimensional Banach spaces.

**Theorem 3.3.** (Modified Farkas’ theorem). Let \( T \in \mathcal{L}(X, Y) \) and let \( S \subseteq X \) be a convex closed cone. Assume that

\[ \exists f^* \in S^* : \text{the set } \{x \in S : \langle f^*, x \rangle \leq r \} \text{ is compact for every } r > 0. \quad (3) \]

Then the inclusion \( b \in T(x) \) has a solution \( x \in S \) if and only if

\[ (T^*(y^*) + \lambda f^*) \cap S^* \neq \emptyset, \quad \lambda \geq 0 \implies \langle y^*, b \rangle + \lambda \delta \geq 0, \]

for some \( \delta > 0 \).
Proof. The inclusion \( b \in T(x) \) has a solution in \( S \) iff the system

\[
\begin{align*}
b &\in T(x), \\
\langle f^*, x \rangle + \lambda &= \delta, \\
x &\in S, \quad \lambda \geq 0.
\end{align*}
\]

has a solution for some \( \delta > 0 \). Consider the set-valued function \( H : X \times R \to Y \times R \)
defined by

\[
H(x, \lambda) = \begin{pmatrix} T(x) \\ \langle f^*, x \rangle + \lambda \end{pmatrix}.
\]

It is easy to see that \( H \in \mathcal{L}(X \times R, Y \times R) \). Therefore, the problem is then equivalent
to the solvability of the system

\[
c \in H(z), \quad z \in S \times R^+,
\]

where \( c = (b, \delta) \), for some \( \delta > 0 \). On the other hand, for every \( \beta \in R^+ \) we can find (see Appendix)

\[
H^*(z^*) = \begin{pmatrix} T^*(y^*) + \beta f^* \\ (\infty, \beta) \end{pmatrix},
\]

where \( z^* = (y^*, \beta) \in Y^* \times R^+ \). We first prove that \( H(S \times R^+) \) is closed. Indeed, let \( \{y_n, \lambda_n\} \in H(S \times R^+) \), such that \( y_n \to y_0, \lambda_n \to \lambda_0 \). Then, there exist sequences

\[
s_n \in S, \beta_n \in R^+
\]
satisfying

\[
y_n \in T(s_n), \quad \langle f^*, s_n \rangle + \beta_n = \lambda_n.
\]

Since the sequence \( \{\langle f^*, s_n \rangle + \beta_n\} \) converges to \( \lambda_0 \), and since \( \langle f^*, s_n \rangle \geq 0, \beta_n \geq 0 \), we obtain that the sequences \( \langle f^*, s_n \rangle, \beta_n \) are bounded. Hence, taking into account the assumption (3), there exist subsequences \( s_{n_k}, \beta_{n_k} \) converging to some \( s_0, \beta_0 \) such that

\[
\langle f^*, s_0 \rangle + \beta_0 = \lambda_0, \quad y_0 \in H(s_0), \quad s_0 \in S.
\]

This proves the closedness of \( H(S \times R^+) \). The proof is then complete by using Theorem 3.2: the system (4) is consistent if and only if

\[
H^*(z^*) \cap S^* \neq \emptyset \implies \langle z^*, c \rangle \geq 0.
\]
or equivalently,
\[(T^*(y^*) + \lambda f^*) \cap S^* \neq \emptyset \implies (y^*, \delta) + \lambda \delta \geq 0,\]
as desired.

**Remark 3.1.** Note that a sufficient condition for (3) is that

\[S = \text{sp } B, \quad B \text{ is a convex compact set and } 0 \notin B.\]

Indeed, by a Hahn-Banach separation theorem, there exist a nonzero functional \(f^* \in X^*\) and a number \(\beta > 0\) such that
\[
\langle f^*, x \rangle \geq \beta, \quad \forall x \in B.
\]

Since \(S = \text{sp } B\), we have \(f^* \in S^*\). On the other hand, we can easily verify that
\[
\{x \in S : \langle f^*, x \rangle \leq r \} \subseteq [0, r/\beta]B.
\]

Since \(B\) is a convex compact set, we conclude that the condition (3) holds.

From the foregoing we see that without the closedness condition an additional condition is needed in the convex cone constraint and in the dual condition. Moreover, the case \(\lambda = 0\) covers the usual Farkas theorem related to the weaker solvability whereas the case \(\lambda > 0\) covers the additional condition required for the exact solvability of the system.

If the system is finite-dimensional, i.e., \(X = \mathbb{R}^n, Y = \mathbb{R}^m\), we will show that the condition (3) can be replaced by a more explicit assumption as follows.

Let \(\phi(x) : X \to R\) be a strongly convex continuous function (i.e., \(\lambda \phi(x) = \phi(\lambda x), \lambda \geq 0, \phi(x_1 + x_2) \leq \phi(x_1) + \phi(x_2), x_1, x_2 \in X\)) satisfying
\[
\exists \phi(x) : X \to R : \|x\| \leq \phi(x), \quad \forall x \in S.
\]

In this case, the convex set-valued function \(H(.)\) can be choosen by
\[
H(x, \lambda) = \begin{pmatrix} T(x) \\ \phi(x) + \lambda \end{pmatrix},
\]
and hence (see Appendix)

\[ H^*(z^*) = \left( T^*(y^*) + \lambda \partial \phi(0) \right), \]

where \( \partial \phi(0) \) is the subdifferential of \( \phi(.) \) at zero, \( z^* = (y^*, \lambda) \). Therefore by the same arguments that used in the proof of Theorem 3.3 we obtain

**Theorem 3.4.** Let \( T \in \mathcal{L}(R^n, R^m) \) and let \( S \subseteq R^n \) be a convex closed cone. Assume that the condition (5) holds. Then the system \( b \in T(x) \) has a solution \( x \in S \) if and only if

\[ (T^*(y^*) + \lambda \partial \phi(0)) \cap S^* \neq \emptyset, \lambda \geq 0 \Rightarrow \langle y^*, b \rangle + \lambda \delta \geq 0, \]

for some \( \delta > 0 \).

It is obvious that instead of the strongly convex function \( \phi(x) \) we can always take \( \phi(x) = \|x\| \), the condition (5) is satisfied and \( \partial \phi(0) = B^*(0) \), where \( B^*(0) \subseteq X^* \) is the unit ball. Therefore, combining with Remark 3.1 we have

**Corollary 3.2.** Let \( T \in \mathcal{L}(R^n, R^m) \) and let \( S \subseteq X \) be a convex closed cone. The linear system \( b \in T(x), x \in S \) is consistent if and only if

\[ (T^*y^* + \lambda B^*(0) \cap S^* \neq \emptyset, \lambda \geq 0 \Rightarrow \langle y^*, b \rangle + \lambda \delta \geq 0, \]

for some \( \delta > 0 \).

**Remark 3.2.** For linear systems with positive semidefinite matrices considered in [8], the condition (5) holds for \( \phi(x) = \langle I_X, x \rangle, x \in S \), where \( S \) is a cone of positive semidefinite matrices, \( I_X \) is the identity matrix of vector space \( X \) of symmetric real matrices with the duality bracket \( \langle x, y \rangle = \text{trace} \ (x.y), \ (x, y) \in X \times X \).

### 4. Applications

The main applications of Farkas theorem and its generalizations have been found in various problems of nonlinear programming and duality theory. In this section we will provide some further useful applications to obtaining the generalized Gale alternative therem, to constrained control problems and to some multiobjective optimization problem.
4.1. Generalized Gale alternative theorem

As is well known, the generalized Gale alternative theorem is derived by the Farkas theorem. In contrast to [5], we will provide another approach to obtaining this theorem by an easy consequence of Theorem 3.4.

**Theorem 4.1.** (Generalized Gale alternative theorem). Let $T : R^n \to R^m$ be a convex process and $z \in R^m$. Then exactly one of the following two conditions holds:

(a) System $z \in T(x)$ has a solution $x \in R^n$.

(b) $(0,-1) \not\in \text{cl} \bigcup_{y^* \in R^m} [T^*(y^*) \times \langle y^*, z \rangle]$.

**Proof.** If (a) does not hold, we prove that (b) holds. Indeed, by the modified Farkas theorem, Theorem 3.4, applied to $S = R^n$, not (a) is equal to:

$$0 \in T^*(y^*) + \lambda f^*, \quad \lambda \geq 0 \implies \langle y^*, z \rangle + \lambda \delta < 0,$$

If $\lambda = 0$ it is easily seen that $(0,-1) \in \text{Im} T^* \times \langle y^*, z \rangle$ which means (b). Now let $\lambda > 0$. Fix $f^*, \delta > 0, y^*$, taking a sequence $\lambda_n > 0$ tending to zero we have

$$0 \in T^*(y^*) + \lambda_n f^* \implies \langle y^*, z \rangle + \lambda_n \delta = \beta_n < 0.$$

Setting

$$\epsilon_n = \frac{1}{-\beta_n + \lambda_n \delta}, \quad f^*_n = -\epsilon_n \lambda_n f^*, \quad y^*_n = \epsilon_n y^*,$$

we see that $\epsilon_n > 0$ and

$$f^*_n \in T^*(y^*_n) \implies \langle y^*_n, z \rangle = -1,$$

where $\{\epsilon_n\}$ is bounded and $f^*_n \to 0$.

Therefore,

$$(0,-1) \not\in \text{cl} \bigcup_{y^* \in R^m} [T^*(y^*) \times \langle y^*, z \rangle].$$

Let now (a) be true. If (b) is also true, we will arrive at a contradiction. Indeed, from (b) it follows that there exist sequences $x^*_n, y^*_n$ such that

$$\lim x^*_n = 0, \quad \lim \langle y^*_n, z \rangle = -1, \quad x^*_n \in T^*(y^*_n).$$

Since (a) holds, $z \in T(x)$ we have

$$\langle y^*_n, z \rangle \geq \langle x^*_n, x \rangle.$$
Letting $n \to 0$ gives

\[-1 = \lim \langle y_n^*, z \rangle \geq \lim \langle x_n^*, x \rangle = 0,\]

which is a contradiction.

**Remark 4.1.** We can derive a generalization of the Gale alternative theorem in infinite-dimensional spaces using Theorem 3.2. In this case we require the closedness of $T(X)$, but the closure in condition (b) of Corollary 3.3 can be removed, i.e., we have

\[(0, -1) \in \bigcup_{y^* \in Y^*} [T^* (y^*) \times \langle y^*, z \rangle].\]

### 4.2. Constrained controllability

In this section we will show how the generalized Farkas’ theorem is applied to obtaining constrained controllability conditions for linear control systems.

Let $A(k) \in L(X, X), B(k) \in L(U, X), k = 0, 1, \ldots$. Consider a control system described by the following discrete-time equations

\[x(k + 1) = A(k)x(k) + B(k)u(k), \quad k = 0, 1, \ldots \tag{6}\]

where $u(k) \subseteq \Omega \subseteq U, \Omega$ is a nonempty set.

We say that control system (6) is globally null-controllable after $N$ steps if for every $x_0 \in X$ there exist controls $u(0), u(1), \ldots, u(N - 1), u(k) \in \Omega$ such that the solution $x(k)$ of (6) satisfies:

\[x(0) = x_0, \quad x(N) = 0.\]

Let us denote

\[P^{i-1}_j = A(i - 1)A(j + 1), i > j; \quad P^i_0 = I, \quad A_i = A(i) \ldots A(1)A(0).\]

Then for every controls $u(k)$ and initial state $x_0$, the solution $x(k)$ of (6) at $k^{th}$ step is given by

\[x(k) = A_kx_0 + \sum_{i=0}^{k-1} P^{k-i-1}_i B(i)u(i). \tag{7}\]

Let us set

\[Z = U \times \underbrace{U \times \cdots \times U}_{N \text{times}}, \quad \Omega^N = \Omega \times \underbrace{\Omega \times \cdots \times \Omega}_{N \text{times}}, \quad F = -A_N.\]
Define a linear operator $T \in L(Z,X)$ by setting

$$T = \sum_{i=0}^{N-1} P_i^{N-1} B(i).$$

**Theorem 4.2.** Assume that $\Omega \subseteq U$ is a convex set such that $T(\Omega^N)$ is a closed set. Then the control system (6) is globally null-controllable after $N$ steps if and only if

$$\bigcap_{k=0}^{N-1} (P_k^{N-k-1} B(k) \Omega)^* \subseteq \text{Ker } (-A_N)^*$$

(8)

**Proof.** From (7) it follows that the system (6) is globally null-controllable after $N$ steps iff

For every $x \in X$ there exists $u \in \Omega^N : \ F x \in T(u).$

(9)

Since the set $T(\Omega^N)$ is convex and closed, Theorem 3.2 is then applied to $K = X, S = \Omega^N,$ (9) holds iff

$$T^*(x^*) \cap S^* \neq \emptyset \implies F^* x^* = 0,$$

because of $X^* = \{0\}.$ Moreover, we note that the adjoint $T^* : X^* \to Z^*$ is defined by

$$T^*(x^*) = [B^*(N-1), B^*(N-2) A^*(N-1), ..., B^*(0) A^*(1) ... A^*(N-1)](x^*).$$

Consequently, the condition $T^*(x^*) \cap S^* \neq \emptyset$ is equal to the following relations:

$$B^*(N-1)x^* \cap \Omega^* \neq \emptyset,$$

$$B^*(N-2)A^*(N-1)x^* \cap \Omega^* \neq \emptyset,$$

$$\ldots$$

$$B^*(0)A^*(1) ... A^*(N-1)x^* \cap \Omega^* \neq \emptyset$$

or equivalently,

$$x^* \in \bigcap_{k=0}^{N-1} [P_k^{N-k-1} B(k) \Omega]^*.$$  

(11)

On the other hand, the condition $F^* x^* = 0$ is equal to

$$x^* \in \text{Ker } (-A_N)^*.$$  

(12)

Combining (11) and (12) proves (8).
Remark 4.2. If we assume that \( A(k), k = 0, 1, \ldots \) are surjective, i.e., \( \text{Im } A(k) = X \), then since \( \text{Ker } A^*(k) = 0 \), the condition (8) becomes

\[
\bigcap_{k=0}^{N-1} [P_k^{-N-k-1}B(k)\Omega]^* = \{0\}.
\]

(13)

In this case, it can be easily verified that the condition (13) is just the generalized controllability rank Kalman condition

\[
\text{sp } \{BU, ABU, \ldots, A^{N-1}BU\} = X,
\]

if \( \Omega = U, A(k) = A, B(k) = B \), and

\[
\text{rank } [B, AB, \ldots, A^{n-1}B] = n
\]

if \( \Omega = \mathbb{R}^n \) and \( A, B \) are constant matrices of appropriate dimensions.

We now consider a problem of controllability to a target set (see, e.g., [12]). Let \( M \subseteq X \) be a given nonempty target set.

We say that control system (6) is controllable to \( M \) after \( N \) steps if for every \( x_0 \in X \) there exist controls \( u(0), u(1), \ldots, u(N-1), u(k) \in \Omega \), such that the solution \( x(k) \) of (6) satisfies

\[
x(0) = x_0, \quad x(N) \in M.
\]

We define a set-valued function \( \mathcal{T} : Z \to X \) by setting

\[
\mathcal{T} = - \sum_{i=0}^{N-1} P_i^{N-i-1}B(i) + M.
\]

If \( M \) is a convex and closed cone, \( \Omega \) is a convex cone, then \( \mathcal{T} \in \mathcal{L}(Z, X) \). Let \( F = A_N \).

Control system (6) is controllable to \( M \) iff

\[
F(X) \subseteq \mathcal{T}(\Omega^N).
\]

Therefore, applying the generalized Farkas theorem in the same way as above we obtain

**Theorem 4.3.** Assume that \( M, \Omega \) are convex closed cones such that \( \mathcal{T}(\Omega^N) \) is a closed set. The the control system (6) is controllable to \( M \) after \( N \) steps if and only if

\[
\bigcap_{k=0}^{N-1} [-P_k^{-N-k-1}B(k)\Omega]^* \cap M^* \subseteq \text{Ker } (A_N)^*.
\]
Remark 4.3. It should be noted that to obtain the above controllability conditions we do not require any assumption on the surjectivity of $A(k)$ as well as on the interior or relative interior of $\Omega$ as in [10–12].

4.3. Multiobjective optimization

In this section we will give an application to some multiobjective optimization problem. For this, we first consider the solvability of a system of inequalities of the form:

$$Bx \leq b, \quad Ax \in M,$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n; B \in \mathbb{R}^{qn}; A \in \mathbb{R}^{pn}; M \subset \mathbb{R}^p$ is a convex cone; $b \in \mathbb{R}^q$.

The vector ordering relations $\leq, \preceq, <$ as in [9], are defined as

$$x \preceq y \iff x^i \leq y^i, \quad i = 1, 2, \ldots, n,$$

$$x \leq y \iff x \preceq y, \quad x \neq y,$$

$$x < y \iff x^i < y^i, \quad i = 1, 2, \ldots, n.$$

It is obvious that if system (14) has a solution then the system

$$Bx \leq b, \quad Ax \in M$$

has a solution. Setting

$$T(x) = \{y : Bx \leq y\}, \quad S = \{x : Ax \in M\},$$

if the system (14) has a solution then the system

$$b \in T(x), \quad x \in S$$

has a solution. We see that $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^q)$ and $S$ is a convex closed set, then by the generalized Farkas theorem, Corollary 3.1, applied to the last system, we obtain

$$T^*(q^*) \cap S^* \neq \emptyset \implies \langle q^*, b \rangle \geq 0,$$

where $^*$ denotes the transpose. Since

$$T^*(q^*) = \{B^*q^* : q^* > 0\}, \quad S^* = \{A^*m^* : m^* \in M^*\}$$

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we obtain

**Theorem 4.4.** If the system $(14)$ has a solution, then

$$B^*q^* - A^*m^* = 0, \quad \langle q^*, b \rangle \geq 0$$

for some $m^* \in M^*, q^* > 0$.

We now consider a multiobjective optimization problem of the form

$$\text{max } f(x) : x \in S = \{x : Ax \in M\}. \quad \text{(MOP)}$$

where $f(x) : R^n \to R^q$ is a convex continuously differentiable function; $A \in R^{pn}; M$ is a convex cone. Here "max" means finding efficient (Pareto optimal) solutions in the sense: a point $x_0$ is said to be an efficient solution of (MOP) if there is no point $x \in S$ such that $f(x) > f(x_0)$. It is obvious that if the problem (MOP) has an efficient solution $x_0$, then the system

$$\Delta f(x_0)^*(x - x_0) \leq 0, \quad x \in S \quad (15)$$

has a solution, where $\Delta f(x_0)$ denotes the Jacobian $(n \times q)$ matrix whose $i$th column is the gradient $\partial f_i$ at $x_0$. Therefore, from Theorem 4.4 applied to

$$B = \Delta f(x_0)^*, \quad b = \Delta f(x_0)^* x_0$$

it follows that if the system (15) is inconsistent then there are $q^* > 0, m^* \in M^*$ such that

$$B^*q^* - A^*m^* = 0,$$

$$\langle q^*, \Delta f(x_0)^*x_0 \rangle \geq 0.$$

Consequently, we have

**Theorem 4.5.** Under the assumptions of Theorem 4.4, if $x_0$ is an efficient solution of (MOP), then there are $m^* \in M^*, q^* > 0$ such that

$$\Delta f(x_0)^*q^* - A^*m^* = 0,$$

$$\langle q^*, \Delta f(x_0)^*x_0 \rangle \geq 0.$$

**Remark 4.4.** It is worth to note that to obtain necessary conditions for the problem (MOP) we do not require $S$ to be a cone, it is enough a convex set containing zero due to (ii) of Proposition 2.1. Hence, if we take $S = \{x : g(x) \leq 0\}$, or $S = \{x : Ax \leq a\},$
then from Theorems 4.4 and 4.5 we can derive some results obtained in [15] as special cases.

5. CONCLUSIONS

We have established some generalizations of Farkas’ theorem for set-valued convex closed functions with arbitrary convex cones in infinite-dimensional Banach spaces. A modified Farkas theorem is obtained without the closedness assumption. The results are applied to controllability problems of discrete-time systems with constrained controls and to some multiobjective optimization problem. Generalizations of the Gale alternative theorem in nonlinear programming are derived.

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APPENDIX

In this appendix we illustrate an approach to finding the adjoint of a class of convex set-valued functions. Let us consider a convex closed set-valued function $H(x, \alpha) : X \times R \to Y \times R$ defined by

$$H(x, \alpha) = \left( \begin{array}{c} T(x) \\ \phi(x) + \alpha \end{array} \right),$$

where $T(x) : X \to Y$ is a convex closed set-valued function, $0 \in T(0)$, and $\phi(x) : X \to R$ is a convex continuous function, $0 \in \phi(0)$. We show that for every $\beta \in R^+$

$$H^*(y^*, \beta) = \left( \begin{array}{c} T^*(y^*) + \beta \partial \phi(0) \\ (-\infty, \beta] \end{array} \right).$$

Indeed, let

$$(x^*, \lambda) \in \left( \begin{array}{c} T^*(y^*) + \beta \partial \phi(0) \\ (-\infty, \beta] \end{array} \right).$$
Then \( \lambda \in (-\infty, \beta] \) and \( \exists \phi^* \in \partial \phi(0) : \ x^* \in T^*(y^*) + \beta \phi^* \). By definition of the adjoint \( T^*(.) \), we have
\[
\langle y^*, y \rangle + \beta \langle \phi^*, x \rangle \geq \langle x^*, x \rangle, \quad \forall x \in X, y \in T(x).
\]
Since \( \phi(x) \geq \langle \phi^*, x \rangle, \beta \geq 0, \lambda \leq \beta \), for all \( x \in X, \alpha \geq 0 \), we have
\[
\langle y^*, y \rangle + \beta (\phi(x) + \alpha) \geq \langle x^*, x \rangle + \lambda \alpha,
\]
or equivalently, \( (x^*, \lambda) \in H^*(y^*, \beta) \).

Now let \( (x^*, \lambda) \in H^*(y^*, \beta) \). By definition of the adjoint \( H^*(.) \) we have
\[
\langle y^*, y \rangle + \beta (\phi(x) + \alpha) \geq \langle x^*, x \rangle + \lambda \alpha, \tag{16}
\]
for all \( \alpha \in R^+, x \in X, y \in T(x) \). Since \( 0 \in T(0), 0 = \phi(0) \), letting \( y = x = 0 \), from (16) it follows that \( \beta \alpha \geq \lambda \alpha \), i.e., \( \lambda \in (-\infty, \beta] \). On the other hand, (16) holds for all \( \alpha \in R^+ \), hence taking \( \alpha = 0 \), we have
\[
\beta \phi(x) \geq -\langle y^*, y \rangle + \langle x^*, x \rangle, \quad \forall x \in X, y \in T(x).
\]
Setting
\[
M = \{(x, y, t) \in X \times Y \times R : \ t > \beta \phi(x)\},
\]
\[
N = \{(v, z, s) \in X \times Y \times R : \ s \leq -\langle y^*, z \rangle + \langle x^*, v \rangle, \ z \in X, z \in T(v)\},
\]
we see that \( M, N \) are convex sets and \( \mathrm{int} M \neq \emptyset \) due to \( \phi(.) \) is a convex, continuous function. Besides, \( M \cap N \neq \emptyset \). Therefore, by a Hahn-Banach separation theorem of convex sets, there exist \( (v^*, z^*, q) \in X^* \times Y^* \times R \) not all zero such that
\[
\langle v^*, x \rangle + \langle z^*, y \rangle + qt \geq \langle v^*, v \rangle + \langle z^*, z \rangle + qs, \tag{17}
\]
for all \( (x, y, t) \in M; (v, z, s) \in N \). We claim that \( q > 0 \). If \( q < 0 \), then letting \( t \to +\infty \) the left hand side of (17) tends to \(-\infty \) which is impossible, so \( q \geq 0 \). If \( q = 0 \), then (17) becomes
\[
\langle v^*, x \rangle + \langle z^*, y \rangle \geq \langle v^*, v \rangle + \langle z^*, z \rangle, \tag{18}
\]
for all \( (x, y) \in X \times Y; (v, z) \in \text{gr} T \). There are two cases:

a) If \( z^* = 0 \). Letting \( v = 0 \in T(0) \) from (18) we have \( \langle v^*, x \rangle \geq 0 \), for all \( x \in X \) which follows \( v^* = 0 \). Thus, we arrive at a contradiction since \( (v^*, z^*, s) \) are not all zero.
b) If $z^* \neq 0$, then
\[
\langle v^*, x \rangle + \langle z^*, y \rangle \geq \langle v^*, v \rangle + \langle z^*, z \rangle
\]
for all $y \in Y, x \in X, z \in T(v)$. Letting $x = z = v = 0$, from (18) we have $\langle z^*, y \rangle \geq 0$, for all $y \in Y$, which follows $z^* = 0$. We also arrive at a contradiction.

Therefore, in both cases the condition $q = 0$ does not hold, so we have $q > 0$. Without loss of generality, we can set $q = 1$ and (17) becomes
\[
\langle v^*, x \rangle + \langle z^*, y \rangle + t \geq \langle v^*, v \rangle + \langle z^*, z \rangle + s,
\]
for all $(x, y, t) \in M, (v, z, s) \in N$. Using the above definition of the sets $M, N$ we have
\[
\langle v^*, x \rangle + \langle z^*, y \rangle + \beta \phi(x) \geq \langle v^*, v \rangle + \langle z^*, z \rangle + \langle x^*, v \rangle - \langle y^*, z \rangle.
\]
(19)
We see that the relation (19) holds for all $x \in X, y \in Y, z \in T(v)$. Letting $y = v = z = 0$ we obtain $\langle v^*, x \rangle + \beta \phi(x) \geq 0$, for all $x \in X$, or equivalently,
\[
\phi(x) \geq \langle -\frac{v^*}{\beta}, x \rangle, \quad \forall x \in X.
\]
By definition of the subdifferential of $\phi(x)$ at zero we obtain that
\[
-v^* \in \beta \partial \phi(0).
\]
(20)
Now letting $x = 0, y = z$ in (19) we have
\[
\langle v^*, v \rangle + \langle x^*, v \rangle \leq \langle y^*, z \rangle, \quad \forall z \in T(v),
\]
and hence, by definition of the adjoint $T^*(.), v^* + x^* \in T^*(y^*)$. Setting $x_0^* = v^* + x^*$ and combining with (20) we have
\[
x_0^* \in T^*(y^*), \quad -u^* \in \beta \partial \phi(0),
\]
which implies $x^* \in T^*(y^*) + \beta \partial \phi(0)$. Hence, we obtain
\[
(x^*, \lambda) \in \begin{pmatrix} T^*(y^*) + \beta \partial \phi(0) \\ (-\infty, \beta) \end{pmatrix}
\]
as desired.
REFERENCES


15. XU ZENGFUN, Generalization of nonhomogeneous Farkas' lemma and applications.