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ON THE SLOPES OF THE MODULI SPACES OF CURVES

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0. Introduction

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g and $\overline{\mathcal{M}}_g$ the moduli space of stable curves of genus g. Then $\overline{\mathcal{M}}_g = \mathcal{M}_g \cup \Delta$, where $\Delta = \sum_{i=0}^{[g/2]} \Delta_i$ and Δ_i is the locus of the closure of the stable curves of type $(i, g-i)$. Let λ be the class of the Hodge line bundle on $\overline{\mathcal{M}}_g$ and δ_i the class of Δ_i. Let $\delta = \delta_0 + \frac{1}{2}\delta_1 + \delta_2 + \cdots + \delta_{[g/2]}$. In the study of the geometry of $\overline{\mathcal{M}}_g$, an important divisor class is $a\lambda - b\delta$, e.g., the canonical class $K_{\overline{\mathcal{M}}_g} \equiv 13\lambda - 2\delta$. This class is called effective if $n(a\lambda - b\delta)$ is effective for sufficiently large n. Harris and Morrison [9] define the slope of the moduli space as

$$s_g = \inf\left\{ \frac{a}{b} \,\middle|\, a\lambda - b\delta \text{ is effective, } a,\ b > 0 \right\}.$$

They proposed an important conjecture on this number (see also [14]):

Slope conjecture.

$$s_g \geq 6 + \frac{12}{g+1},$$

with equality if and only if $g+1$ is composite.

This conjecture would have a number of important consequences. It would show that the Kodaira dimension of $\overline{\mathcal{M}}_g$ is $-\infty$ if and only if $g \leq 22$. For the other applications, see [9, 13, 14].

On the other hand, Eisenbud and Harris prove in [8] that $s_g \leq 6 + 12/(g+1)$ if $g+1$ is composite. In [10], Harris and Mumford prove that $s_g \leq 13/2$ if $g \geq 23$. As to the lower bound of s_g, Harris and Morrison [9] prove that there is a positive constant c such that $s_g > c/g$ for all $g \geq 2$. Chang and Ran prove that $s_g \geq 13/2$ if $g \leq 16$. But the exact values of s_g are only known for $g \leq 6$ (see [9, 6]): $s_1 = 12$, $s_2 = 10$, $s_3 = 9$, $s_4 = 17/2$, $s_5 = 8$ and $s_6 = 47/6$. Hence the conjecture is confirmed in these cases. In [2–6], some other lower bounds (not the best possible ones) are also obtained for $7 \leq g \leq 9$, $g = 15$ and 16.

In the study of the pencils of curves (or fibered surfaces), the number s_g is also studied from a different point of view by different methods (cf. [1, 11, 12, 17, 21]). In fact, the upper bounds of s_g are also obtained for $g \leq 5$ by this method.

The main purpose of this short note is to give the exact value of s_g for $g = 7$, 8, 9 and 11. We show that for these g, $s_g = 6 + 12/(g+1)$. Therefore the slope conjecture is confirmed for $g \leq 9$ and $g = 11$. We prove also that $s_{10} \geq 7$ and $s_{12} \geq 41/6$. As in [9] and [4], we shall give some upper bounds on the slope of a divisor so that it must contain some special subvarieties of $\overline{\mathcal{M}}_g$. In particular, we show that any divisor in the n-canonical system $|nK|$ of $\overline{\mathcal{M}}_g$ contains Δ_i with multiplicity $\geq (21-i)n$ for some i. All these bounds are better than the known ones.

1. Technical Computations and Constructions

Let $f : S \to B$ be a fibration of a complex nonsingular projective surface S over a curve B. Assume that the generic fiber of f is a curve of genus g and there is no (-1)-curves in a fiber. Then f induces a map $\rho : B \to \overline{\mathcal{M}}_g$.

On the moduli space $\overline{\mathcal{M}}_g$, there are three important classes: λ, δ, κ, where λ is the Hodge divisor class, δ the boundary class and $\kappa = 12\lambda - \delta$. (For the original definition of κ, see [16]). In the study of the geometry of $\overline{\mathcal{M}}_g$, we usually need to compute the following three numbers for the pencil $f : S \to B$:

$$\lambda B = \deg \rho^* \lambda, \quad \delta B = \deg \rho^* \delta, \quad \kappa B = \deg \rho^* \kappa.$$

Now we consider the computation of these numbers.

For this purpose, we consider the standard relative invariants of f:

$$\begin{aligned} \chi_f &= \deg f_* \omega_{S/B} = \chi(\mathcal{O}_S) - (g-1)(b-1), \\ K_f^2 &= K_{S/B}^2 = c_1^2(S) - 8(g-1)(b-1), \\ e_f &= \sum_F (\chi_{\rm top}(F) + 2g - 2) = c_2(S) - 4(g-1)(b-1), \end{aligned}$$

where $b = g(B)$. It is well-known that if f is semistable, then we have

$$\lambda B = \chi_f, \quad \kappa B = K_f^2, \quad \delta B = e_f.$$

In order to do the computation for the non-semistable case, we have to use some new invariants $I_\chi(f)$, $I_e(f)$ and $I_K(f)$, which are introduced in [20] (see also [19]).

First we associate to each singular fiber F three nonnegative numbers $c_1^2(F)$, $c_2(F)$ and χ_F which can be computed easily by the embedded resolution of F. One of the three numbers vanishes if and only if F is semistable. Then we can define

$$\begin{aligned} I_\chi(f) &= \chi_f - \sum_F \chi_F, \\ I_K(f) &= K_f^2 - \sum_F c_1^2(F), \\ I_e(f) &= e_f - \sum_F c_2(F), \end{aligned}$$

where F runs over all the singular fibers. They are nonnegative. $I_\chi(f)$ (or $I_K(f)$) vanishes if and only if f is isotrivial. If $\widetilde{f} : \widetilde{S} \to \widetilde{C}$ is the pullback fibration of $f : S \to C$ under a base change $\pi : \widetilde{B} \to B$ of degree d, then

$$I_\chi(\widetilde{f}) = dI_\chi(f), \quad I_K(\widetilde{f}) = dI_K(f), \quad I_e(\widetilde{f}) = dI_e(f).$$

Now we have

Lemma 1.1. *For any fibration $f : S \to B$ of genus g, we have*

$$\lambda B = I_\chi(f), \qquad \kappa B = I_K(f), \qquad \delta B = I_e(f).$$

Proof. It is well-known that there exists a finite base change $\pi : \widetilde{B} \to B$ such that the pullback fibration $\widetilde{f} : \widetilde{S} \to \widetilde{B}$ of f is semistable. Therefore, the lemma holds for $\widetilde{f}$. On the other hand, by definition above and [20, Theorem A′], all of the invariants in this lemma are multiples of the corresponding invariants of $\widetilde{f}$ by $1/\deg \pi$. Hence the lemma holds for f. □

Our method is as follows. For some g, we try to find a set Σ_g of non-isotrivial pencils $f : S \to B$ such that any generic nonsingular curve of genus g can be the fiber of a pencil in Σ_g. Then for any effective divisor $D \equiv a\lambda - b\delta$ on $\overline{\mathcal{M}}_g$, there is a pencil $f : S \to B$ in Σ_g such that the image of B is not contained in D. This means that the image of the pencils in Σ_g is dense in $\overline{\mathcal{M}}_g$, we shall say that these pencils fill up $\overline{\mathcal{M}}_g$. Then we have

$$BD =: \deg \rho^* D = a\lambda B - b\delta B \geq 0,$$

hence $a/b \geq s(B) =: \delta B / \lambda B$, thus we have $s_g \geq s(B)$ for some pencils $f : S \to B$ in Σ_g.

In our cases, any pencil $f : S \to B$ of genus i in Σ_i has at least two disjoint (-1)-curves E and E' as its sections. From them we can lift this pencil to two pencils of higher genus.

Firstly, if $g > i$ and C is a general curve of genus $g - i$ with a fixed point $p \in C$, then we can construct a pencil $f_1 : S_1 \to E$ of genus g. The fiber of f_1 over $e \in E$ is the curve $f^{-1}(f(e)) \cup C$ obtained by identifying e with p. Now we have some standard formulas (cf. [15 or 6]),

$$E\lambda = B\lambda, \quad E\delta_i = -1, \quad E\delta_j = B\delta_j, \quad (j \neq i,\ g - i).$$

Note that the λ and δ_i in $E\lambda$ and $E\delta_i$ are the classes on $\overline{\mathcal{M}}_g$. So we have

$$s(E) = s(B) - \frac{1}{\lambda B}.$$

We denote by $\Sigma^1_{i,g}$ the set of pencils constructed by this method from all pencils in Σ_i and all general curves of genus $g - i$. Then we can see that the pencils in $\Sigma^1_{i,g}$ fill up $\Delta_i \subset \overline{\mathcal{M}}_g$.

The second pencil is constructed by gluing the two sections E and E' of $f : S \to B$. Then we obtain a pencil $f_2 : S_2 \to B_2$ of genus $i + 1$ in $\Delta_0 \subset \overline{\mathcal{M}}_{i+1}$, where B_2 is isomorphic to B. We have

$$B_2\lambda = B\lambda, \quad B_2\delta_j = B\delta_j, \quad j \neq 0,\ 1,\ i,$$
$$B_2\delta_1 = 0, \quad B_2\delta_0 = B\delta_0 - 2.$$

Note that the left hand sides of the equalities are computed in $\overline{\mathcal{M}}_{i+1}$. So we have

$$s(B_2) = s(B) - \frac{2}{\lambda B}.$$

We denote by $\Sigma^2_{i,i+1}$ the set of all the pencils constructed by the second method from Σ_i. Now we can see easily that the pencils in $\Sigma^2_{i,i+1}$ fill up $\Delta_0 \subset \overline{\mathcal{M}}_{i+1}$.

2. On the Slope Conjecture

In this section, we shall confirm the slope conjecture for $g \leq 9$ and $g = 11$.

Theorem 2.1. *If $3 \leq g \leq 9$ or $g = 11$, then $s_g \geq 6 + 12/(g+1)$, with equality if and only if $g + 1$ is composite.*

Proof. We only need to prove this theorem for $g = 7, 8, 9$ and 11. Note that a generic curve H of genus $g = 3, \cdots, 9$ or 11 can be the hyperplane section of a K3 surface X of degree $2g - 2$ in $\mathbb{P}^g$. We consider a general (Lefschetz) pencil in $|H|$. By blowing up X at some $2g - 2$ points, we can obtain a semistable pencil $f : S \to B$ of genus g whose singular fibers are irreducible curves with only one node. Note that

$$c_1^2(S) = 2 - 2g, \quad \chi(\mathcal{O}_S) = 2, \quad c_2(S) = 22 + 2g, \quad g(B) = 0.$$

By the computation formulas given in previous section, we have

$$\lambda B = g + 1, \quad \delta B = 6(g + 3).$$

We can see easily that these pencils fill up $\overline{\mathcal{M}}_g$, thus we have $s_g \geq s(B) = 6 + 12/(g + 1)$. Note that if $g = 7, 8, 9$ or 11, then $g + 1$ is composite. By a result of Eisenbud and Harris in [8], we have $s_g \leq 6 + 12/(g + 1)$, so $s_g = 6 + 12/(g + 1)$. $\square$

We denote by Σ_g the set of pencils constructed in the above proof.

If $4 \leq g \leq 10$ or $g = 12$, then we can define

$$t_g = \min \left\{ s(B) \,\middle|\, B \text{ is a pencil in } \Sigma^2_{g-1,g} \text{ or } \Sigma^1_{i,g},\ i = 1, \cdots, [g/2] \right\}.$$

Lemma 2.2. *If $4 \leq g \leq 10$ or $g = 12$, then $s_g \geq t_g$.*

Proof. Let $D \equiv a\lambda - b\delta$ be an effective divisor on $\overline{\mathcal{M}}_g$. We need to prove that $a/b \geq t_g$. Indeed, if D does not contain Δ_i for some $0 \leq i \leq [g/2]$, then there exists a pencil B in $\Sigma^2_{g-1,g}$ or $\Sigma^1_{i,g}$ for some i such that B is not contained in D. Then $DB \geq 0$, i.e., $a/b \geq s(B) \geq t_g$. If D contains all of the Δ_i, $0 \leq i \leq [g/2]$, then there exists a positive integer n such that $D' \equiv a\lambda - (b + n)\delta$ is effective and does not contain $\Delta = \sum_i \Delta_i$. Hence

$$\frac{a}{b} > \frac{a}{b + n} \geq t_g.$$

This completes the proof. $\square$

Note that $t_g = 6 + 10/g$. Hence we have

Corollary 2.3.

$$s_{10} \geq 7, \qquad s_{12} \geq \frac{41}{6}.$$

Note that there is an effective divisor $\mathcal{K}$ on $\overline{\mathcal{M}}_{10}$ with

$$\mathcal{K} \equiv 7\lambda - \delta_0 - b_1\delta_1 - \cdots - b_5\delta_5.$$

See [7] for the construction of this divisor.

3. Base Locus of Effective Divisors

Theorem 3.1. *Fix integer $i = 3, \cdots, 9$, or 11 and $g \geq i + 1$. Let $D \equiv a\lambda - b\delta$ be an effective divisor on $\overline{\mathcal{M}}_g$. If $a/b < 6 + 11/(i + 1)$, then D must contain Δ_i with multiplicity $\geq (6i + 17)b - (i + 1)a$.*

Proof. Let m be the multiplicity of Δ_i in D. Then $D' = D - m\Delta_i$ does not contain Δ_i. We consider a general pencil E in $\Sigma^1_{i,g}$. Then E is not contained in D. Hence $DE \geq 0$,

$$m\Delta_i E \leq a\lambda E - b\delta E.$$

Since $\Delta_i E = -1$, $\lambda E = i + 1$, $\delta E = 6i + 17$, we have

$$m \geq (6i + 17)b - (i + 1)a.$$

This is what we desired. $\square$

Corollary 3.2. *Let K denote the canonical divisor of $\overline{\mathcal{M}}_g$. Then for i, g as in Theorem 3.1, any divisor in the n-canonical system $|nK|$ must contain Δ_i with multiplicity $\geq (21 - i)n$.*

Proof. This corollary follows from $K \equiv 13\lambda - 2\delta$. □

Note that Chang and Ran [4] have also given a lower bound on the multiplicity for $1 \leq i \leq 5$ and $i = 15$. For $i = 3, 4$ and 5, their bounds are respectively 13n, 14n and 11n, and our bounds are respectively 19n, 18n and 17n.

4. Trigonal Locus and Slopes of Effective Divisors

Let $D \equiv a\lambda - b\delta$ be an effective divisor on $\overline{\mathcal{M}}_g$. We define the slope of D as $s(D) = a/b$. Harris and Morrison find a function $s_g(k)$ in [9] such that if $s(D) < s_g(k)$, then D contains the k-gonal locus. For example, $s_g(2) = 8 + 4/g$ and $s_g(3) \sim 72/11$ when g is big enough. Our next purpose is to give a better bound for the trigonal case.

Theorem 4.1. *Let $g \geq 3$ and let D be an effective divisor on $\overline{\mathcal{M}}_g$. If $s(D) < 7 + 6/g$, then D contains the trigonal locus.*

Proof. We shall use a result of Petri on trigonal curves. Petri proves that all trigonal curves lie on a rational scroll (cf. [18]). If C is a trigonal curve of genus g, then it is a trisection in $\mathbb{F}_k = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-k))$ with $k \geq 0$. So C is contained in the linear system $|3C_0 + tF|$, where C_0 is the unique irreducible curve such that $C_0^2 = -k$, F is a fiber of the ruling $\mathbb{F}_k \to \mathbb{P}^1$, and $2t - 3k = g + 2$. We consider the generic trigonal curves of genus g, then the linear system above is very ample. By choosing a general pencil in $|C|$ and blowing up $\mathbb{F}_k$ at $C^2 = 6k - 9t = 3g + 6$ points, we get a genus g semistable fibration $f : S \to B$. We have

$$c_1^2(S) = 2 - 3g, \quad c_2(S) = 10 + 3g, \quad \chi(\mathcal{O}_S) = 1, \quad g(B) = 0.$$

Hence we obtain

$$K_f^2 = 5g - 6, \quad e_f = 7g + 6, \quad \chi_f = g.$$

Consequently,

$$s(B) = 7 + \frac{6}{g}.$$

We let Σ_g be the set of all the pencils constructed above. Then these pencils fill up the trigonal locus of $\overline{\mathcal{M}}_g$.

Now if D does not contain the trigonal locus, then for a generic pencil B in Σ_g, the image of B in $\overline{\mathcal{M}}_g$ is not contained in D. Thus $DB \geq 0$, i.e., $s(D) \geq s(B) = 7 + 6/g$. This completes the proof. □

Note that one may prove this theorem by using the generic triple covers of $\mathbb{F}_k$ such that the induced fibrations are of genus g. Then all the pencils fill up the trigonal locus.

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