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MIXED SPIN ISING MODEL WITH FOUR-SPIN INTERACTION

Adam Lipowski[1]
International Centre for Theoretical Physics, 34100 Trieste, Italy.

ABSTRACT

We show that the mixed spin Ising model with four-spin interaction on the union jack (centered square) lattice is equivalent to the eight-vertex model. In certain subspaces of parameters the model is solvable and we can locate exactly the coexistence surface between two ordered phases of the model (symmetric case) and some critical points (free-fermion case).

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[1]Permanent address: Magnetism Theory Division, Department of Physics, A. Mickiewicz University, Ul. Matejki 48/49, Poznań, Poland.

In this Letter we consider the mixed spin Ising model on the union jack (centered square) lattice. In this model at the eight- (four-) coordinated sites there are spin 1/2 (1) spin operators. Such models have been introduced in the context of describing certain ferrimagnetic systems [1]. However, competing interactions and higher spin operators make the behaviour of such models more intricate and sometimes only extensive numerical calculations can provide the correct description [2].

In our previous paper [3] (hereafter referred to as I) it has been shown that, when formulated on the union jack lattice, such models are solvable due to the relation with the symmetric eight-vertex model [4]. In the present Letter we consider a more general version namely a model with four-spin interaction. The Hamiltonian of our model is written as

$$H = -J_1 \sum \mu_i \mu_j - J_2 \sum \mu_i S_j - D \sum S_i^2 - G \sum \mu_i \mu_j \mu_k \mu_l \tag{1}$$

where $\mu_i = \pm 1$ and $S_i = 0, \pm 1$ are spin operators placed on the eight-and four-coordinated sites respectively. The first and second summations are over nearest neighbours, the third is over four-coordinated sites and the last one is over the plaquettes. In the following we put $J_2 = 1$. The case $G = 0$ has been considered in I.

The ground-state of this model can be easily found by comparing energies of the corresponding configurations. Since the parameter space of the model is still very large in graphical presentations we will restrict ourselves to the $J_1 = -0.25$ case. In such a case the ground state is shown in Fig. 1. In region I the ground state is ferromagnetic with all μ's and S's having the same values. In region II μ's are antiferromagnetically ordered and $S_i = 0$. Regions III and IV are strongly degenerate: μ's on each plaquette have to satisfy the condition that their product is negative: $\mu_i \mu_j \mu_k \mu_l = -1$. In region III we have $S_i = 0$ and in region IV S_i takes the value of the majority of spins forming the plaquette: $S_i = \mathrm{sign}(\mu_i + \mu_j + \mu_k + \mu_l)$. Models with ground states like these in regions III and IV have recently been considered in some other context [5].

Following the same procedure as that applied in I we decimate over all four-coordinated sites S_i. The resulting model is equivalent to the eight-vertex model with the weights:

$$\omega_1 = e^{\beta(G+2J_1)}[1 + 2e^{\beta D}\cosh(4\beta)] \qquad \omega_2 = e^{\beta(G-2J_1)}(1 + 2e^{\beta D}) \tag{2}$$

$$\omega_3 = \omega_4 = e^{\beta G}(1 + 2e^{\beta D}) \qquad \omega_5 = \omega_6 = \omega_7 = \omega_8 = e^{-\beta G}[1 + 2e^{\beta D}\cosh(2\beta)] \tag{3}$$

where $\beta = 1/T$ and we put $k_{\mathrm{B}} = 1$. Spin configurations corresponding to these weights are shown in Fig. 2.

The further analysis is based on the observation that in certain subspaces of parameter space the eight-vertex model (2)-(3) is solvable. One branch of solutions corresponds to the symmetric case [4]. Since in our model we have $\omega_3 = \omega_4$, $\omega_5 = \omega_6$ and $\omega_7 = \omega_8$ thus the symmetric case is obtained imposing the condition that $\omega_1 = \omega_2$ or equivalently:

$$e^{2\beta J_1}[1 + 2e^{\beta D}\cosh(4\beta)] = e^{-2\beta J_1}[1 + 2e\beta D] \tag{4}$$

Let us note that this condition is G-independent and thus is exactly the same as that obtained in I. The real solutions of (4) exist only for $-1 < J_1 < 0$ and a plot of (4) for $J_1 = -0.25$ is shown in Fig. 3.

On the manifold (4) the vertex model becomes critical when $\omega_1 = \omega_3 + \omega_5 + \omega_7$ or equivalently:

$$e^{\beta(G+2J_1)}[1 + 2e^{\beta D}\cosh(4\beta)] = e^{\beta G}(1 + 2e^{\beta D}) + 2e^{-\beta G}[1 + 2e^{\beta D}\cosh(2\beta)] \tag{5}$$

For a fixed J_1 a practical way to solve equations (4)-(5) is first to determine D from (4) as a function of T and then one can determine G from (5). In such a way we obtain the critical line in the parameter space (D, G, T) which is parametrized by T. The projection of this line into the $T = 0$ plane is shown in Fig. 1. Along this line critical exponents change continuously [4]. Moreover, using the same arguments as in I we can show that the solvable manifold (4) in the ordered region corresponds actually to the coexistence surface of the ferromagnetic phase (I) and the antiferromagnetic phase (II) and thus it gives the location of the first-order transition between these two phases. The critical line gives actually the location of the bicritical points in our model.

From Fig. 3 it follows that for $J_1 = -0.25$ in the range $-3.7562... < D < -3$ the manifold (4) is a double-valued function. A consequence of this fact is that in this range of D the phase diagram in the (G, T) plane has the form as shown in Fig. 4. The location of bicritical points and the horizontal lines of first-order transitions can easily be determined exactly for a given J_1 and D. For example, for $J_1 = -0.25$ and $D = -3.5$ solving (4)-(5) we obtain that the bicritical points are located at (1) $G = 0.98052..., T = 1.25366...$ (2) $G = 3.69970..., T = 2.98374...$ The vertical coordinates of these points determine the location of first-order transition lines. We expect that transitions to the paramagnetic phase will be continuous and of 2D Ising universality class. For $G = 0$ such scenario has been confirmed in I. In our opinion, it is quite interesting that the model with such complicated phase diagram, which contains even two consecutive first-order phase transitions, can be studied to some extent exactly. Let us also emphasize that the temperature of first order transitons does not depend on G. This is a consequence of the already mentioned G-independence of the manifold (4).

Analysis of (4)-(5) reveals that for $D > -3$ there is only one phase order transition. An interesting point is that such a transition always exists for sufficiently large G.

The second branch of solutions corresponds to the so-called free-fermion case. In this case the weights ω_i have to satisfy the following equation

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8 \tag{6}$$

or equivalently

$$e^{2\beta G}(1 + 2e^{\beta D})[1 + 2e^{\beta D}\cosh(4\beta)] + e^{2\beta G}(1 + 2e^{\beta D})^2 = 2e^{-2\beta G}[1 + 2e^{\beta D}\cosh(2\beta)]^2 \tag{7}$$

Let us note that this equation is J_1-independent. For some values of G the plot of (7) is shown in Fig. 5. On the solvable manifold (7) there are two branches of critical points. One branch, obtained as a solution of the equation $\omega_1 = \omega_2 + \omega_3 + \omega_4$ corresponds to the decomposition of the ferromagnetic order (I) while the other, $\omega_2 = \omega_1 + \omega_3 + \omega_4$, corresponds to the decomposition of the antiferromagnetic order (II). Using (2)-(3) these equations read

$$e^{2\beta J_1}[1 + 2e^{\beta D}\cosh(4\beta)] = (e^{-2\beta J_1} - 2)(1 + e^{\beta D}) \quad (I) \tag{8}$$

$$e^{2\beta J_1}[1 + 2e^{\beta D}\cosh(4\beta)] = (e^{-2\beta J_1} - 2)(1 + e^{\beta D}) \quad (II) \tag{9}$$

These equations are G-independent and for $J_1 = -0.25$ their plots are shown in Fig. 5. Crossing points of (8)-(9) with (7) determine the critical points.

As it has already been noticed in I, for $G = 0$ the only free-fermion solutions correspond to some limiting, and already well-known cases, namely $D = \infty$ ($S = 1/2$ union jack lattice model) or $J_2 = 0$ ($S = 1/2$ square lattice model). On the contrary, for $G < 0$ this branch gives solutions in quite nontrivial and novel cases. Another interesting point is that in the limit $T \to \infty$ the solvable free-fermion manifold (7) lies entirely in the ordered ground states (I) or (II). This suggests (but obviously not proves) that the disordered ground states (III) or (IV) do not support any kind of long range order. However, as suggested by recent Cluster Variational Method calcultions [5] such ground states might support a weak ferromagnetic order in some other $S = 1$ models with four-spin interactions. The existence of such order would be quite interesting and is certainly worth further study.

In summary, we have shown that the mixed spin Ising model with four-spin interaction on the union jack lattice is equivalent to the eight-vertex model. In the four-dimensional parameter space (T, G, J_1, D) of this model there are two three-dimensional subspaces where this model is solvable. This mixed spin model has a very rich and intricate critical behaviour and the obtained exact solution should contribute to its better understanding.

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Figure captions

Fig. 1: The ground-state configurations for $J_1 = -0.25$. The dotted line is a $T = 0$ projection of the critical line (5).

Fig. 2: Configurations of the eight-vertex model in the spin representation.

Fig. 3: Solvable manifold (4) for $J_1 = -0.25$.

Fig. 4: Qualitative phase diagram for a fixed J_1 and D within a range (-3.7562,-3). Thick vertical lines, whose location can be found exactly, denote first-order phase transitions. Thin lines are conjectured lines of second-order phase transitions.

Fig. 5: Solid lines represent the solvable manifold (7). The dashed lines are the critical lines (8)-(9).

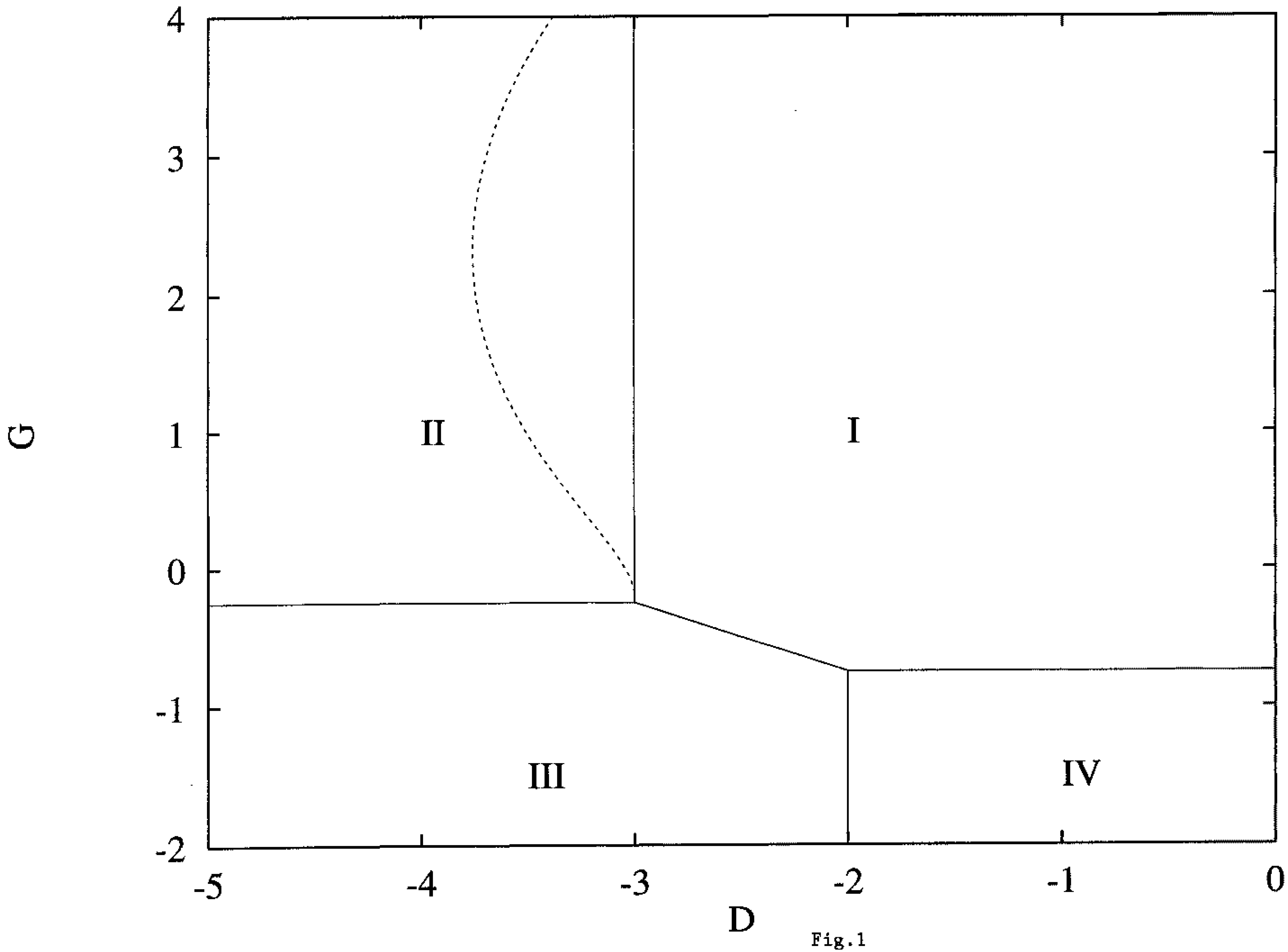

Fig.1

+ +	+ −	+ −	+ +	+ −	− −	− +	− −
+ +	− +	+ −	− −	− −	− +	− −	+ −
1	2	3	4	5	6	7	8

Fig.2

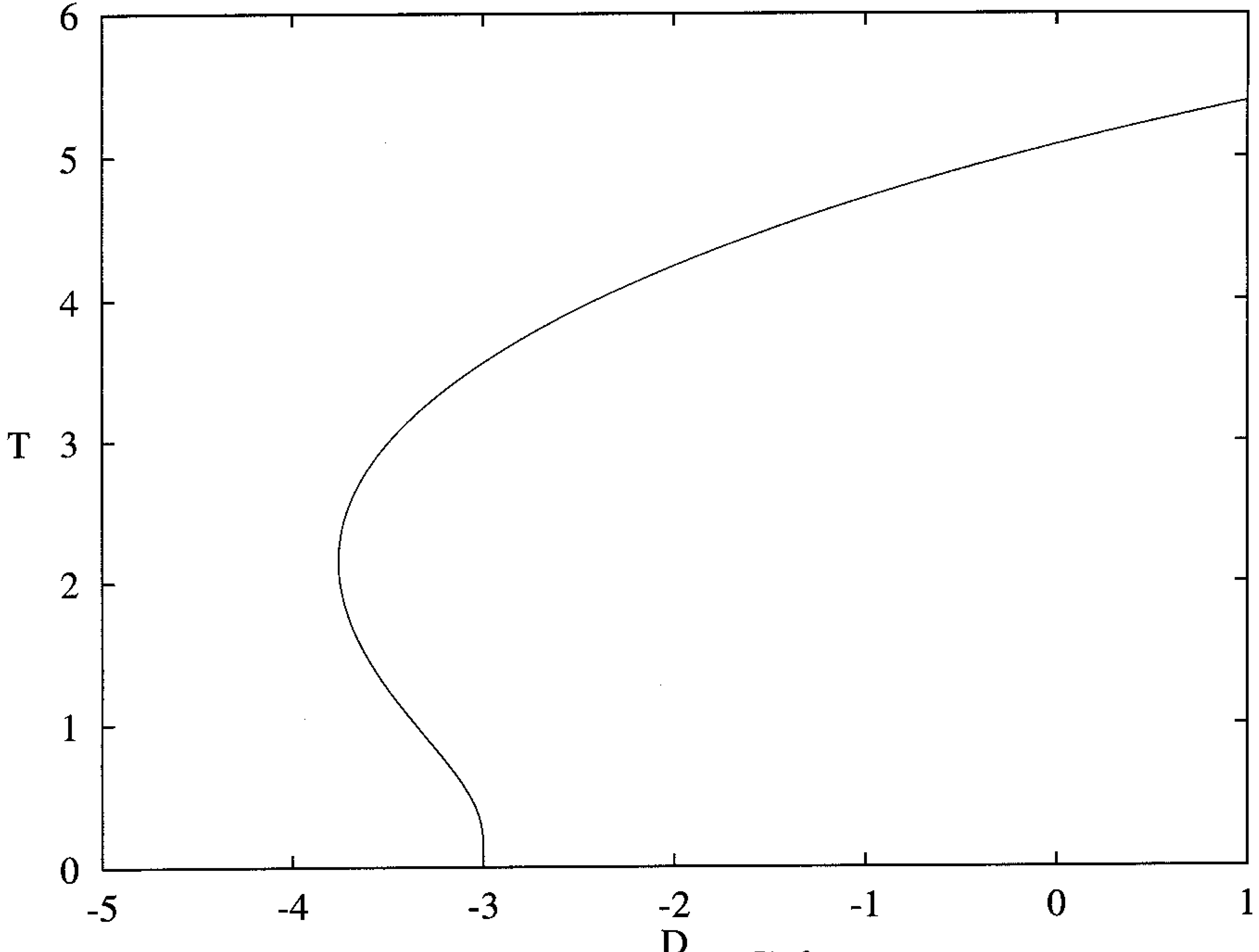

Fig.3

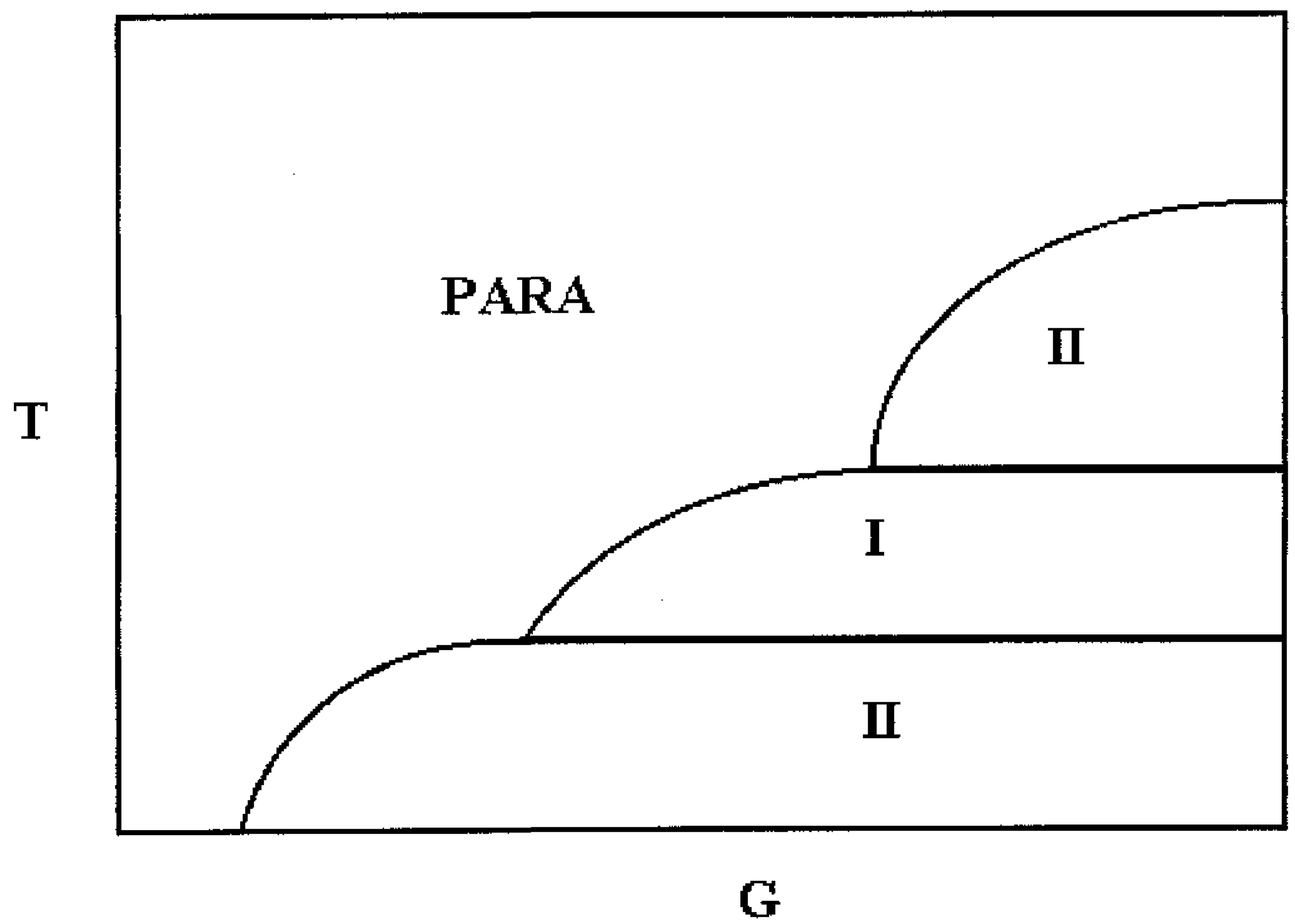

Fig.4

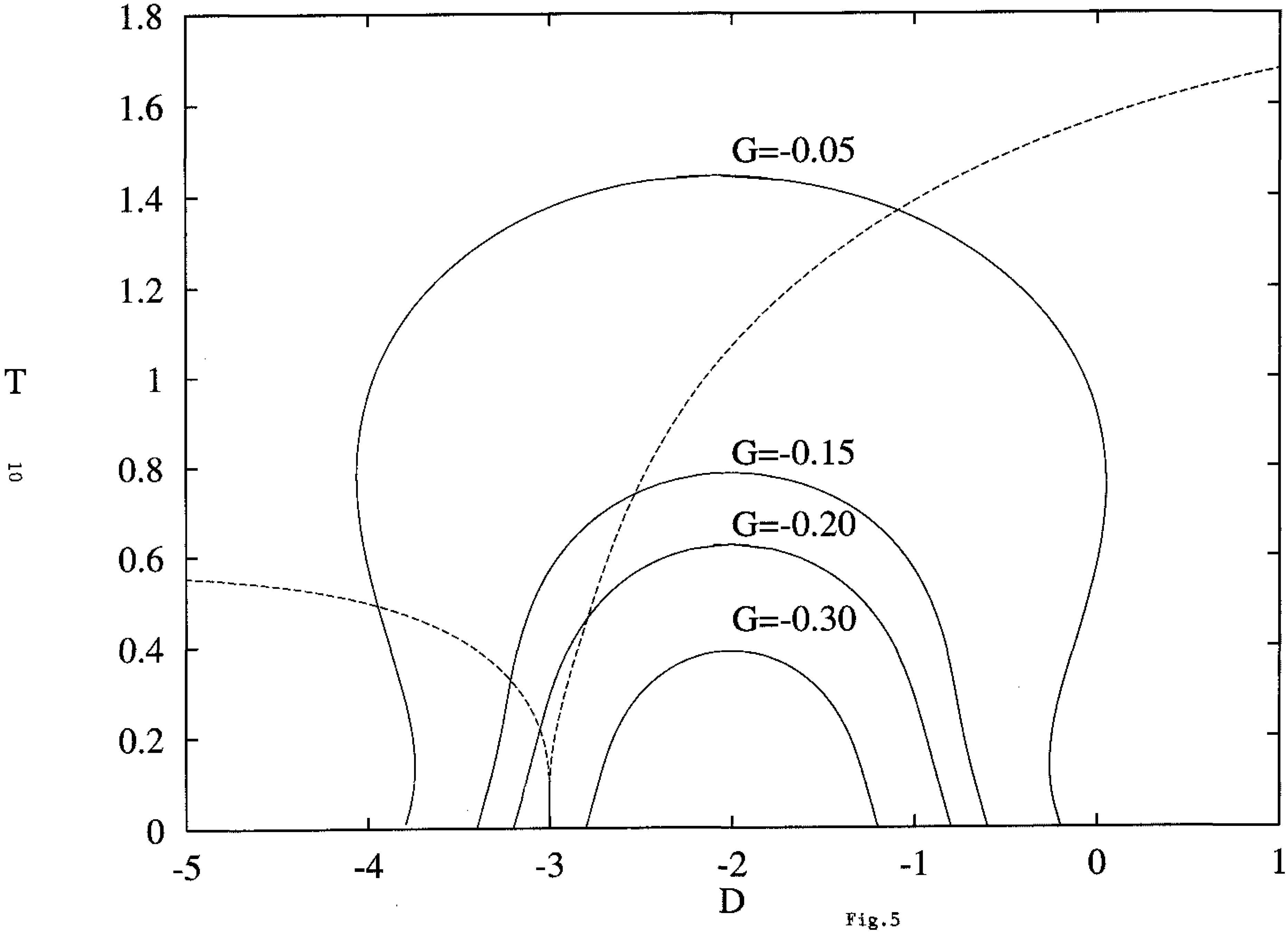

Fig.5