EULIDEAN COHERENT STATES I: GEOMETRIC OPTIC CASE

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ABSTRACT

Manifold of light rays is naturally described in terms of coset spaces of (2+1)-dimensional euclidean group. Starting from unitary irreducible representations of the latter, which are not square integrable, one is led to consider a family of coherent states labelled by the elements of quotients (phase spaces). It is shown that the refractive index representation of the later group is square integrable modulo the optic axis translation subgroup and modulo the flat, paraxial and Seidel-Lie aberration sections.

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1 Introduction

In the original paper ([1]), the author proposed to deal with the Euclidean group than the Weyl-Heisenberg (W-H) one in geometric and wave optics. In fact, the W-H model of Lie-Fourier optics ([2]) builds a phase space where each coordinate (position $q$) and its canonically conjugate coordinate (momentum $p$) ranges over $\mathbb{R}$. This allows the usual quantization schema of the system ([3]) and works well in the paraxial approximation. But this means that we neglect the fact that the optical Hamiltonian momentum ([4], [5], [6]), the ray direction, ranges over the Decartes sphere $S^2$ in 2-dimensional optics, and over the circle $S^1$ in 1-dimensional world of cylindric lenses. In ref. [1], there was considered the mixed W-H group $W \{ZS^2\}$

$$W \{ZS^2\} = \{g | g = (p, q; \varphi), p \in \mathbb{R} \}$$

where the position variable was discrete and $\mathbb{T}$ is the unit circle, $\mathbb{T} = \{z \in \mathbb{C}^2 | |z| = 1\}$. This was still inconsistent to get the canonical coherent states (CS) [7] and leads to the real contention that the basic group of global optic is the three-dimensional group $E(2+1) = ISO(3)$ of rigid motions of three-space.

Recall here that the canonical CS, originally introduced by Schrödinger ([7]) in the context of a harmonic oscillator are related to the W-H group $G_{WH}$

$$G_{WH} = \{ (g | g = (p, q, \varphi), (p, q) \in \mathbb{R}^2, \varphi \in \mathbb{R} \}$$

with the group law

$$gg' = (p + p'; q + q'; \varphi + \varphi' + \frac{1}{2}(pq' - qp'))$$

Later, those canonical CS were popularized by Glauber and Klauder for description of coherent light and have been generalized so that they appear in many physical and mathematical research fields ([8]). The well known application of the canonical CS is the quantum optics ([9], [10], [11], [12]).

Let us write the elements of $G_{WH}$ in (1.2) as $(z; \varphi) \in \mathbb{C} \times \mathbb{T}$ and $(0; \varphi) = Z$ the center of $G_{WH}$. Thus, every unitary irreducible representation (UIR) $U$ of $G_{WH}$ has a restriction to the center $Z$ of the form $U(0, \varphi) = \exp(2\pi i \lambda \varphi), \lambda \in \mathbb{R}$. The Stone-von Neumann uniqueness theorem asserts that, for fixed $\lambda \neq 0$, all UIR's of $G_{WH}$ are unitarily equivalent and have the form

$$U^\lambda(z; \varphi) = e^{2\pi i ph} D^\lambda(z) = W(p, q; \varphi)$$

where $D^\lambda(z) = M(p)T(q), z = ip + q$.

Here we follow the ancient convention used by opticians: for example, instead of denoting the north and south poles of $S^2$ by (1,0) and (-1,0) respectively, we consider the forward pole of the Decartes sphere (0,1) and the Euclidean group will be denoted by $E(N + 1) = ISO(N + 1)$, where $N \in \mathbb{N}$ is the dimension of the screen coordinates, pointing out the 1-dimensional optical axis ([3]), playing the similar role of time in the usual dynamics.
where $D^\lambda(z)$ is the displacement operator in the phase space $\mathbb{C}$, with components $M(p)$ and $T(q)$, the modulation and shift operators respectively ((14), (13), (15)). Given such a representation $U^\lambda$ in a Hilbert space $\mathcal{H}$ and a fixed vector or ket-ray $|\eta\rangle$ in $\mathcal{H}$, the orbit $G_{\mathcal{H}}\eta$ of $G_{\mathcal{H}}$ in $\mathcal{H}$

$$G_{\mathcal{H}}\eta = \{D^\lambda(z)|\eta\rangle = \eta(z), \ z \in \mathbb{C}\}$$

is the canonical CS's family. From here, we see that the construction of coherent states system is a group representation study. Another high fact is that every UIR $U^\lambda$ is square integrable in the following sense:

$$\int_{G_{\mathcal{H}}/\mathbb{Z}} |D^\lambda(z)|^2 < \infty \ \forall s \in \mathcal{H}$$

The integral in Eq. (1.6) runs over the quotient space $G_{\mathcal{H}}/\mathbb{Z}$ i.e. the representation $U^\lambda$ of $G_{\mathcal{H}}$ is square integrable modulo the center $\mathbb{Z}$ of $G_{\mathcal{H}}$. This observation is the foundation and leit-motiv of many works actually. Perelomov extended this concept to other groups (16) and its generalization was done by Ali, Antoine and Gazeau [17], [18]. More recently, there was obtained a quantization method using coherent states frames [19]. The way how far one can go in constructing CS is described in [20]. In this optic too, many groups were examined: the relativistic Poincaré and non relativistic Galilean groups ([17], [21], [22]), the affine or weveled group $\alpha + \beta$ ([23], [24]), the affine diamond group ([25]), the similetude groups in the nonrelativistic and relativistic cases ([26], [27]).

The present work fits in this program with applications in optics.

Recently the CS relative to Euclidean groups, Wigner representations of which were studied within the localizability of quantum mechanical systems (28), where described from the differential geometry point of view (29). More recently (30), CS on the unit circle were obtained from the canonical CS on the line by a direct integral decomposition (of Bloch type) of $L^2(\mathbb{R})$ into copies of $L^2(\mathbb{S}^1)$. The present work, far from to be the repetition of [30], is non trivial in the sense that now the CS on the circle (Descartes sphere in 3-dimensional case (6), (31)) are obtained as an orbit of a fixed vector in $\mathcal{H} = L^2(\mathbb{S}^1)$ under the Euclidean group action.

Coming back to the work [1], there was examined a 1-dimensional W-H group where the momentum parameter $p$ was cyclic (that is true since direction lies on a circle as it will be shown later). This entailed that the position parameter be discrete denying thus the possibility of having an infinitesimal translation generator. The central subgroup is also forced to be cyclic - as it should, being a phase. The $N$-dimensional version of that construction yields the direction vector as ranging over $N$-torus, however, instead of an $N$-sphere. In this work we show that there is no more discretization constraints on those group parameters. And, moreover, there is no need to pass by the opening comatic map or its inverse ([32]) in view to get the Euclidean CS.

Another non trivial problem is the next: it is well known that non relativistic quantum mechanics is a tool of much practical use in paraxial optics, and is too a basis for aberration expansions into the metaxial regime; but, really it is a theory globally different from the Euclidean optics. So, how far off the optical axis can we go? This work is an attempt to answer to this problem, in the sense that, choosing and describing some sections it will be possible to get in the metaxial regime.

Finally, the main task of this work is two-forwards: first, to build the UIR's of $E(2+1)$ and get Euclidean CS labelled by points on the cylinder, the new natural phase space of light rays world. Second, to try to answer partially to those optical problems. The paper is organized as follows: Section 2 describes the nature, manifold and the path of the light rays in homogeneous medium and paraxial regime. In Section 3, we give a parametrization of the Euclidean group, its orbits and its refracting index UIR's. The CS formalism is then given in view to get the CS family modulo some affine sections in the Section 4. Finally, the last section summarizes the results. A powerful tool of building the UIR's of $E(2+1)$, the Mackey-Kirillov theory, is briefly done in the Appendix.

2 Manifold and Path of Light Rays

This section will provide a convenient tool for treatment of elementary geometric optic and thus will fix notations used throughout this paper.

2.1 Fiber bundle of light rays

In geometrical optics, the essential concept is the ray of light. To an adequate approximation, we may regard rays as the paths along which the radiation energy travels and this breaks down near the edges of shadows. Let a point source of light be placed in the center of $\mathbb{R}^3$ (see Fig. 2.1).

**Fig. 2.1:** Light rays, from the point source $0$, propagate in the direction $\mathbf{n}$ and in a given time they form a spherical surface $S^2$

Since the light is propagating with a finite velocity, we may mark off on these rays sets of points which the light disturbance reach in a given time and join up the sets to form a spherical surface $S^2$. We define the velocity of light in a medium of refractive index $n$ to be $c/n$ ($n = |\mathbf{n}|$). From Fig. 2.1, light travels is from 0 to $s$ in the time

$$t_{ws} = \int_0^s n(s') \, ds' \tag{2.1}$$
where the integration is taken along the ray path.

The quantity \( n \, ds \) is called the optical path length. The point \( s \) runs over the so-called

Dcartes sphere \( S^2 \). When the medium through which the light travels homogeneous, then \( n \) = constant and the value \( n \, ds \) is the usual (1.7) optical distance [33].

We introduce now the coordinate system for the beam of light rays. Let a z-coordinate be the optical axis and \( z = 0 \) the reference plane or standard screen. Coordinates in that plane is \( q \) and the position vector on the screen is defined by

\[
\vec{q} = (q, z) \in \mathbb{R}^{2+1} \tag{2.2}
\]

The projection of the rays direction \( \vec{n} = \vec{n}(q, z) \) on the screen is

\[
\vec{p} = (p, h) \in S^2 \quad h = \sqrt{n^2(q, z) - p^2} \tag{2.3}
\]

here, the vector \( p \in S^2 \) will be called the optical momentum, \( r = \pm 1 \) plus (minus) corresponding to the case when light is moving in the positive (negative) z-direction. So, light rays are modeled as curves or lines in homogeneous media in 3-spaces oriented in all directions. Lines oriented in a chosen direction may be identified by a point \( \vec{n} \in S^2 \); but that is a projection ([34])

\[
\pi : B^2 \to S^2 \tag{2.4}
\]

from the ray bundle \( B^2 \) to the basis \( S^2 \) and the inverse image \( \pi^{-1}(\vec{n}) \) of a point on the Descartes sphere is a set of parallel rays that can be brought onto the other by translation \( T(q) \) within their perpendicular plane \( S^2 \) (see Fig. 2.2). The manifold of light rays is, thus a vector bundle \( B^2 \), with an \( S^1 \) base space of rays direction, and \( R^2 \) the local screen or the standard or typical fiber of the bundle ([35], [36], [37]). Later, this bundle will be obtained naturally in classification of the orbits of the Euclidean group \( E(2+1) \).

2.2 The Hamilton equations from the Snell’s law

We consider a transparent inhomogeneous medium where the refractive index \( n \), in general a function of position \( (n = n(q, z)) \), has a refracting or discontinuity surface \( S \) ([38]) (see Fig. 2.3). We assume that \( S \) separates the former from a second such medium of different index \( n' \); we then choose a point \( s \) on \( S \) and measure the angles \( \theta \) and \( \phi \) between the incident \( \vec{n} \) and the refracted \( \vec{n}' \) rays through the point \( s \) and the normal to surface at \( s \).

![Fig. 2.3: The in and out rays \( \vec{n} \) and \( \vec{n}' \) respectively are related by the conservation of their components tangential to the refracting surface \( S \) (Snell law or Sine law: \( p = |\vec{n}| \sin \theta = |\vec{n}'| \sin \phi \)).](image)

Now, there applies the well known Snell law of refraction ([39]):

\[
n \sin \theta = n' \sin \phi \tag{2.5}
\]

and the two rays and the normal \( \vec{N} \) lies in a plane. Eq. (2.1) may be viewed as a local conservation statement: the quantity \( p = n \sin \theta \) before the refraction must equal the quantity \( p' = n' \sin \phi \), after the refraction: in other words, the two rays \( \vec{n} \) and \( \vec{n}' \) are related by the conservation of their component tangential to the refracting surface \( S \).

First the \( \vec{n} = (p, h) \), tangent to \( \vec{q}(z) \), is parallel to \( dq/dz \) hence with Eq.(2.3),

\[
\frac{dq}{dz} = \frac{p}{z} = -\frac{dh}{dp} \tag{2.6}
\]

On the other hand ([32]), the gradient of the refractive index

\[
\vec{\nabla} n = \vec{\nabla} n(q, z) = \left( \frac{dn}{dq}, \frac{dn}{dz} \right) \tag{2.7}
\]

is parallel to the normal \( \vec{N} \), thus parallel to the change of direction vector

\[
\frac{dn'}{dz} = \left( \frac{dp}{dz}, \frac{dh}{dz} \right) = \alpha \vec{\nabla} n \tag{2.8}
\]
where the ratio \( \alpha = \alpha(p, q; z) \) is defined by the constraint that the direction vector remains on its Descartes sphere \( \vec{n} \cdot \vec{n} = n^2 = p^2 + h^2 \) and
\[
\frac{d}{dz} n^2 = 2n \cdot \frac{d}{dz} n = 2n \cdot \nabla n
\]  
(2.9)

Now, using (2.6) and the chain rule,
\[
\frac{d}{dz} n^2 = 2n \left( \frac{\partial n}{\partial p} + \frac{\partial n}{\partial q} \right) = 2n \left( \frac{\partial n}{\partial p} + \frac{\partial n}{\partial q} \right)
\]  
(2.10)

From (2.9-10) the ratio \( \alpha = \frac{\alpha}{\gamma} \) and
\[
\frac{d\alpha}{dz} = \frac{\alpha}{\gamma} \cdot \frac{\nabla n}{h} - \frac{\alpha}{\gamma} \frac{\partial n}{\partial q}
\]  
(2.11)

The next equalities, readily obtained from (2.6) and (2.11)
\[
\frac{dp}{dz} = -\frac{\alpha}{\gamma} \frac{\partial n}{\partial p}
\]  
(2.12)

constitute the Hamilton equations of the ray path along the optical axis \( z \), with the optical Hamiltonian
\[
H = -\hbar = -i\hbar \frac{p^2}{2m} - q \]  
(2.13)
Here \( H \) is taken to be negative for the reason that, when expanded in series of \( p^2 \), it reads
\[
H = -\sqrt{n^2 - p^2} = -n - \frac{p^2}{2n} - \frac{p^4}{8n^3} + \frac{p^6}{16n^5} + \ldots, |p| < n
\]  
(2.14)

and we get a positive kinetic energy \( E_k = p^2/2m \), with the mass \( m = n \) and the potential \( V(q, z) = -n \); but this is just a formal comparison with the classical mechanics ([40]), and as recalled in (2.6), we will stay with \( h = -\hbar \) as the component of \( \hbar \) along the optical axis, with \( h > 0 \) for right-moving rays and \( h < 0 \) for left-moving ones. If not specified, we will mainly consider the forward case i.e. \( h > 0 \) or \( z = +1 \).

### 2.3 Elementary examples

In the case of homogeneous medium, \( n = \) constant and the equations of motion become
\[
\frac{dp}{dz} = 0, \quad \frac{dq}{dz} = -\frac{p^2}{\sqrt{n^2 - p^2}}
\]  
(2.15)

The solution of these equations, in terms of the initial \( (z = 0) \) screen values \( p_0 \) and \( q_0 \), is the ray path
\[
p(z) = p_0 \quad q(z) = q_0 + z \frac{p_0}{\sqrt{n^2 - p_0^2}}
\]  
(2.16)

or equivalently,
\[
z = \phi(p, q) + \phi(p) \cdot q
\]  
\[
\phi(p) = -h_0 p \cdot q_0
\]  
\[
h_0 = |p|^2 / \sqrt{n^2 - p^2}
\]  
(2.17)

In the case of the paraxial optical regime, i.e. the approximation that the angle \( \theta \) between \( \hbar \) and the optical axis \( z \) (see Fig. 2.1) is small, we select a subspace of optical phase space restricted by:
\[
|\theta| \ll \pi, \quad |p| \ll n
\]  
(2.18)

For such rays, the Hamiltonian (2.14), expanded in power series of \( p^2/n^2 \) has next paraxial form:
\[
H = \frac{p^2}{2n} - q
\]  
(2.19)

If the refractive index \( n \) is again constant, then equation (2.19) gives the Hamiltonian for free motion in 2-dimensional space. The corresponding phase space is 4-dimensional and the basic group is the W-H one defined in (1.2-3). Here we emphasize that (2.19) is a particular case of the global or metaxial form of \( h \) in (2.13), where yet there are no assumptions on the vicinity of the optical axis \( z \); this was the stimulation of the present work.

If we want to compare trajectories from the Euclidean Hamiltonian (2.13) to those given by the paraxial or W-H Hamiltonian (2.19): we use the comatic procedure ([32]), which may be summarized in next steps: first, we regress the light ray back to the standard or object screen \( z = 0 \); second, we perform the opening or spraying transformation from variables in global regime (2.13) to paraxial regime (2.19) and third, we evolve the ray from the screen to general \( z \), but now in the paraxial regime. Recall that the Lie transformation ([41]) help us to solve the system (2.12): when \( h \) is defined by (2.13), we have in the backward (to the object screen) evolution \( (z) \):
\[
p_0 = p(0) = \exp(+zH)p(z)
\]  
(2.20)

where \( f \) being any function on the phase space and \( h \) is the optical hamiltonian ([5]). In the forward evolution \( (z) \) for \( h \) is (2.10), we get:
\[
p_0 = p(0) = \exp(-zH)p(z)
\]  
(2.21)

where \( p(0) \) and \( q(0) \) are the initial momenta and positions respectively in the W-H phase.
space. The essential ingredient is the next geometric map \( C \) at the screen (Fig. 2.4):

\[
\begin{array}{c}
|p| < n \\
|q| < n
\end{array}
\]

Fig. 2.4: The geometric map \( C \) opens the compact momentum range \( |p| < n \) to the full momentum range \( p \in \mathbb{R}^2 \). From which we obtain:

\[
\begin{align*}
p &\rightarrow C_p = \tilde{p} = \frac{\gamma p}{\sqrt{\gamma^2 + \beta}(|q| - n^2 p \cdot q)} \\
q &\rightarrow C_q = \tilde{q} = \frac{\gamma q}{\sqrt{\gamma^2 + \beta}(|q| - n^2 p \cdot q)}
\end{align*}
\]  
(2.22)

This map opens the compact momentum range \( |p| < n \) to the full W-H momentum range \( \mathbb{R}^2 \) and it is called **comatic** because it is generated by the comatic Hamiltonian \( H_{com} = p^2 q \), according to [42] and [43]:

\[
\begin{align*}
\tilde{p} &= \exp(\gamma H_{com})p \\
\tilde{q} &= \exp(\gamma H_{com})q
\end{align*}
\]  
(2.23)

for \( \gamma = \frac{1}{2n^2} \). To third degree in the phase space variables, we have the third-order Siedel coma of optics, and the full Lie series define the Seidel-Lie global coma aberration.

Applying now the coma procedure pointed out before, we have:

\[
\begin{align*}
\tilde{p}(z) &= \exp(-zH)C \exp(+z\tilde{H}) \left( \begin{array}{c} p(z) \\ q(z) \end{array} \right) \\
\tilde{q}(z) &= \frac{\gamma z}{z^2 + \beta} \left( q - \frac{n^2 p \cdot q}{\sqrt{\gamma^2 + \beta}} \right)
\end{align*}
\]  
(2.24)

to get, by mean of (2.22-23):

\[
\begin{align*}
\tilde{p} &= \frac{\gamma p}{\sqrt{\gamma^2 + \beta}} \left( q - \frac{n^2 p \cdot q}{\sqrt{\gamma^2 + \beta}} \right) + \frac{\gamma z}{\sqrt{\gamma^2 + \beta}} \left( q - \frac{n^2 p \cdot q}{\sqrt{\gamma^2 + \beta}} \right) p \\
\tilde{q} &= \frac{n^2 p \cdot q}{\sqrt{\gamma^2 + \beta}} (q - n^2 p \cdot q)
\end{align*}
\]  
(2.25)

Returning again to the optical Hamiltonian \( H \) in (2.13), for a classical point particles systems, the latter contains only one form of order two in \( p \); this is the Gaussian or paraxial approximation to optics, as just noticed before. Terms of order higher than two in (2.14) generate nonlinear transformations in optical phase space, which are defined as aberrations ([44]). Free propagation in homogenous medium may itself aberrate; the relation (2.16) produces next linear transformation of phase space (inverse regime of (2.20)):

\[
\begin{align*}
\left( \begin{array}{c} p \\ q \end{array} \right) &\rightarrow S_{\alpha}(p, q) = \left( \begin{array}{c} p \\ q + \frac{n^2 p \cdot q}{\sqrt{\gamma^2 + \beta}} \end{array} \right) \\
&= \left( \begin{array}{c} p \\ q + \frac{n^2 p \cdot q}{\sqrt{\gamma^2 + \beta}} p \end{array} \right)
\end{align*}
\]  
(2.26)

with \( \omega_{0}(p) = (1 - p^2/n^2)^{1/2} \).

The coefficient \( z/2n \) of \( |p|^3 \) represents the third-order spherical aberration of simple propagation.

We summarize the results in this section, by a classification of the next propagation regimes, interacting in applications at the end of section 4.

(i) Paraxial case:

Relation (2.21) may rewritten as follows:

\[
\begin{align*}
p &= p_0 \\
q &= q_0 + z \tilde{p}(p, q_0) \\
z &= \varphi(p, q_0) \equiv \varphi_1(p) + \varphi_3(p, q_0)
\end{align*}
\]  
(2.27)

with,

\[
\varphi_1(p) = \frac{n^2 p \cdot q_0}{|p|^2}, \quad \varphi_3(p, q_0) = -\varphi_1(p)q_0
\]  
(2.28)

(ii) Seidel-Lie coma aberration case:

In the same manner, from (2.25) we get, for a new vectorial function \( \varphi_2(p) \) and a scalar \( \varphi_3(p, q_0) \), next relations:

\[
\begin{align*}
\varphi_2(p) &= \frac{\omega_0}{1 - \omega_0^2} \varphi_1(p) \\
\varphi_3(p, q_0) &= -\frac{\omega_0^2}{1 - \omega_0^2} \varphi_2(p)
\end{align*}
\]  
(2.29)

with \( \omega_0(p) = (1 - p^2/n^2)^{1/2} \).

(iii) Seidel-Lie spherical aberration case:

The formula (2.26), looked at from the same point of view, leads to:

\[
\varphi_3(p) = \omega_0^2(p) \varphi_2(p) \\
\varphi_3(p, q_0) = -\omega_0^2(p)q_0 \cdot \varphi_2(p)
\]  
(2.30)

where \( \omega_0(p) = (1 + \frac{2}{n^2})^{-1} \).

(iv) Global or metaxial homogeneous case:

When now all terms in (2.26) are accounted for, we see that:

\[
\varphi_3(p) = \omega_0(p) \varphi_2(p) \\
\varphi_3(p, q_0) = -\omega_0(p)q_0 \cdot \varphi_2(p)
\]  
(2.31)

(v) static or optical center ([45]) case:

Let in (2.26) the reference or object screen values \( p_0 = q_0 = 0 \). Then

\[
\varphi_3(p, q_0) = 0
\]  
(2.32)

Even trivial, this case will be of great interest in the section 4, as it is a starting point in building CS in a more simple way.

To conclude with generalization, all the functions

\[
\varphi(p) = \varphi_3(p, q_0)
\]  
(2.33)

where \( \varphi_3(p) \) are defined in (2.27-32) belong to the so-called **affine** sections ([17]) and obviously

\[
|\varphi_1| > |\varphi_2| > |\varphi_3| > |\varphi_4| = 0
\]  
(2.34)

Further, the family of functions \( \varphi(p) \) will be called the optical one since still we deal with the geometric optic ([39]).
3 Refractive Index UIR’s of E(2+1)

The Euclidean group, denoted \( E(2+1) \), following notations in (2.2.3) and the footnote (2), is the group of rigid motions in usual 3-spaces ([46],[28]). Its subgroup \( E(1+1) \) is well known in the “helicity role” or “little group” in classifying the massless UIR’s of the Poincaré group \( P(3,1) \) ([47],[48]). In this section we deal directly with \( E(2+1) \).

3.1 The structure of the Euclidean group

The group \( E(2+1) \) acts on the real vector space \( \mathbb{R}^{2+1} \) by a rotating element \( R \in SO(2+1) \).

\[ \mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3 \]  

We write the element of \( E(2+1) \) by \( g = (R; b) \), where \( b \) now is the space translation vector or in matrix notation

\[ g = (R; b) \equiv \begin{pmatrix} R & b \\ 0 & 1 \end{pmatrix} \]  

The group law and the inverse element are, in light of (3.2),

\[ g \cdot g' = (R; b) \cdot (R'; b') = \left( RR'; Rb + b' \right) \]  

\[ g^{-1} = (R^{-1}; -R^{-1}b) \]  

The group law (3.3) shows clearly that \( E(2+1) \) is a semi-direct product of groups

\[ E(2+1) = SO(2+1) \ltimes \mathbb{R}^{2+1} \]  

This group is unimodular i.e. the left and right Haar measures are equal:

\[ d\mu_L(R; b) = d\mu_R(R; b) = dm(R)db \]  

where \( dm(R) \) is the usual measure on \( SO(2+1) \).

The Lie algebra \( \mathfrak{e}(2+1) \) of \( E(2+1) \) can be represented as a Lie algebra of matrices, as follows: let \( \mathfrak{y} \in \mathfrak{so}(2+1) \) and \( \mathbf{v} \in \mathbb{R}^{2+1} \), then the generic element \( \mathfrak{y} \in \mathfrak{e}(2+1) \) is written:

\[ \mathfrak{y} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{v} \end{pmatrix} \]  

and

\[ \mathfrak{e}(2+1) \simeq \mathfrak{so}(2+1) \ltimes \mathfrak{r} \]  

where \( \{J,T\} \) is the usual span or basis of \( \mathfrak{e}(2+1) \). A simple calculation shows that

\[ Ad(R; b) \cdot Y = gYg^{-1} = \begin{pmatrix} Y_R & RY_Rb \\ 0 & 1 \end{pmatrix} \]  

So that \( Ad(R; b) \) has the following matrix realization in the above basis:

\[ Ad(R; b) = \begin{pmatrix} R & 0 \\ -R^{-1}b & R^{-1} \end{pmatrix} \]  

and from (A.5),

\[ ad(Y_R; \pi_0) = \begin{pmatrix} Y_R & 0 \\ -\mathbf{v} & Y_R \end{pmatrix} \]  

where \( \{\cdot,\cdot\} \) is the usual Lie bracket.

It is then simple to derive the matrix form of the coadjoint action, with respect to the contragradient basis element \( F = (Y_R; \pi_0) \in \mathfrak{e}^*(2+1) \) defined by the coupling,

\[ (F, Y) = F(Y) = tr(Y_R\pi_0 \mathbf{v}) + \mathbf{k} \cdot \mathbf{v} \]  

namely

\[ Ad^*(R; b) = \begin{pmatrix} R^{-1} \cdot R & 0 \\ -R^{-1}b \cdot R^{-1} \end{pmatrix} \]  

Now, the associated coadjoint orbits read:

\[ Ad^*(R; b)F = Ad^*(R; b)(Y_R; \pi_0) = (Y'_R; \pi') : Y'_R = -R^{-1}Y_RR + R^{-1}b \otimes R^{-1} \]  

where \( \mathbf{k} \otimes \mathbf{b} \) is the usual diadic product. On substituting (3.11) in (A5-8), we obtain

\[ ad^*(Y_R; \pi_0)(Y_R; \pi_0) = \begin{pmatrix} [Y_R, \pi_0] & -\pi_0 \\ 0 & -Y_R \end{pmatrix} \]  

\[ ad^*(Y_R; \pi_0)(Y_R; \pi_0) = \begin{pmatrix} [Y_R, \pi_0] & -\pi_0 \\ 0 & -Y_R \end{pmatrix} \]  

It is convenient to rewrite elements of \( E(2+1) \), \( \mathfrak{e}(2+1) \) and \( \mathfrak{e}^*(2+1) \) respectively by:

\[ g = (R; b) = (x, z; b, y), x \in S^1, z \in \mathbb{S}(2) \]  

\[ F = (Y_R; \pi) = (\xi, \mathbf{v}, \pi, \mathbf{v}_0), \xi \in \mathbb{R}^2 \]  

that follows from the usual Cartan decomposition of \( R \in SO(2+1) \):

\[ R = PK = (x, z; b, y) \]  

where

\[ p(x) = l_2 - \alpha x \otimes x, \quad \alpha = (x_1 + 1)^{-1}, \quad l_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]  

In terms of the above notations in (3.15), (3.3) and (A.8) read:

\[ g' = (xz, z^2 x, s, z, -s, x^2, -x, 0, 0, 0) \]  

\[ g'^{-1} = (-x, s, 0, -z, 0, 0, 0, 0, 0, 0) \]  

\[ B_{R}(y, \mathbf{v}) = -k \cdot \mathbf{v} + tr(\mathbf{k} \otimes \mathbf{v}) + (k \otimes \mathbf{v})Y_R^* + \xi \cdot \mathbf{v} + \xi \cdot \mathbf{k} \cdot \mathbf{v} \]  

We finally have three distinct families of orbits uniquely characterized by the points \( (k, k_1) \in \mathbb{R}^{2+1} \) according to (3.14):
(i) degenerate orbits: \((k = 0, k_1 = 0)\)

(ii) refractive index or Stone-von Neumann orbits \([25]\): they are related to a non-vanishing point \(F_0\).

\[
F_0 = (\mathbf{o}, n) \in \mathbf{c}^*(2 + 1)
\]  

(3.18)

where \(n\) is the refractive index defined in the last section. These orbits may be compared to the Wigner massive ones of the Poincaré group \([47]\).

(iii) Wavefront orbits, similarly to the massless case \([47, 48]\), are those corresponding to the point

\[
F_0 = (k_0, n), \quad k_0 = (k, 0), \quad k \in \mathbb{R}
\]  

(3.19)

with \(n\) as before and \(|k|\) the wavenumber \([6]\). We will mainly deal with them in the next section.

3.2 Polarization and inducing procedure

As just said, we consider the second type of orbits, the degenerate case being trivial. The stabilizer algebra of \(F_0\) is readily seen from (3.17-18):

\[
\ker \mathcal{B}_{F_0} = \{ Y(F_0) \in \mathbb{E}(2 + 1) | Y(F_0) \} \]  

(3.20)

We are now in a position to show that, naturally,

\[
\mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{Z}_2 (2 + 1) \]  

(3.21)

is the phase space of the light rays, as seen in section 2.1.

Indeed, \(G(F_0) \cong \mathbb{E}(2 + 1) / \mathbb{E}(2) \cong \mathbb{S}^2 \) which is the symmetry group of the light rays \([49]\). Moreover, \(\dim G / G(F_0) \) and \(\dim G / G(F_0) \) are even as it was expected \([29]\).

Applying the definition A.1 and the relation A.13-14 to (3.20-21), we get the next polarization:

\[
\mathbb{S}^2 \times \mathbb{Z}_2 (2 + 1) \]  

(3.22)

Thus,

\[
\mathbb{C}^2 \times \mathbb{S}^2 \times \mathbb{Z}_2 (2 + 1) \]  

(3.23)

Having disposed of the preliminary step in A.27-28, we have next natural section \([50]\):

\[
\sigma_{n^{-1}} : \mathbb{C}^2 \times \mathbb{S}^2 \times \mathbb{Z}_2 (2 + 1) \to \mathbb{Z}_2 (2 + 1)
\]  

(3.24)

where \(\mathcal{S}(k) \in \mathbb{S}(2)\) is chosen such that

\[
\sigma_n^{-1} = \mathbb{C}^2 \times \mathbb{S}^2 \times \mathbb{Z}_2 (2 + 1) \to \mathbb{C}^2 \times \mathbb{S}^2 \times \mathbb{Z}_2 (2 + 1)
\]  

(3.25)

that is, \(\sigma_n^{-1}\) takes \(F_0\) to \(k\). The point \(g \cdot |k|\) is

\[
g \cdot |k| = \frac{1}{n} (x_k + r(x)k_0), \quad g \in \mathbb{E}(2 + 1), \quad k \in \mathbb{S}^2
\]  

(3.26)

and

\[
\sigma_n(g \cdot |k|) = (-n^{-1} \frac{1}{n} (x_k + r(x)k_0), \mathcal{S}(k); o, 0)\]  

(3.27)

From the above, we compute

\[
\sigma_n^{-1}(k) g \sigma_n(g \cdot |k|) = (o, \mathcal{S}(g, k); b, b')\]  

(3.28)

where

\[
b' = -n^{-1} |k|_0 = (-n^{-1} |k|_0 \otimes k) b
\]

\[
b'' = n^{-1} (k_0 b + k \cdot b) = n^{-1} \mathcal{S}(k_0 b + k \cdot b)
\]

\[
\mathcal{S}(g, k) = \mathcal{S}(k) \mathcal{S}(x_k + r(x)k_0), \quad \forall \mathcal{S}(2)
\]  

(3.29)

The last rotating element \(\mathcal{S}(g, k)\) is \(\mathcal{S}(x, k) \in \mathbb{S}(2)\) known as the Wigner rotation \([51, 52]\). As we consider the scalar field of light rays, we will put \(\mathcal{S}(k) = 1\).

Now, the element in eq.(3.28) belongs to the polarization subgroup at \(F_0\). The later is abelian and has all its IUR's one dimensional. Using Eqs. A.17-18, and (3.28), we thus arrive at the following expression for the "cocycle" \([50]\) or character \(\chi_{F_0}\):

\[
\chi_{F_0}(\mathcal{S}(g, k)) = \exp\left[2\pi i (k_0 b + k \cdot b)\right]
\]  

(3.30)

With this form of the cocycle, we write the final form of the induced IUR's of \(E(2 + 1)\), called Refractive Index according to (3.18):

\[
\eta_0 = \mathcal{S}(x, k) \in \mathbb{S}(2) \]  

(3.31)

where \(\eta(k)\) is a square integrable on \(S^2\) i.e. the Descartes sphere.

Using Eqs. A.17-18, and (3.28), we thus arrive at the following expression for the "cocycle" \([50]\) or character \(\chi_{F_0}\):

\[
\chi_{F_0}(\mathcal{S}(g, k)) = \exp\left[2\pi i (k_0 b + k \cdot b)\right]
\]  

(3.32)

With this form of the cocycle, we write the final form of the induced IUR's of \(E(2 + 1)\), called Refractive Index according to (3.18):

\[
\eta_0 = \mathcal{S}(x, k) \in \mathbb{S}(2) \]  

(3.33)

where \(\eta(k)\) is a square integrable on \(S^2\) i.e. the Descartes sphere.

The inner product of two functions in that space is

\[
\langle \eta_0 | \eta_0 \rangle = \int_{S^2} \mathcal{S}(g, k) \mathcal{S}(g, k) \mathcal{S}(k, k) \]  

(3.34)

here, the direction sphere \(\mathbb{S}(2)\) is projected (twice) on its equatorial screen plane, the disk \(\mathbb{D}(n)\), where \(|k| < n\) and the boundary \(|\eta| = n\) works the disks; the function \(\eta_k(k)\) is written as \(\eta_k(k)\) independent of \(k\) for \(k_i > 1\) in the "forward" hemisphere and \(k_i < 1\) in the "backward" hemisphere. The inner product in (3.32) is invariant under rotations of \(S^2\), as well as under translations, since the latters cancels on account of the sesquilinearity of the inner product.

In the next section, we consider functions \(\eta_k(k)\) for \(\eta_k(k)\) and so consider the forward hemisphere: then (3.32) is simply written:

\[
\langle \eta_0 | \eta_0 \rangle = \int_{\mathbb{D}(n)} \mathcal{S}(g, k) \mathcal{S}(g, k) \mathcal{S}(k, k) \]  

(3.35)

or, in polar coordinates

\[
\langle \eta_0 | \eta_0 \rangle = \int_{\mathbb{D}(n)} \frac{d\mathcal{S}(k, k)}{\sqrt{\mathcal{S}(k, k)^2}} \int_{\mathcal{S}(k, k)} \mathcal{S}(g, k) \mathcal{S}(g, k) \mathcal{S}(k, k) \]  

(3.36)
4 Geometric Optical Coherent States

4.1 Coherent states formalism

Here we review the standard method of coherent states construction ([53]). Let $G$ be a locally compact group, with left Haar measure $d
u_k(g)$, and $U$ a continuous unitary irreducible representation of $G$ into a Hilbert space $\mathcal{H}$. Then $U$ is said to be "square integrable" if there exists a vector $\zeta \in \mathcal{H}$ for which

$$d(\eta, \zeta) = \int G |(U(g)\eta|\zeta)^2d\nu_k(g) < \infty, \forall \zeta \in \mathcal{H}$$

or, equivalently, the linear map

$$W_\eta : \mathcal{H} \to L^2(X, d\nu), (W_\eta \zeta)(x) = \langle \eta, \zeta \rangle$$

is an isometry onto a closed subspace $\mathcal{H}_\eta \subset L^2(X, d\nu)$:

$$W_\eta^* W_\eta = I$$

(ii) The projection operator

$$P_\eta = W_\eta W_\eta^*$$
on $\mathcal{H}_\eta$ is an integral operator with kernel

$$K(x, x') = \langle \eta, \eta_x \rangle$$

and $\mathcal{H}_\eta = P_\eta \mathcal{H}$ is a "reproducing kernel Hilbert space" of functions. (iii) Since $W_\eta$ is an isometry, then it may be inverted on its range $\mathcal{H}_\eta$ and the inverse is simply the adjoint operator

$$W_\eta^{-1} = W_\eta^*$$
on and thus takes place the "inversion formula":

$$W_\eta^{-1} \zeta = \int_X \zeta(x)\eta_x d\nu(x), \zeta \in \mathcal{H}$$

Here, we recall the quantum mechanical notations used in the next subsections:

$$\mathcal{C}(\mathcal{H}) \ni \langle \xi | W_\eta \rangle = \langle W \xi | \eta \rangle = \langle \xi | W_\eta^* \rangle = \langle \xi | \zeta \rangle = \langle W \xi | \eta \rangle = \langle \xi | W_\eta \rangle$$

where, for instance, $W = L^* U_\eta$ is the space of the representation $U_\eta$ of $G$, and $V$ the configuration space of the considered system. The formalism from 4.5-15 may be more generalized in the sense of [17]. Let $G, U, \eta, \eta_x$ be as before, $H$ a closed subgroup of $G, X = G/H$ with invariant measure $\nu$ and section $\sigma : X \to G/H \to G$ a Borel section.

**Definition 4.1:** $U$ is "square" integrable modulo $(H, \sigma)$ for the ray $\eta \in \mathcal{H}$ if the integral

$$\int_X \nu(\sigma(x))d\nu_k(\eta_x U(\sigma(x))d\nu(x)$$

converges weakly to a "bounded positive" invertible operator $A_\eta$ on $\mathcal{H}$ i.e.,

$$0 < \int_X (|U(\sigma(x))\eta|)^2d\nu(x) = \int_X (|P_\eta \zeta(x)|^2d\nu(x) = \langle \zeta|A_\eta \zeta \rangle < \infty, \forall \zeta \in \mathcal{H}$$

Notice that $A_\eta^{-1}$ may be unbounded in general. The family of coherent states is defined as before

$$O_\eta(\sigma) = \{\eta_x \in \mathcal{H}| \eta_x = U(\sigma(x))\eta, x \in X\}$$

or, equivalently, the linear map

$$W_\eta : \mathcal{H} \to L^2(X, d\nu), (W_\eta \zeta)(x) = \langle \eta, \zeta \rangle$$

is an isometry onto a closed subspace $\mathcal{H}_\eta \subset L^2(X, d\nu)$:

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$$W_\eta : \mathcal{H} \to L^2(X, d\nu), (W_\eta \zeta)(x) = \langle \eta, \zeta \rangle$$

is an isometry onto a closed subspace $\mathcal{H}_\eta \subset L^2(X, d\nu)$:

$$W_\eta^* W_\eta = I$$

(iii) Since $W_\eta$ is an isometry, then it may be inverted on its range $\mathcal{H}_\eta$ and the inverse is simply the adjoint operator

$$W_\eta^{-1} = W_\eta^*$$

and $\mathcal{H}_\eta = P_\eta \mathcal{H}$ is a "reproducing kernel Hilbert space" of functions. (iii) Since $W_\eta$ is an isometry, then it may be inverted on its range $\mathcal{H}_\eta$ and the inverse is simply the adjoint operator

$$W_\eta^{-1} = W_\eta^*$$

and thus takes place the "inversion formula":

$$W_\eta^{-1} \zeta = \int_X \zeta(x)\eta_x d\nu(x), \zeta \in \mathcal{H}$$

Here, we recall the quantum mechanical notations used in the next subsections:

$$\mathcal{C}(\mathcal{H}) \ni \langle \xi | W_\eta \rangle = \langle W \xi | \eta \rangle = \langle \xi | W_\eta^* \rangle = \langle \xi | \zeta \rangle = \langle W \xi | \eta \rangle = \langle \xi | W_\eta \rangle$$

where, for instance, $W = L^* U_\eta$ is the space of the representation $U_\eta$ of $G$, and $V$ the configuration space of the considered system. The formalism from 4.5-15 may be more generalized in the sense of [17]. Let $G, U, \eta, \eta_x$ be as before, $H$ a closed subgroup of $G, X = G/H$ with invariant measure $\nu$ and section $\sigma : X \to G/H \to G$ a Borel section.

**Definition 4.1:** $U$ is "square" integrable modulo $(H, \sigma)$ for the ray $\eta \in \mathcal{H}$ if the integral

$$\int_X \nu(\sigma(x))d\nu_k(\eta_x U(\sigma(x))d\nu(x)$$

converges weakly to a "bounded positive" invertible operator $A_\eta$ on $\mathcal{H}$ i.e.,

$$0 < \int_X (|U(\sigma(x))\eta|)^2d\nu(x) = \int_X (|P_\eta \zeta(x)|^2d\nu(x) = \langle \zeta|A_\eta \zeta \rangle < \infty, \forall \zeta \in \mathcal{H}$$

Notice that $A_\eta^{-1}$ may be unbounded in general. The family of coherent states is defined as before

$$O_\eta(\sigma) = \{\eta_x \in \mathcal{H}| \eta_x = U(\sigma(x))\eta, x \in X\}$$
with the following properties ([17]):

(i) Define the map

\[ W_0 : \mathcal{H} \to L^2(X, dv), \quad \langle W_0 \zeta \rangle(x) = \langle \eta_{res} \zeta \rangle \]  

Then the range \( \mathcal{H}_0 \) of \( W_0 \) is complete with respect to the scalar product

\[ (4.21) \]

and \( W_0 \) is "unitary" operator from \( \mathcal{H}_0 \) onto \( \mathcal{H}_0 \), i.e. one has the resolution (not to the identity!)

\[ (4.22) \]

(ii) In addition, the orthogonal projection from \( L^2(X, dv) \) onto \( \mathcal{H}_0 \) is an integral operator \( K_0 \), thus \( \mathcal{H}_0 \) is a reproducing kernel Hilbert space of functions, and

\[ (4.23) \]

(iii) As before, the map \( W_0 \) may be inverted on its range by the adjoint operator on \( \mathcal{H}_0 \),

\[ (4.24) \]

The kernel is given explicitly by

\[ (4.25) \]

The function is called "weighted coherent states" if the vectors

\[ (4.26) \]

called "weighted coherent states" are introduced, \( W_0(x) \) are suitable bounded operator, essentially \( A_0^{-1/2} \) acting "fiberwise":

\[ (4.27) \]

4.2 Gaussian sections

The representation obtained in (3.30) is not square integrable in \( \mathcal{H} = L^2(S^2, du(\hat{k})) \), according to the relation (4.1): indeed, integration over the variable \( b \), splits into infinity ([54]), Thus, we naturally kill that group parameter and work in the phase space \( B^2 \) in (3.21), just applying the same techniques done in the relativistic case ([17]).

To this end, we see how the coset spaces appear with the group action therein and their quasi-invariant measures (see A 20-21), before we restrict the Refractive Index Representation (3.30) to the left coset.

In Section 2.2, the light rays were described as "points" \((p, q)\) in an optical "phase space" \( B^2 \), evolving along the "optical axis" \( z \) of the system or "elementary object". If such an object is rotated around its axis or translated in its direction, it is still the same elementary object. These are the "symmetry transformations" of the light rays, which coincide with the stabilizer subgroup of \( E(2+1) \), \( G(F_0) = \{ g(F_0) \in GL(F_0) \mid g(o, s, o, b) \}

\[ (4.28) \]

The Euclidean transformations \((x, z, b, h)\) of the rays in \( \Gamma = \Gamma_L (\Gamma_R) \) can be found by acting from the left (right) on \( [p, q] \):

\[ (4.29) \]

\[ (4.30) \]

From these relations, we see that, unless \( \Gamma_L \) and \( \Gamma_R \) are equal, they have two different quasi-invariant measures:

\[ (4.31) \]

\[ (4.32) \]

On \( B^2 = \Gamma = \Gamma_L \), we choose next section:

\[ (4.33) \]

This section will be called the flat ([25]), basic or Galilean ([54], [27]). Now, any other measurable section then may have the form

\[ (4.34) \]

where \( z(p, q) \) is a \( R \)-valued function on the phase space \( B^2 \) and, according to section 2, may be taken, for instance to be the solution of the Hamiltonian equation in (2.17-17):

\[ (4.35) \]

\[ (4.36) \]

The family of such sections was called, according to section 2, an optical one and in the next, we will simply denote it by \( z(p, q) = \sigma(p, q) \).
4.3 De Bièvre-Klauder coherent states

We define the Euclidean coherent states as orbit in $\mathcal{H} = L^2(\mathbb{R}^2, du(\vec{k}))$, according to (4.20) and (4.23):

$$\eta_{\sigma(p,q)} = U(\sigma(p,q))\eta,$$  

(4.37)

where $U(\sigma(p,q))$ is the restriction of (3.30) to the section and the corresponding resolution of the operator

$$\mathcal{A}_H^2 = \int_S |\eta_{\sigma(p,q)}\rangle \langle \eta_{\sigma(p,q)}| dp dq$$  

(4.38)

The fundamental or crucial ([54]) exercise is to evaluate the integrals of the type (see 4.19): \( \forall \zeta_1, \zeta_2 \in \mathcal{H} \)

$$I_{\zeta_1}^{\zeta_2}(\zeta_1, \zeta_2) = \langle \zeta_1 | \mathcal{A}_H^2 | \zeta_2 \rangle = \int_S \langle \zeta_1 | \mathcal{A}_H^2(p, q) | \zeta_2 \rangle dp dq$$  

(4.39)

We start with the more general integral

$$I_{\zeta_1}^{\zeta_2}(\zeta_1, \zeta_2) = \int_S \langle \zeta_1 | \mathcal{A}_H^2 | \zeta_2 \rangle dp dq$$  

(4.40)

from which $I_{\zeta_1}^{\zeta_2}(\zeta_1, \zeta_2)$ is obtained by setting $\eta_1 = \eta_2 = \eta$.

It is pedagogically useful to treat the simplest case when $\sigma(p, q)$ is the flat section ($\nu = 0$) i.e. $\sigma_0(p) = 0$ and to extend the construction to a more general case when $\eta_1(p) \neq 0$. Therefore (4.40) becomes

$$I_{\zeta_1}^{\zeta_2}(\zeta_1, \zeta_2) = \int_S dp dq \int_{\mathbb{R}^2} \frac{dk}{k_1} \frac{dk'}{k'} \exp[2\pi i \langle k-k' \rangle \cdot \xi] \times \langle \xi | \theta^{-1}(p k_1 + \Gamma(p, k)) \theta^{-1}(p k' + \Gamma(p, k')) \zeta_1(k) \zeta_2(k') \rangle$$  

(4.41)

Replacing now the integral over $q$ of $\exp[2\pi i \langle k-k' \rangle \cdot \xi]$ by $\delta(k-k')$ and performing the $dk'$ integration, (4.40) becomes with $\eta_1 = \eta_2 = \eta, \zeta_1 = \zeta_2 = \zeta$

$$I_{\zeta_1}^{\zeta_2}(\zeta) = \int_{\mathbb{R}^2} dp \int_{\mathbb{R}^2} \frac{dk}{k_1} \delta(h^{-1}(p k_1 + \Gamma(p, k)))^2 |\zeta(k)|^2$$  

(4.42)

We consider now the variable change

$$\begin{cases}  
p - p' = n^{-1}(p k_1 + \Gamma(p, k))  
p_1 = n^{-1}(p k_1 - p' k) \end{cases}$$  

(4.43)

and

$$dp = n^{-1}(p k_1 + \Gamma(p, k))^2 \delta(p - p') \big| dp' = n^{-1}(p k_1 - p' k) \big| dp'$$  

(4.44)

and

$$I_{\zeta_1}^{\zeta_2}(\zeta) = I_{\zeta_1}^{\zeta_2}(\zeta) + I_{\zeta_1}^{\zeta_2}(\zeta)$$  

(4.45)

where

$$I_{\zeta_1}^{\zeta_2}(\zeta) = n^{-1} \| \zeta \|^2 \int_{\mathbb{R}^2} \frac{dp}{p_1} \langle \sqrt{p_1} \eta(p) \rangle^2$$  

(4.46)

From (4.46), we may now find a ray (fiducial vector (54)] $\eta$ such that $I_{\zeta_1}^{\zeta_2}(\zeta)$ is finite, that is $\eta \in D(H^{1/2})$, the domain of the square root of the optical Hamiltonian and

$$(H\eta)(k) = k \eta(k)$$  

(4.49)

For fixed $\eta \in D(H^{1/2}) = \{ \eta \in \mathcal{H} = L^2(\mathbb{R}^2, du(\vec{k})); ||\eta||^2<\infty \}$

$$\int_{\mathbb{R}^2} |\eta(p)|^2 \frac{dp}{p_1} = 1$$  

(4.49)

Hence,

$$I_{\zeta_1}^{\zeta_2}(\zeta) = \delta_{\eta}(\eta) ||\zeta||^2, \quad \delta_{\eta}(\eta) = n^{-1} ||H^{1/2}\eta||^2$$  

(4.50)

and

$$\delta_{\eta}(\eta) = \delta_{\eta}(\eta) \big\{ \Omega_{\eta}(\eta) = \{ \delta_{\eta}(\eta) \}^{-1/2} U(\sigma_0(p, q)) \zeta(k) \big\}$$  

(4.51)

where the theory in subsection 4.1 is easily applied. The map in (4.21)

$$W_{\eta}: \mathcal{H} \mapsto L^2(\mathbb{R}^2, dp dq)$$  

(4.52)

is an isometry. Given a vector $\eta \in \mathcal{A}(G(F_0, e_0))$, the set of all admissible vectors dense in $\mathcal{H} = L^2(\mathbb{S}^2, dp dq)$, its orbit under $U(\sigma_0(p, q))$:

$$\{ \Omega_{\eta}(\eta) \}$$  

(4.53)

is overcomplete in $\mathcal{H}$ and one gets finally the resolution of the identity:

$$\int_{\mathbb{R}^2} |\delta_{\eta}(\eta)\rangle \langle \delta_{\eta}(\eta)| dp dq = 1$$  

(4.54)

Let $p_1 = W_{\eta} W_{\eta}^*$ be the projection operator onto the subspace $\mathcal{H}_\eta \subseteq L^2(\mathbb{R}^2, dp dq)$, which is the image of $\mathcal{H}$ under $W_{\eta}$. Thus there exist a reproducing kernel

$$K_\eta: \mathbb{S}^2 \times \mathbb{S}^2 \rightarrow \mathbb{C}$$  

(4.55)

such that

$$K_\eta(p, q; p', q') = \langle \delta_{\eta}(\eta) | \eta_{\sigma_0(q)} \rangle \delta_{\eta}(\eta) | \eta_{\sigma_0(q')} \rangle$$  

(4.56)

$$K_{\eta, \Phi}(p, q) = \int_{\mathbb{R}^2} \langle \eta_{\sigma_0(q')} \rangle dp dq'$$  

(4.57)

$$\int_{\mathbb{R}^2} \langle \eta_{\sigma_0(q')} \rangle dp dq' = \langle \eta_{\sigma_0(q')} \rangle dp dq'$$  

(4.58)

The restriction condition (4.48) means that $\eta(k)$ has its support on the forward hemisphere of $\mathbb{S}^2, SO(2)$-invariant and the growth condition is given by $I_{\zeta_1}^{\zeta_2}(\zeta) < \infty$ in (4.46). As mentioned in the introduction, the family of CS in (4.53) was obtained from differential geometric point of view ([29]) and more recently ([30]) was constructed from the canonical
W-H CS on the line by direct integral decomposition of $L^2(\mathbb{R})$ into copies of $L^2(S^1)$. Here, we get the Euclidean CS as an orbit of a fixed $\eta \in L^2(S^2)$ under the action of $E(2+1)$, modulo the subgroup of symmetry of the light rays, namely $O(K^2) = \{ (0, \varnothing , 0, 0) \}$. And instead of reducing those CS from reducible representation ([54]), here we use the refractive index representation which is irreducible as induced from the maximal isotropy subgroup or the polarization $P$. Thus, we can state that the family of CS associated to the basic or "object screen" (optical center) section $\sigma_p(p, q)$ are those obtained by authors in [29] and [54]. Now, we extend them to the other sections described in the section 4.2.

### 4.4 Optical coherent states

Let us consider the optical section $\sigma_v = \sigma$ (hereafter) in (4.5). The restriction of the UHR in (3.30) to such section is

$$[U(p, q)]\eta(k) = \exp[-2\pi i K(p, k) \cdot \eta]$$

with

$$K(p, k) = (1 - n \cdot p) k + P_1 k_1 \varnothing (4.59)$$

Before doing the usual crucial exercise, we mention here some properties inherent to $K(p)$.

For any $p = \varnothing (p)$, we easily verify that

$$K(p, k) - K(p, k') = 0 \iff k = k' \text{ and } k_1 = k_1' \quad (4.60)$$

Moreover, if

$$k_1 - \varnothing(p)_1 \cdot \hat{k} > 0 \quad (4.61)$$

then

$$\frac{P_1}{n} \varnothing(k)_1 \cdot k < 1 - \frac{P_1}{n} \theta \quad (4.62)$$

As before, we consider the forward hemisphere $S^2 = \{ k \in \mathbb{R}^3 \mid |k|^2 = n^2, k_1 > 0 \}$. For the light rays near the optical axis ($|k| \ll k_1$), we have from (4.64)

$$p \cdot \varnothing < n \quad (4.65)$$

For the oblique or metaxial rays i.e. $k_1^2 k \rightarrow \pm n$ where $\pm u$ is a unit vector in the plane of the screen, then, if $u$ is up oriented,

$$|\varnothing| < \frac{n}{|p_u - p|} \quad (4.66)$$

and where $u$ is down oriented,

$$|\varnothing| > \frac{n}{|p_u + p|} \quad (4.67)$$

That is

$$\frac{n}{|p_u + p|} < |\varnothing| < \frac{n}{|p_u - p|} \quad (4.68)$$

4.4 Optical coherent states

We see that, from here, all sections defined by (4.34), satisfy the last inequality. Finally, we are ready to consider the relation equivalent to (4.41):

$$I_2(\zeta) = \int_{S^2 \times S^2} \frac{dp dk}{k_1} \exp\{2\pi i [K(p, k) - K(p, k')] \cdot \zeta \}$$

And we essentially used the Fubini theorem and the implication relation (4.61). The same variable change (4.43) leads to the next equality:

$$I_2(\zeta) = \int_{S^2 \times S^2} \frac{dp dk}{k_1} \delta(K(p, k) - K(p, k')) [\eta(k)\eta(k') \xi(k') \xi(k)]^2 \quad (4.69)$$

Now, the kernel $A(k, p)$ is given by

$$A(k, p) = \frac{p_1 k - p \cdot k}{k_1 - \varnothing(k)_1 p} = P_1 \left[ 1 - \frac{(k_1 - \varnothing(p)_1 k_1)}{k_1 - \varnothing(k)_1 p} \right] \quad (4.70)$$

and is positive defined for $\forall k, p \in S^2$ and $\varnothing$ obeying (4.66). Substituting (4.69) into (4.68), we obtain the relation:

$$I_2(\zeta) = (H)_{||\zeta||} - \int_{S^2 \times S^2} \frac{dp dk}{k_1} \frac{A(k, p) |\eta(p)|^2 |\xi(k)|^2 (4.71)$$

where

$$(H)_{||\zeta||} = u^{-\frac{1}{2}} \int \frac{dp}{p_n \sqrt{\eta(p)}} \quad (4.72)$$

(see (4.44)).

If we may find $\eta$ in $D(H^{1/2})$ such that the second term in (4.70) vanishes, then we get again the resolution of the identity.

For a fixed $\eta \in D(H^{1/2})$ and $\sigma_v$, we finally obtain the following overcomplete family of vectors (geometric optical CS):

$$\phi_{\sigma_v}(\eta) = \{ \eta \in \mathcal{F} \mid A_v(\eta) = A_v(\sigma_v(p, q)) \} \quad (4.73)$$

with $\sigma_v$ from (4.35) and

$$A_v(\eta)(k) = A_v(\sigma_v)(k) \quad (4.74)$$

where

$$A_v(\eta)(k) = \int_{S^2} \frac{\eta(p) - p \cdot \eta_1 k_1}{k_1 - \varnothing(k)_1 p} \frac{dp}{p_n} \quad (4.75)$$

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The projection operators $P_{\pm}$ are now
\[ P_{\pm}(q) = (|\psi_{\pm}^{(q)}\rangle \langle \psi_{\pm}^{(q)}|) \in \mathbb{A}^2, \quad \psi_{\pm}^{(q)} = \psi_{\pm}^{(q)} \mp \psi_{\pm}^{(q)} \quad (4.74) \]
where $\psi_{\pm}^{(q)}$ is a weighted coherent state.

And, as in (4.54), we have the resolution of the identity
\[ \int_{\mathbb{A}^2} \psi_{\pm}^{(q)} \psi_{\pm}^{(q)} dq = 1 \quad (4.54) \]

5 Conclusion

The Mackey-Kirillov theory of inducing from the polarization representations is a very convenient tool to build and classify the UIR's representations of nilpotents groups \([57],[57],[68]\), or, more generally, of reductive groups \([59]\). We have applied the later in building the refractive Index Representation of \(E(2+1)\).

Sections 2 and 3 show clearly that light rays are naturally associated to the left quotient space of \(E(2+1)\) by the subgroup of symmetry, which is nothing else but the stabilizer of the forward pole \((\alpha, n)\) of the Descartes sphere. The family of Euclidean CS are obtained straightforward as the orbit of a group action on an admissible vector \(t_j\) on the sphere \(S^1\). Identification of some sections leading to those CS is done from elementary optical considerations. Summarizing, formulae (4.53) and (4.71) are the main results of this paper.

Next step should be to realize the “wavization” program ([69]) i.e. to do a quantization scheme using those optical geometric coherent states. The dual space \(g^*\) of \(g\) is defined by a real linear functional \(F\) on \(g\) by next coupling of \(g^* \times g \to \mathbb{R}\):
\[ (F, Y) \equiv F(Y), \quad F \in g^*, \quad Y \in g \quad (A.3) \]
which readily now defines the coadjoint action as follows
\[ (F, Ad^{-1}(g)Y) = (Ad(g) \cdot F, Y), \quad \forall g \in G \quad (A.4) \]
The differential \(d(Ad(g))_{g_{xx}}\) of the coadjoint map at the unit element \(e \in G\) is written and given by
\[
\text{ad}^{-1}(g^*) = \text{End}(g^*)
\]
\[
\text{ad}^{-1}(Y \cdot F, Y, Y) = F(\text{ad}^{-1}(Y)Y)
\]
where \(Y, Y \in g, \quad F \in g^*\) and \(\text{ad}^{-1}(Y) = d(Ad(g))_{g_{xx}}\).

Now \(G\) acts on \(g^*\) by the coadjoint representation \(Ad^*(g)\) in (A.4). If \(F \in g^*\), we denote by \(G(F) \subseteq G\) the isotropy group of \(g\) at \(F:\)
\[ G(F) = \{ g \in G | Ad(g)F = F \} \quad (A.5) \]
\(G(F)\) is called also the stabilizer of \(F\) and is related the Lie subalgebra \(g(F)\): \(g(F) = \{ Y \in \text{gl}(Y)F = 0 \} \quad (A.6)\)

From (A.3), we define an alternating bilinear form \(B_{F}\) on \(g\) by
\[ B_{F}(Y, Y') = \{ F, [Y, Y'] \}, \quad Y, Y' \in g \quad (A.8) \]
where \([Y, Y']\) is the bracket in \(g\). This form is antisymmetric and its \textit{radical} is, by definition,
\[ \text{Ker}B_{F} = \{ Y \in g | B_{F}(Y, Y) = 0, \forall Y \in g \} \quad (A.9) \]
\(\text{Ker}B_{F}\) is, following (A.5) and (A.7),
\[ \text{Ker}B_{F} \subseteq \{ Y \in g | \text{ad}^{-1}(Y)F, Y = 0, \forall Y \in g \} \quad (A.10) \]
Obviously,
\[ \text{Ker}B_{F} = \{ F \} \quad (A.11) \]

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Appendix

Representations associated to polarizations

In this appendix, we define the concept of a polarization and apply it to the Mackey-Kirillov theory \([56],[57],[64],[65],[66],[63],[58]\).

Let \(G\) be a real Lie group with Lie algebra \(g\). If \(g \in G\), the map
\[ \alpha(g) : \rightarrow g \rightarrow g^{-1}, g \in G \quad (A.1) \]
is an inner automorphism of \(G\), and its differential at the unit element \(e\).
\[ \text{Ad}(g) = \frac{d}{dt} \alpha(e^{t}g), \quad Y \in g \quad (A.2) \]
is an automorphism of \(g\). The dual space \(g^*\) of \(g\) is defined by a real linear functional \(F\) on \(g\) by next coupling of \(g^* \times g \to \mathbb{R}\):
\[ (F, Y) \equiv F(Y), \quad F \in g^*, \quad Y \in g \quad (A.3) \]
which readily now defines the coadjoint action as follows
\[ (F, Ad^{-1}(g)Y) = (Ad(g) \cdot F, Y), \quad \forall g \in G \quad (A.4) \]

The differential \(d(Ad(g))_{g_{xx}}\) of the coadjoint map at the unit element \(e \in G\) is written and given by
\[ \text{ad}^{-1}(g^*) = \text{End}(g^*)
\]
\[ \text{ad}^{-1}(Y \cdot F, Y, Y) = F(\text{ad}^{-1}(Y)Y)
\]
where \(Y, Y \in g, \quad F \in g^*\) and \(\text{ad}^{-1}(Y) = d(Ad(g))_{g_{xx}}\).

Now \(G\) acts on \(g^*\) by the coadjoint representation \(Ad^*(g)\) in (A.4). If \(F \in g^*\), we denote by \(G(F) \subseteq G\) the isotropy group of \(g\) at \(F:\)
\[ G(F) = \{ g \in G | Ad(g)F = F \} \quad (A.5) \]
\(G(F)\) is called also the stabilizer of \(F\) and is related the Lie subalgebra \(g(F)\): \(g(F) = \{ Y \in \text{gl}(Y)F = 0 \} \quad (A.6)\)

From (A.3), we define an alternating bilinear form \(B_{F}\) on \(g\) by
\[ B_{F}(Y, Y') = \{ F, [Y, Y'] \}, \quad Y, Y' \in g \quad (A.8) \]
where \([Y, Y']\) is the bracket in \(g\). This form is antisymmetric and its \textit{radical} is, by definition,
\[ \text{Ker}B_{F} = \{ Y \in g | B_{F}(Y, Y) = 0, \forall Y \in g \} \quad (A.9) \]
\(\text{Ker}B_{F}\) is, following (A.5) and (A.7),
\[ \text{Ker}B_{F} \subseteq \{ Y \in g | \text{ad}^{-1}(Y)F, Y = 0, \forall Y \in g \} \quad (A.10) \]
Obviously,
\[ \text{Ker}B_{F} = \{ F \} \quad (A.11) \]
and the isotropic subspaces of \( \text{Ker}B \) for \( B \) have codimension

\[ k = \frac{1}{2} \dim(g/g(F)) \quad (A.12) \]

In general, \( g/g(F) \) is a symplectic manifold with even dimension and plays the role, in physics, of phase space of a considered system (see sections 2 and 4).

It is of great importance for representation theory of groups that there exist (not always [64]) subalgebras \( p \subseteq g \) that are isotropic and have the maximal isotropic dimension \((N + 1 - k)\), where \( N + 1 = \dim g \). In the real case, they are defined as follows ([64]):

**Definition A.1.** Let \( g \) a Lie algebra and consider \( F \in g^* \) as a real valued linear functional on \( g \). A polarization at \( F \) is a real subalgebra \( P \subset g \) such that

(i) \( g(F) \subset P \) and \( P \) is "stable" under \( \text{Ad}(g(F)) \)

(ii) \( \dim P = \dim g - \frac{1}{2} \dim(g/g(F)) \)

(iii) \( \langle F, [F, F] \rangle = 0 \)

If a group \( G \) acts on a set \( V_0 \) ([30]), the orbit of \( G \) will be denoted by \( O_G(V_0) \) i.e. the orbit of the point \( v_0 \in V_0 \) generated by \( G \). Following (A.4), the coadjoint orbit is simply given by the coadjoint action of \( G \) on \( g^* \):

\[ O_G(F) = \{ F' \in g^*: F = \text{Ad}(g)F, g \in G, F \in g^* \} \quad (A.13) \]

where \( p^\perp \subseteq g^* \) is the annihilator of \( p \subseteq g \) or its orthogonal subspace with respect to the bilinear form \( B \), i.e.

\[ p^\perp = \{ F^\perp \in g^*: BF^\perp, Y_p = 0, \forall Y \in g, Y_p \in P \} \quad (A.14) \]

**Definition A.2.** The point \( F \in g^* \) is integral if there exists a character

\[ \chi_F: G(F) \to T, \quad T = \{ z \in \mathbb{C} | z = 1 \} \]

whose differential is

\[ \dot{\chi}_F = \frac{d}{dt} \chi_F(\exp t Y) |_{t=0} = 2\pi i(F, Y) \quad (A.15) \]

If the Pukanzky condition for \( p \), a polarization at \( F \), is satisfied then \( \chi_F \) extends to a unique character

\[ \chi_F : P \to T \]

whose differential is

\[ \dot{\chi}_F = 2\pi i(F, Y_p), \quad Y_p \subseteq p \quad (A.16) \]

So, \( \chi_F \) defines a 1-dimensional representation of the subgroup \( P = \exp(p) \) and the maximal isotropy insures that \( \chi_F \) induces to an UIR of \( G \) as we see it in what follows.

Let \( G \) be a locally compact group, \( H \) its closed subgroup and \( \pi_0 \) a unitary representation of \( H \) in Hilbert space \( \mathcal{H}_0 \)

\[ \pi_0 : H \to \text{Aut}(\mathcal{H}_0) \quad (A.19) \]

Let \( X = G/H \) be a left coset or homogeneous space having a quasi-invariant measure \( \mu(x) \), that is the shifted measure \( \mu(x) \) is equivalent to \( \mu(x) \):

\[ \mu(x) = \rho(x)\mu(x) \quad (A.20) \]

The next criterion is sufficient to \( X \) to have an invariant measure

\[ h_B(h) = \Delta_G(h), \quad \forall h \in H \quad (A.21) \]

where \( \Delta_H \) and \( \Delta_G \) are the modular functions on \( H \) and \( G \) respectively. In our case, the euclidean \( E(2 + 1) \) will share an invariant measure on \( G/H \) since \( G \) and \( H \) will be unimodular.

Consider a set of functions \( \psi_0(g) \) defined on \( G \) with values in \( \mathcal{H}_0 \) and verifying next conditions:

(i) \( \forall \alpha, \beta, \gamma \in \mathcal{H}_0 \), the function \( g \mapsto (\psi_0(g)|\alpha) \) is measurable for a Haar measure \( \mu(g) \) on \( G \).

(ii) \( \psi_0(g) = \psi^{-1}_0(g), \forall h \in H, \forall g \in G \)

(iii) \( \int_{G/H} \psi_0(g)\psi_0^*(g)\mu(g) < \infty \)

where (|) is the scalar product in \( \mathcal{H}_0 \), whereas \(| \rangle \) is the scalar product \( \mathcal{H}_0 \). Then, the induced representation \( U_{\psi_0} \) is defined as follows

\[ U_{\psi_0} : \mathcal{H}_0 \oplus L^2(G/H, \mu) \to \mathcal{H}_0 \]

where \( \oplus \) is the direct sum of \( \mathcal{H}_0 \) and \( L^2(G/H, \mu) \), with values on \( \mathcal{H}_0 \).

The isomorphism

\[ B : G \to \text{Aut}(\mathcal{H}_0) \]

\[ B(g)\psi(x) = \psi_0(g) \]

is unitary

\[ \int_{G/H} ||\psi(x)||^2d\mu(x) = \int ||\phi(x)||^2d\mu(x) \quad (A.23) \]

from (A.23), the induced representation takes the form

\[ U_{\psi_0}(g)\psi(x) = B^{-1}(g)B(g^{-1})\psi(g^{-1}x) \quad (A.24) \]
It remains to define $B(g)$ from (A.25). To this end, choose a point $x_0 \in G/H$ with a stabilizer subgroup $\sigma(x)$ such that

$$\sigma(x)x_0 = x$$  \hspace{1cm} (A.27)

or, from the relation (A.4)

$$\text{Ad}'(\sigma(x)) \cdot x_0 = x$$  \hspace{1cm} (A.28)

Here, $\sigma$ is section from the basis $G/H$ of $G$ considered as a bundle, with the natural projection $\pi: G \to G/H$ (see fig. A.1).

**Fig. A.1:** $\sigma$ is a Borel section from the basis $X = G/H$ of the bundle $G$ with the natural projection $\pi: G \to X$ and $H$ as the standard fiber.

In general, there are many such sections. Suppose we have chosen $\tau(x)$, for any $x \in \pi^{-1}(H) = \mathcal{O}$. Since any $g$ may be written $g = \sigma(x)(\sigma(x)^{-1}g)$, with $\sigma^{-1}(x)g$ as a stabilizer element of $G$ then

$$B(g) = \pi_0^{-1}(\sigma^{-1}(x))B(\sigma(x) = \pi_0^{-1}(\sigma(x))B(\sigma(x))$$

and the general form for the induced representation is then, with (A.26)

$$[U_{\phi}(g)\psi](x) = B^{-1}(\sigma(x))\pi_0(\sigma^{-1}(x)g(x^{-1})B(\sigma_0^{-1}(x))\psi(g^{-1}x)$$  \hspace{1cm} (A.29)

This form depends mainly on the choice of $\sigma(x)$ and $B(g)$. In any case, we may always take it to be

$$B(\sigma(x)) = 1 \quad \forall x \in G/H$$

without losing the generality.

Finally we get the standard or Wigner form of $U_{\phi}$ associated to the choice of $\sigma(x)$:

$$[U_{\phi}(g)\psi](x) = \pi_0(\sigma^{-1}(x)g(x^{-1})\psi(g^{-1}x)$$  \hspace{1cm} (A.30)

This is the starting point for inducing the representation (3.25) where $H = P$, the polarization at the point $F \in \mathcal{F}$, $\pi_0 = \chi_F$ the character associated to $F$ according to (A.17), $g^{-1}x$ is the coadjoint action of $G$ on $x \in G/P$ and $\psi \in \mathcal{H}(G/P, d\mu(x))$. With these elements we will in the next paper be able to build and classify any representations of $G = E(N + 1)$.

### References


