MULTISITE ANTIFERROMAGNETIC ISING SPIN MODEL: PHASE TRANSITION THROUGH DOUBLING BIFURCATION

N.S. Ananikian

and

K.A. Oganessyan

MIRAMARE-TRIESTE
1 Introduction

It is well known that critical phenomena are closely connected with the behavior of non-linear dynamic systems [1, 2].

In our previous paper we have considered the three-site antiferromagnetic Ising spin model (TSAI) on Husimi tree and have received the quantitative picture of full doubling bifurcations diagram including chaos for the magnetization of the base site of this system [3], whereas in antiferromagnetic Potts model only one period doubling occurred [4]. In particular, it was shown that the numerical values of Feigenbaum constants \( \alpha \) and \( \beta \) for TSAI system on Husimi tree coincide with the famous Feigenbaum constants with high accuracy.

On the other hand, a multisite interaction system on Husimi tree approximation was investigated also in ref. [5]. It was shown, that this approach yields good approximation for the phase diagrams, which closely match the exact results obtained on a Kagome lattice [6].

The aim of our paper is to obtain the line of the second order phase transition for TSAI system on Husimi tree, occurred through so-called period doubling bifurcation.

2 Husimi tree and recursion relation

The pure Husimi tree [7] (see Fig.1) is characterized the \( \gamma \)-the number of the triangle neighbors. The 0th-generation is a single central triangle.
The TSAI model in the magnetic field defined by the Hamiltonian

$$H = -J_3 \sum_\triangle \sigma_i \sigma_j \sigma_k - h' \sum_i \sigma_i,$$

where $\sigma_i$ takes values $\pm 1$, the first sum goes over all triangular faces of the Husimi tree and the second over all sites. Besides we denote $J_3 = \beta J_0$, $h = \beta h'$, $\beta = 1/kT$, where $h$-external magnetic field, $T$-temperature of the system and $J_3 < 0$ corresponds to the antiferromagnetic case.

The partition function will be written as

$$Z = \sum_{\{\sigma\}} \exp \left\{ J_3 \sum_\triangle \sigma_i \sigma_j \sigma_k + h \sum_i \sigma_i \right\},$$

where the summation goes over all configurations of system.

The advantage of the Husimi tree introduced is that for the models formulated on it, exact recursion relation can be derived. When "cutting apart" the Husimi tree at the central triangle it separates into 3 identical branches and each of them contains $\gamma - 1$ branches. Then the partition function may be written

$$Z = \sum_{\{\sigma_0\}} \exp \left\{ J_3 \sum_\triangle \sigma_0^{(1)} \sigma_0^{(2)} \sigma_0^{(3)} + h \sum \sigma_0^{(i)} \right\} [g_{n-1}(\sigma_0^{(1)})]^{\gamma-1} [g_{n-1}(\sigma_0^{(2)})]^{\gamma-1} [g_{n-1}(\sigma_0^{(3)})]^{\gamma-1},$$

where $\sigma_0^{(i)}$ are spins of central triangle, $n$-number of shells and the equation for one of branches can be written:

$$g_n(\sigma_0) = \sum_{\{\sigma\}, \sigma_0 \sigma_1} \exp \left\{ J_3 \sum_\triangle \sigma_0 \sigma_1 + h \sum \sigma_0 + J_3 \sum_\triangle \sigma_1 \sigma_2 + h \sum \sigma_1 \right\}.$$  

One of branches, in its turn, can be cut on the site of 1st-generation, which is the nearest to the central site. Therefore, the expression for $g_n(\sigma_0)$ can be rewritten in the form:

$$g_n(\sigma_0) = \exp \left\{ J_3 \sum_\triangle \sigma_0 \sigma_1 + h \sum \sigma_0 \right\} [g_{n-1}(\sigma_0^{(1)})]^{\gamma-1} [g_{n-1}(\sigma_0^{(2)})]^{\gamma-1}. $$

From eq.(5) one can easily obtain:

$$g_n(+) = e^{\Delta + 2h} g_{n-1}(+) + 2e^{-h} g_{n-1}(+)g_{n-1}(-) + e^{-\Delta - 2h} g_{n-1}(-),$$

$$g_n(-) = e^{-\Delta + 2h} g_{n-1}(+) + 2e^h g_{n-1}(+)g_{n-1}(-) + e^{-\Delta + 2h} g_{n-1}(-).$$

Let the following variable be introduced:

$$x_n = \frac{g_n(+)}{g_n(-)}.$$ 

Then for $x_n$ we can obtain the following recursion relation:

$$x_n = f(x_{n-1}), \quad f(x) = \frac{z^{\mu} x^{2(\gamma-1)} + 2 \mu x^{\gamma-1} + z}{\mu^2 x^{2\gamma-1} + 2 \mu x^{\gamma-1} + 1},$$

where $z = e^{\Delta - h}$, $\mu = e^{2h}$ and $0 \leq x_n \leq 1$. The eq.(7) coincides with that obtained by Monroe [4], when pair interaction absents.

The $x_n$ has no direct physical meaning but through it one can express the magnetization of the central base site:

$$m = \langle \sigma \rangle = \frac{e^h g_0^+(+) - e^{-h} g_0^-(+)}{e^h g_0^+(+) + e^{-h} g_0^-(+)} = \frac{x_n^\gamma - 1}{x_n^\gamma + 1},$$

and other thermodynamic parameters, since we can say that the $x_n$ determine the states of the system.
3 Phase transition through doubling bifurcation

The fixed points of the iteration sequence $x_n$ in the thermodynamic limit $n \to \infty$ are the solutions of the equation:

$$x = f(x, h, T).$$

(9)

However at low temperatures $T < T_c$ this single fixed point becomes unstable, and a so-called period doubling bifurcation occurs: the recursive sequence eq.(7) converges now not to the single fixed point but to the stable two-cycle $x_1, x_2$. This face should be explained as an arising of a two-sublattice phase such that $x_1$ and $x_2$ determine the states on each sublattices.

Let us consider the following system of equations:

$$\begin{cases}
    f(x) - x = 0 \\
    f'(x) = -1
\end{cases},$$

(10)

which determined the point, where the first period doubling bifurcation began - the point of the second order phase transition from the disordered to the two ordered phase.

For recursion function of eq.(7) when $\gamma = 3$, the eq.(10) will have the following form:

$$\begin{cases}
    \mu^2 x^5 - z \mu x^4 + 2 \mu x^3 - 2 \mu x^2 + x - z = 0 \\
    3 \mu^2 x^4 - 4 \mu^2 x^3 + 2 \mu^4 x^2 - 4 \mu x - 1 = 0
\end{cases}.$$

(11)

With excepting the x from eq.(11) one can obtain:

$$16z^4(3z^2 - 1) + 8z^2(34z^2 - z^4 - 1) + 8z(9 - z^2) + 16 = 0,$$

(12)

which determined the points, at which the TSAI system undergoes the second order phase transition from the disordered to the two-sublattice ordered phase. The whole line of the second order phase transition corresponds to eq.(12) for the case $J_3 = -1$ is shown in Fig.2.

One can see from Fig.2, that there is some $T_c$ - the upper bound of the temperature at which for TSAI system the second order phase transition appeared.

For obtaining this critical temperature, let us consider the following system of equations:

$$\begin{cases}
    \mu^2 x^5 - z \mu x^4 + 2 \mu x^3 - 2 \mu x^2 + x - z = 0 \\
    3 \mu^2 x^4 - 4 \mu^2 x^3 + 2 \mu^4 x^2 - 4 \mu x - 1 = 0
\end{cases}$$

(13)

where the second equation is derivative of eq.(12) with respect to $\mu$, with taking into account that $T_c$ corresponds to maximum of function $T(h)$: $dT/dh = 0, \partial z/\partial h = 0, \partial \mu/\partial h = 2\mu/T$.

With excepting the $\mu$ from eq.(13) on can obtain:

$$16z^2(Q(z))² - A(z)B(z)² - [A(z)C(z) - z²Q(z)D(z)][Q(z)C(z) + D(z)B(z)] = 0,$$

(14)

where

$$A(z) = 15z^2 - z^4 + 2, \quad B(z) = 2z^4 + 15z^2 - 1,$$

$$C(z) = 23z^4 + 42z^2 - 1, \quad D(z) = 42z^2 - z^4 + 23, \quad Q(z) = 3(1 - z^2).$$
We solve numerically the eq. (14) for the case when $J_3 = -1$ and obtain:

$$T_c = 1.38536, \quad (15)$$

upper of which there are not phase transitions of the second order for TSAI system.

It is interesting to note, that if one lets $\gamma = 2$ in eq. (7) rather then $\gamma = 3$, the above mentioned situation changes dramatically.

For recursion function of eq. (7) when $\gamma = 2$, eq. (10) will have the form:

$$\begin{cases} 
\mu^2 x^2 + \nu(2 - \mu)x^2 + (1 - 2\mu)x - x = 0, \\
\nu x^2 - 2\nu x - (1 + 2\mu) = 0 
\end{cases} \quad (16)$$

which for any $T$ and $h$ have only nonphysical solutions. Therefore, for TSAI system when $\gamma = 2$ the period doubling bifurcations picture absents. It means that for this statistical physical system there is not phase transition of second order when $\gamma = 2$.

4 Conclusion

As it is mentioned in the introduction, in our previous paper [3] we have investigated the three-site antiferromagnetic Ising spin model on Husimi tree and a strong connection with results from the theory of dynamical systems have made, which gives possibility to study the statistical physical systems in a new context and in a simple manner. In particular, in this paper with using the well known technique for dynamical systems, we obtain (when $\gamma = 3$) the exact equation, which determined the points, at which the TSAI system undergoes the second order phase transition from the disordered to the two-sublattice ordered phase and obtain also the critical temperature - $T_c$, at which the second order phase transitions appeared. We show, that for TSAI system there are not phase transitions of the second order when $\gamma = 2$.

We think, that obtained results are interesting and we plan to continue to investigate this line of approach for TSAI system and for several other systems.

On the other hand, the study of chaotic statistical physical system has opened new challenges for theories of stochastic processes, especially in the direction of stochasticity of vacuum in QCD [8]. In this direction the interesting results for Z(Q) gauge model on generalized Bethe lattice was obtained [9]. It gives bases to suppose that TSAI Ising spin model on Husimi tree approximation can be connected with double plaquette representation of the gauge theory.

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References


Fig. 1. Husimi tree with $\gamma = 3$. 
Fig. 2. The line of the second order phase transition from disordered to the two-sublattice ordered phase for TSAI system \((J_3 = -1)\). The point C describes the upper bound of the temperature \(T_c\), the numerical value of which is given in eq.(15).