TEST-PARTICLE SCATTERING
ON A MAGNETICALLY CHARGED DILATONIC BLACK HOLE

I.E. Pris
Institute of Physics, 220072 Minsk, Republic of Belarus,
Ya.M. Shnir
International Centre for Theoretical Physics, Trieste, Italy

ABSTRACT

We consider motion of test particles on magnetically charged dilatonic black hole background. The differential cross-section for small-angle scattering is obtained.
At present, the so-called dilatonic black holes (DBH) are being investigated intensively. (See, for example, [1-5].) Solutions of this type appear in some models connected with string theory, conformal field theory and general relativity. The presence of a dilatonic field leads to essential change of some properties of general relativity’s BH. At the same time, DBH can be either electrically or magnetically charged as well as in general relativity. However, solutions which carry both type of charges can be constructed only under condition of introducing an interaction with axionic field.

Further progress in understanding DBH’s properties can be reached by studying their influence on dynamics of test objects. Recently, in the work [5] motion of test classical particles assumed to have electric and dilatonic charges, around an electrically charged DBH has been investigated.

In the present work an analogous problem is considered for a magnetically charged DBH. Since a magnetic monopole is undetectable object up to now, the investigation of magnetic DBHs is more interesting problem rather than their electric analogs. In particular, the differential cross-section of test particles for small-angle scattering is obtained.

The action including the dilaton coupling has the form:

\[
S = \int d^4x \sqrt{-g} \left( R - 2(\nabla \phi)^2 - e^{-2a\phi} F^2 \right)
\]

where \( g = \det g_{\mu\nu} \), \( g_{\mu\nu} \) is the metric tensor, \( R \) is the Ricci scalar, \( F_{\mu\nu} \) is the electromagnetic tensor, \( \phi \) is the dilaton and \( a \) is an arbitrary constant.

Since the corresponding field equations are invariant under the following generalized discrete duality transformation (see, for example, [2] for the particular case \( a = 1 \)):

\[
F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu} = \frac{1}{2} e^{-2a\phi} \epsilon_{\mu\nu} F_{\lambda\rho} \quad \phi \rightarrow \phi' = -\phi \quad g_{\mu\nu} \rightarrow g_{\mu\nu}
\]

we obtain the solution \((g_{\mu\nu}, \phi', F)\) for a magnetically charged DBH from the corresponding one for DBH with electric charge \( Q = g \) [5]:

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\Delta(r)\sigma^{-2} dt^2 + \sigma^2(\Delta(r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2))
\]

\[
e^{-2a\phi'} = e^{2a\phi} = \sigma^2(r)
\]

and

\[
F = dA \quad A = g \cos \theta d\varphi
\]

where \( A \) is the Dirac potential, \( g \) is the magnetic charge of DBH,

\[
\Delta(r) = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)
\]

\[
\sigma^2(r) = \left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}}.
\]

In (5), (6), \( r_+ \) and \( r_- \) are constants, which are related to the mass \( M \) and the charge \( g \) of the DBH:

\[
2M = r_+ + \frac{1}{1+a^2}r_- \quad g^2 = \frac{r_+r_-}{1+a^2}
\]
We take a test particle action in the form [5]

\[
S = -\int dt \left( m e^{b\phi} \sqrt{-g_{\mu\nu} \frac{\partial x^\mu}{\partial t} \frac{\partial x^\nu}{\partial t} + \frac{\partial A_\mu}{\partial t} \frac{\partial x^\mu}{\partial t}} \right)
\]  

(8)

where \( m, e \) are the mass and the electric charge of the test particle, respectively, and the constant \( b \) is considered as its dilatonic charge. The action (8) satisfies to the Hamilton-Jacobi equation:

\[
g^{\mu\nu} (\partial_\mu S + e A_\mu) (\partial_\nu S + c A_\nu) + m^2 e^{2b\phi} = 0.
\]  

(9)

Solving this equation by separation of variables, we get

\[
S = -Et + J\phi + S_1(\theta) + S_2(r)
\]  

(10)

where

\[
S_1(\theta) = \sqrt{L^2 - \left( \frac{J + eg \cos \theta}{\sin \theta} \right)^2} d\theta
\]  

(11)

\[
S_2(r) = \int r^2 \Delta \sigma^2 \sqrt{E^2 - \frac{\Delta}{r^2} \left( m^2 \sigma^{2\phi} / a^2 + a^4 \frac{L^2}{r^2} \right)} dr. \quad b' = -b.
\]  

(12)

As follows from the consideration of the flat space case (\( \Delta \rightarrow 1, \sigma^2 \rightarrow 1 \)), the constants \( J \) and \( L^2 \) have the meaning of the total angular-momentum projection on the axe z and square of the orbital angular-momentum at spatial infinity, respectively.

In particular case \( \alpha = 0 (\sigma^2 = 1) \) from (10)-(12) we obtain the action of a test particle moving around the usual magnetically charged BH [6].

Since \( S_1(\theta) \) does not depend on \( \Delta \) and \( \sigma^2 \) it is easy to conclude that the test particle moves on the surface of a cone with the apex angle \( \psi \), where

\[
\cos \psi = -\frac{eg}{\sqrt{L^2 + (eg)^2}} = -\frac{eg}{J}.
\]  

(13)

We recall that the total angular-momentum in a charge-monopole system has the form (in our case at spatial infinity) \( J = L - eg^2 r \) [7]. Let the axe z is directed along the \( J \), i.e. \( J_z = J \). Then

\[
\left( \frac{J + eg \cos \psi}{\sin \psi} \right)^2 = L^2 \quad \frac{L}{\sin \psi} = J
\]  

(14)

where \( J + eg \cos \psi = L_z \). Introducing the velocity of the particle at infinity \( v \) and the impact parameter \( d \), we also have

\[
E = \frac{m}{\sqrt{1 - v^2}} \quad L = \frac{mv d}{\sqrt{1 - v^2}}.
\]

The first order equations describing the test particle motion are derived from (10)-(12):

\[
\dot{\theta} = 0 (\theta = \psi)
\]  

(15)

\[
-m e^{b\phi} g_{\mu} \frac{dt}{dr} = m \Delta \sigma^{2\phi / a^2} = E
\]  

(16)
\[-me^6 g_{\varphi \varphi} \frac{d\varphi}{d\tau} + eg \cos \psi = -m\alpha^2 + \frac{b'}{r^2} \sin^2 \psi \frac{d\varphi}{d\tau} + eg \cos \psi = -J,\]

or
\[-m\alpha^2 + \frac{b'}{r^2} \sin^2 \psi \frac{d\varphi}{d\tau} = -\frac{J + eg \cos \psi}{\sin^2 \psi} = -\frac{L_z}{\sin^2 \psi} = -\frac{L}{\sin \psi} = -J \quad (17)\]

(see (14)), and
\[
\left( \frac{dr}{d\psi} \right)^2 = \frac{\sigma - 2b'/a}{m^2} - \Delta \sigma^2 \left( 1 + \frac{L^2 \sigma - 2b'/a}{m^2 r^2} \right). \quad (18)
\]

We see that unlike an electrically charged DBH case a new constant of motion $J$ appears in the motion equations.

Let us consider the case when $d >> r_+, r_-$ and the kinetic energy of the test particle is much more than the absolute value of its potential energy. This corresponds to a small-angle scattering. Then from (17)-(18) at first order approximation we get the equation for the trajectory:

\[
\left( \frac{dr}{d\psi} \right)^2 = \sin^2 \psi \frac{r^4}{d^2} \left( \frac{1}{1 + \frac{B}{r}} - \frac{1}{r^2} \right), \quad \frac{B}{d} << 1 \quad (19)
\]

where
\[
B = \frac{1}{v^2} \left( 1 - v^2 \right) \left( r_+ + \frac{1 + 3a^2 + 2ab'}{1 + a^2} r_- \right) - \frac{4a^2}{1 + a^2} r_-. \]

Note that when the space-time is flat and the dilaton field vanishes, a non-zero constant $B$ arises only for dyon, which carries both electric and magnetic charges. Therefore, the test particle scattering on the magnetically charged DBH must be similar with the scattering on a dyon. (See, for example [8] for a non-relativistic case.)

The solution of (19) has the form
\[
\frac{1}{r} = \sqrt{\frac{1}{d^2} + \left( \frac{B}{2d^2} \right)^2} \cos(\sin \psi (\varphi - \varphi_0)) + \frac{B}{2d^2}
\]

where $\varphi_0$ is an arbitrary constant. Taking $r = \infty$, we find the change of the $\varphi$-coordinate during the scattering of the particle:

\[
\Delta \varphi = \frac{2}{\sin \psi} \arccos \left( -\frac{B}{2d} \frac{1}{\sqrt{1 + \left( \frac{B}{2d} \right)^2}} \right) \quad (20)
\]

Then the scattering angle is expressed as follows [7]:

\[
\cos \theta = -\sin^2 \psi \cos \Delta \varphi - \cos^2 \psi. \quad (21)
\]

From (20)-(21) we find

\[
\cos \theta = -\sin^2 \psi \cos \left( \frac{2}{\sin \psi} \arccos \left( -\frac{B}{2d} \frac{1}{\sqrt{1 + \left( \frac{B}{2d} \right)^2}} \right) \right) - \cos^2 \psi. \quad (22)
\]

For the differential cross-section

\[
\frac{d\sigma}{d\Omega} = d \left( -\frac{\partial \cos \theta}{\partial d} \right)^{-1}
\]
from (22), taking into account $d(\frac{\partial \sin \psi}{\partial d})^{-1} = \frac{d^2 \sqrt{1 - v^2}}{mv(eg)^2}$, we have

$$
\frac{d\sigma}{d\Omega} = \left(\frac{eg}{mv}\right)^2 (1 - v^2) \sin \psi \cos^4 \psi \left| 2 \sin \psi \left\{ 1 - \cos \left( 2 \sin^{-1} \psi \arccos C \right) \right\} - 2 \sin(2 \sin^{-1} \psi \arccos C) \right) \left( \arccos C - \sin \psi \left( \frac{\partial \sin \psi}{\partial d} \right)^{-1} \frac{\partial}{\partial d} \arccos C \right)^{-1}
$$

where $C = \frac{B}{d} \sqrt{1 + \left(\frac{B}{d}\right)^2}$.

To simplify further calculations it is convenient to introduce the following notation:

$$
\gamma = \frac{1}{\sin \psi} = \sqrt{1 + \left(\frac{eg}{L}\right)^2}.
$$

Then

$$
\frac{d\sigma}{d\Omega} = \left(\frac{eg}{mv}\right)^2 (1 - v^2) \gamma^4 (\gamma^2 - 1)^{-\frac{1}{2}} \left| 2 \sin''(\gamma \arccos C) - \gamma \sin(2 \gamma \arccos C) \right|
$$

$$
\left( \arccos C + \frac{d\gamma^2}{1 - \gamma^2} \frac{B}{2d^2} \right)^{-1}
$$

and the connection between the scattering angle $\theta$ and the apex angle of cone $\psi$ is given by (see (22))

$$
\cos \frac{\theta}{2} = \frac{1}{\gamma} \sin(\gamma \arccos C).
$$

For small angle scattering case, which is under consideration, $\frac{eg}{L} << 1$ ($\gamma \rightarrow 1$, the motion of the test particle is almost flat), $\frac{B}{d} << 1$, and the formulas (25), (26) are considerably simplified.

From (26) at first order approximation for the scattering angle we obtain:

$$
\theta = 2 \sqrt{\left(\frac{eg}{L}\right)^2 + \left(\frac{B}{2d}\right)^2} + O\left(\frac{B}{d} \frac{1}{L} \left(\frac{eg}{L}\right)\right).
$$

This yields the following differential cross-section:

$$
\frac{d\sigma}{d\Omega} = \frac{d^4}{4 \left(\frac{eg}{mv}\right)^2 (1 - v^2) + B^2} = \frac{4 \left(\frac{eg}{mv}\right)^2 (1 - v^2) + B^2}{\sin^4 \frac{\theta}{2}}
$$

which has the same form as the Rutherford one.

The next approximation is found from (25):

$$
\frac{d\sigma}{d\Omega} = \frac{d^4}{\left(\frac{eg}{mv}\right)^2 (1 - v^2) \left(4 + \pi \frac{B}{d}\right) + B^2}.
$$

In conclusion we note that the presence of dilaton leads to existence of equilibrium points for a test particle in case of magnetically charged BH, too. Thus, the dilatonic force behaves as an attractive one. The corresponding conditions have the form:

$$
0 < D < \frac{r_+}{r_-}, \quad \frac{r_+}{r_-} < 1
$$

5
where \( D = \frac{a^2 + 2ab - 1}{1 + a^2} \), and a point of equilibrium is given by

\[
r = \frac{r_+ r_- (1 - D)}{r_+ - Dr_-}.
\]

Obviously, that for corresponding values of \( a \) and \( b \) any point \( r > r_+ r_- \) can be transformed into a point of equilibrium. In case the dilaton is absent there are no points of equilibrium. However, when the dilatonic charge of the test particle is equal zero, points of equilibrium exist under condition

\[
1 < a^2 < \frac{r_+ r_-}{r_+ - r_-}, \quad \frac{r_+}{r_-} < 1
\]

and are given by

\[
r = \frac{2r_+}{(a^2 + 1) \frac{r_+}{r_-} - (a^2 - 1)}.
\]

At last, for the energy of the test particle in the equilibrium points we have

\[
E = m \left| \frac{r_+ - r_-}{\sqrt{r_- r_+}} \right| \left( \frac{D(r_+ - r_-)}{r_+ (1 - D)} \right)^\frac{\sqrt{a - 1}}{1 + a^2}.
\]

Acknowledgments

One of the authors (Y.M.S.) would like to thank the International Centre for Theoretical Physics, Trieste, for hospitality, where this work was completed. He would also like to thank Prof. E. Gava for helpful discussions.

References