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0. Introduction

Throughout this paper, A denotes a Noetherian local ring with maximal ideal m and M a finitely generated A module with $d := \dim_A M \geq 1$.

Let $x = x_1, ..., x_d$ be a system of parameters (abbr. s.o.p) of M and $n = n_1, ..., n_d$ be a d- tuple of positive integers. We consider the following two problems:

i) When is the length of the Koszul homology

$$\ell(H_k(x_1^{n_1}, ..., x_d^{n_d}\ ;\ M))$$

a polynomial in n for all $k = 0, ..., d$ and $n_1, ..., n_d$ sufficiently large (abbr. $n >> 0$) ?

ii) Is the length of the generalized fraction $1/(x_1^{n_1}, ..., x_d^{n_d}, 1)$ in $U_{d+1}^{-d-1}(M)M$ a polynomial in n for $n >> 0$?

Note that the theory of generalized fractions has been introduced by R.Y.Sharp and H.Zakeri in [S-Z1] and problem (ii) is asked first in [S-H]. An affirmative answer is also proved for the problem (ii) in [S-H] when M is a generalized Cohen - Macaulay (abbr. g.CM) modules. On the other hand, the problem (i) is investigated in [C1] for the case $k = 0$ and a necessary and sufficient condition for x is given in this case so that $\ell(H_0(x_1^{n_1}, ..., x_d^{n_d}\ ;\ M))$ is a polynomial for $n >> 0$. It is also well-known that for a g.CM module M, $\ell(H_k(x_1^{n_1}, ..., x_d^{n_d};\ M))$ $(k > 0)$ is a constant for $n >> 0$ (cf. [C-S-T]). Thus, the above both problems are solved for the case that M is a g.CM module.

The purpose of this paper is to study the above problems for a class of modules which strictly contains all of g.CM modules, namely, the class of modules having the *polynomial type* at most 1. Recall that the polynomial type of a module M is defined first in [C2] as follows:

Consider the difference between the length and the multiplicity

$$\ell(M/(x_1^{n_1}, ..., x_d^{n_d})M)\ -\ n_1...n_d e(x;\ M)$$

as a function in n. Then the least degree of all polynomials in n bounding above this function is independent of the choice of the s.o.p x. This invariant is called the polynomial type of M and denoted by $p(M)$. It is easily to see that M is g.CM if and only if $p(M) \leq 0$.

Now, let us to give a summarize of this paper. In Section 1, we prove some preliminary result which will be used for next sections. In Section 2, we give some results about local cohomology modules in case $p(M) \leq 1$. In Section 3, we show that if $p(M) \leq 1$ then there exist is a s.o.p $x_1, ..., x_d$ of M such that, for $n >> 0$, $\ell(H_i(x_1^{n_1}, ..., x_j^{n_j}\ ;\ M))$ is a polynomial of degree $p(M)$ for $i = 1, j = d$ and constant for the other cases. As a consequence, we get the same results for higher Euler - Poincare characteristics

$$X_k(x_1^{n_1}, ..., x_j^{n_j}; M) = \sum_{i \geq k} (-1)^{i-k} \ell(H_i(x_1^{n_1}, ..., x_j^{n_j}\ ;\ M)).$$

With same assumptions as in Section 3 for a s.o.p x, we prove in Section 4 that $\ell(M(1/(x_1^{n_1}, ..., x_d^{n_d}, 1)))$ is a polynomial for $n >> 0$ and this polynomial can be computed by the lengths of local cohomology modules.

1. Preliminaries

From now on, for convenience, we will write

$$I_M(n;x) = \ell(M/(x_1^{n_1},...,x_d^{n_d})M) - e(x_1^{n_1},...,x_d^{n_d};\ M)$$

or $I(n;x)$ for a fixed module M. Denote by $a_i(M)$ the annihilator of the i-th local cohomology module $H^i_m(M)$ of M with respect to the maximal ideal m and put $a(M) = a_0(M)...a_{d-1}(M)$.

First of all, we recall a result of P. Schenzel which will be often used in this paper.

Lemma 1.1. *(see [S, Theorem 3]). Denote by $r(M)$ the intersection of all annihilators of modules*

$$(x_1,...,x_{i-1})M\ :\ x_i\ /(x_1,...,x_{i-1})\ M,$$

where $x\ =\ x_1,...,x_i$ runs through all subsets of systems of parameters of M. Then $a(M) \subseteq r(M)$.

Recall that a system of elements in A, $x\ =\ x_1,...,x_t$ is said to be a d-sequence of M if

$$(x_1,...,x_{i-1})M\ :\ x_i\ =\ (x_1,...,x_{i-1})M\ :\ x_ix_j$$

for all $i\ =\ 1,...,t$ and $j \geq i$. Then, as an immediate consequence of Lemma 1.1, we have the following lemma.

Lemma 1.2. *Let $x\ =\ x_1,...,x_d$ be a s.o.p for M such that $(x_2,...,x_d)A \subseteq$ Rad $a(M)$. Then $x_1^{n_1},x_2^{n_2},...,x_d^{n_d}$ is a d- sequence of M for $n >> 0$.*

Proof. Choose $n >> 0$ so that $(0:x_1^{n_1})_M = (0:x_1^{2n_1})_M$ and $(x_2^{n_2},...,x_d^{n_d})A \subseteq a(M)$. Then by Lemma 1.1, $(0:x_1^{n_1})_M \subseteq (0:x_i^{n_i})_M$ for all $i > 0$ and

$$(x_1^{n_1},...,x_{i-1}^{n_{i-1}})M\ :\ x_i^{n_i}\ =\ (x_1^{n_1},...,x_{i-1}^{n_{i-1}})M\ :\ a(M)$$

$$=\ (x_1^{n_1},...,x_{i-1}^{n_{i-1}})M\ :\ x_i^{n_i}x_j^{n_j}$$

for all $i\ =\ 1,...,d$ and $i \leq j \leq d$. Therefore, $x_1^{n_1},x_2^{n_2},...,x_d^{n_d}$ is a d- sequence of M for $n >> 0$.

Lemma 1.3. *Let $x\ =\ x_1,...,x_d$ be a s.o.p for M such that $x_d \in a(M)$. Then*

$$I(n_1,...,n_d;x)\ =\ I(n_1,...,n_{d-1},1;x)$$

for all $n_1,...,n_d \geq 1$.

Proof. By [A-B,Corollary 4.3], we have

$$I(n;x)\ =\ \ell((x_1^{n_1},...,x_{d-1}^{n_{d-1}})M\ :\ x_d^{n_d}\ /\ (x_1^{n_1},...,x_{d-1}^{n_{d-1}})M)$$

$$+\sum_{i=1}^{d-1} e(x_{i+1}^{n_{i+1}},...,x_d^{n_d};(x_1^{n_1},...,x_{i-1}^{n_{i-1}})M\ :\ x_i^{n_i}\ /\ (x_1^{n_1},...,x_{i-1}^{n_{i-1}})M).$$

Since

$$x_d^{n_d}((x_1^{n_1},...,x_{i-1}^{n_{i-1}})M\ :\ x_i^{n_i}\ /\ (x_1^{n_1},...,x_{i-1}^{n_{i-1}})M)\ =\ 0$$

by Lemma 1.1, we get

$$e(x_{i+1}^{n_{i+1}},...,x_d^{n_d};(x_1^{n_1},...,x_{i-1}^{n_{i-1}})M\ :\ x_i^{n_i}\ /\ (x_1^{n_1},...,x_{i-1}^{n_{i-1}})M)\ =\ 0.$$

for $i = 1,...,d-1$. Thus

$$I(n;x)\ =\ \ell((x_1^{n_1},...,x_{d-1}^{n_{d-1}})M\ :\ x_d^{n_d}\ /\ (x_1^{n_1},...,x_{d-1}^{n_{d-1}})M)$$

$$= \ell((x_1^{n_1},...,x_{d-1}^{n_{d-1}})M\ :\ x_d\ /\ (x_1^{n_1},...,x_{d-1}^{n_{d-1}})M)\ =\ I(n_1,...,n_{d-1},1;x)$$

by Lemma 1.1 again.

The lemma 1.3 enables us to get a result in [C3] as follows.

Corollary 1.4. $\dim A/a(M) \geq p(M)$.

Proof. Set $k\ =\ \dim A/a(M)$. Since $p(M) \leq d$, then the inequality holds for $k\ =\ d$. For $k\ <\ d$, we can choose a s.o.p $x\ =\ x_1,...,x_d$ such that $x_{k+1},...,x_d \subseteq a(M)$. Hence, by Lemma 1.3,

$$I(n;x)\ =\ I(n_1,...,n_k,1,...,1;x).$$

This implies that $p(M) \leq k$.

Lemma 1.5. *Let $y \in a(M)$ and set $k = \dim A/a(M)$. Then we have the exact sequence*

$$0 \longrightarrow\ H^i_m(M)\ \longrightarrow H^i_m(M/yM)\ \longrightarrow H^{i+1}_m(M)\ \longrightarrow 0,$$

for $i\ =\ k,...,\ d-2$.

Proof. Consider the following exact sequences

$$0 \longrightarrow (0:y)_M \longrightarrow M \longrightarrow M/(0:y)_M \longrightarrow 0\ ,$$

$$0 \to M/(0:y)_M \to M \to M/yM \to 0\ .$$

Note that $\dim(0\ :\ y)_M\ \leq\ \dim A/a(M)\ =\ k$ by Lemma 1.1. The first exact sequence yields isomorphisms

$$f_i : H^i_m(M) \cong H^i_m(M/(0:y)_M)$$

for all $i > k$ and an epimorphism

$$f_k : H^k_m(M) \longrightarrow H^k_m(M/(0:y)_M) \longrightarrow 0.$$

Furthermore, since $yH^i_m(M) = 0$, for $i = 0, ..., d-1$ and f_i are surjective for all i with $i \geq k$, from the commutative diagram

$$\begin{array}{ccc} H^i_m(M) & \xrightarrow{y} & H^i_m(M) \\ & f_i \searrow \quad \nearrow g_i & \\ & H^i_m(M/(0:y)_M) & \end{array}$$

we deduce that g_i is zero for all $i = k, ..., d-1$. Using the isomorphisms f_i and the second exact sequence we get the following exact sequence

$$0 \longrightarrow H^i_m(M) \longrightarrow H^i_m(M/yM) \longrightarrow H^{i+1}_m(M) \longrightarrow 0,$$

for all $i = k, ..., d-2$ as required .

2. Local cohomology modules

By [C-S-T] we know that M is a g.CM module if and only if $\dim A/a(M) = 0$. From this section on, we will consider a class of modules M satisfying the condition $\dim A/a(M) \leq 1$. Then, every g.CM module is contained in this class. First of all we have the following result.

Proposition 2.1. *Suppose that* $\dim A/a(M) \leq 1$ *then* $\ell(H^i_m(M)) < \infty$, $i = 0, ..., d-2$.

Proof. We prove by induction on d.

It is clear for $d = 2$. For $d > 2$, let $y \in a(M)$ such that y is an element of parameter of M. Then $\dim A/a(M/yM) = \dim A/a(M)$ by [C2, Lemma 3.3]. Therefore,

$$\ell(H^i_m(M/yM)) < \infty, \ i = 0, ..., d-3$$

by induction hypothesis. By Lemma 1.5 we have the exact sequence

$$0 \longrightarrow H^i_m(M) \longrightarrow H^i_m(M/yM) \longrightarrow H^{i+1}_m(M) \longrightarrow 0,$$

for $i = 1, ..., d-2$. Hence $\ell(H^i_m(M)) < \infty$, $i = 0, ..., d-2$ as required.

Lemma 2.2. *Suppose that* $\dim A/a(M) \leq 1$. *Let* $x = x_1, ..., x_d$ *be a s.o.p for* M *such that* $(x_2, ..., x_d)A \subseteq \operatorname{Rad} a(M)$. *Then the following exact sequences hold:*

$$0 \longrightarrow H^i_m(M) \longrightarrow H^i_m(M/x_1^{n_1}M) \longrightarrow H^{i+1}_m(M) \longrightarrow 0, \ i = 0, ..., d-3$$

and

$$0 \longrightarrow H^{d-2}_m(M) \longrightarrow H^{d-2}_m(M/x_1^{n_1}M) \longrightarrow (0 : x_1^{n_1})_{H^{d-1}_m(M)} \longrightarrow 0$$

for $n_1 >> 0$.

Proof. Set $\overline{M} = M/(0 : x_1^{n_1})_M$. For $n_1 >> 0$, by Lemma 1.2 we get $\dim(0 : x_1^{n_1})_M = 0$, therefore $H^i_m(M) \cong H^i_m(\overline{M})$ for $i > 0$. Thus, from the exact sequence

$$0 \longrightarrow \overline{M} \longrightarrow M \longrightarrow M/x_1^{n_1}M \longrightarrow 0,$$

we get the following exact sequences

$$0 \longrightarrow H^0_m(M) \longrightarrow H^0_m(M/x_1^{n_1}M) \longrightarrow (0 : x_1^{n_1})_{H^1_m(M)} \longrightarrow 0$$

and

$$0 \longrightarrow H^i_m(M)/x_1^{n_1}H^i_m(M) \longrightarrow H^i_m(M/x_1^{n_1}M) \longrightarrow (0 : x_1^{n_1})_{H^{i+1}_m(M)} \longrightarrow 0$$

for all $i > 0$. In virtue of Lemma 2.1 we deduce that $m^{n_1} H^i_m(M) = 0$, $i = 0, ..., d-2$ for $n_1 >> 0$. Hence $x_1^{n_1}H^i_m(M) = 0$, $i = 0, ..., d-2$ and the lemma follows from the above exact sequences.

Theorem 2.3. *Suppose that* $\dim A/a(M) \leq 1$. *Let* $x = x_1, ..., x_d$ *be a s.o.p for* M *such that* $(x_2, ..., x_d)A \subseteq \operatorname{Rad} a(M)$. *Then, for* $n >> 0$, *we get*

i) $I_M(n, x)$ *is a polynomial.*

ii) $M/x_1^{n_1}M$ *is a g.CM module and* $x_2^{n_2}, ..., x_d^{n_d}$ *is a standard s.o.p of* $M/x_1^{n_1}M$, *i.e,*

$$I_{M/x_1^{n_1}M}(n_2, ..., n_d;\ x_2, ..., x_d) = I_{M/x_1^{n_1}M}(2n_2, ..., 2n_d;\ x_2, ..., x_d).$$

iii)

$$\sup_{n_1 \geq 1} \ell(H^i_m(M/x_1^{n_1}M)) < \infty, \ i = 0, ..., d-3$$

and $\ell(H^{d-2}_m(M/x_1^{n_1}M))$ *is a polynomial in* n_1 *of degree* $p(M)$.

Proof. (i). It follows from the hypothesis that $(x_2^{n_2}, ..., x_d^{n_d})A \subseteq a(M)$ for $n_2, ..., n_d >> 0$. Therefore $I(n; x) = I(n_1, 1, ..., 1; x)$ by Lemma 1.3. Using Corollary 4.3 of [A-B] we get

$$I(n_1, 1, ..., 1; x) = \ell((x_2, ..., x_d)M : x_1^{n_1}/(x_2, ..., x_d)M)$$

$$+ n_1 \sum_{i=2}^{d} e(x_{i+1}, ..., x_d, x_1; (x_2, ..., x_{i-1})M : x_i / (x_2, ..., x_{i-1})M).$$

Thus $I(n;x)$ is a polynomial for $n >> 0$ as required.

(ii). Put $\overline{M} = M/x_1^{n_1}M$, $x' = x_2,...,x_d$ and $n' = (n_2,...,n_d)$. Since $\dim(0 : x_1^{n_1})_M = 0$ for $n_1 >> 0$ by Lemma 1.2, then

$$I(n;x) = I_{\overline{M}}(n';x') - e(x_2^{n_2},...,x_d^{n_d};\ (0 : x_1^{n_1})_M) = I_{\overline{M}}(n';x').$$

Therefore, for $n >> 0$, we deduce by Lemma 1.3 that

$$I_{\overline{M}}(n_2,...,n_d;x') = I_M(n;x) = I_M(n_1,2n_2,...,2n_d;x)$$
$$= I_{\overline{M}}(2n_2,...,2n_d;x').$$

So $x_2^{n_2},...,x_d^{n_d}$ is a standard s.o.p of $\overline{M}$.

(iii). By Lemma 2.2, we get the exact sequence

$$0 \longrightarrow H^i_m(M) \longrightarrow H^i_m(M/x_1^{n_1}M) \longrightarrow H^{i+1}_m(M) \longrightarrow 0\ ,\ i = 0,...,d-3$$

for $n_1 >> 0$. Hence

$$\sup_{n_1 \geq 1} \ell(H^i_m(M/x_1^{n_1}M)) < \infty,\ i = 0,...,d-3.$$

On the other hand, by (ii) and [C-S-T],

$$I_M(n;x) = I_{M/x_1^{n_1}M}(n_2,...,n_d;\ x_2,...,x_d) = \sum_{i=0}^{d-2}\binom{d-2}{i}\ell(H^i_m(M/x_1^{n_1}M))$$

for $n >> 0$. It follows from (i) that $\ell(H^{d-2}_m(M/x_1^{n_1}M))$ is a polynomial in n_1 of degree $p(M)$.

3. Lengths of the Koszul homology modules

We begin with the following lemma.

Lemma 3.1. *Let $n = n_1,...,n_d$ a d-tuple of positive integers and $x = x_1,...,x_d$ be a s.o.p for M such that $x_1^{n_1},...,x_d^{n_d}$ is a d- sequence for all $n_1,...,n_d \geq 1$. Then the length of the Koszul homology module $H_i(x_1^{n_1},...,x_j^{n_j}\ ;\ M\ )$ is finite and given by*

$$(*)\quad \ell(H_i(x_1^{n_1},...,x_j^{n_j}\ ;\ M\ )) = \begin{cases} 0 & \text{if } j < i \\ \sum_{s=0}^{j-i}\binom{j-s-1}{i-1} l(H^0_m(M/(x_1^{n_1},...,x_s^{n_s})M)) & \text{if } j \geq i \end{cases}$$

for $i,j = 1,...,d$.

Proof. Since $x_1^{n_1},...,x_d^{n_d}$ is a d-sequence, by [G-Y, Corollary 1.5],

$$\Big((x_1^{n_1},...,x_d^{n_d})A + \mathrm{Ann}_A(M)\Big)H_i(x_1^{n_1},...,x_j^{n_j}\ ;\ M\ ) = 0,$$

for $i,\ j = 1,...,d$. Thus $\ell\big(H_i(x_1^{n_1},...,x_j^{n_j}\ ;\ M)\big) < \infty$, for $i,\ j = 1,...,d$. Furthermore, since $x_1^{n_1},...,x_d^{n_d}$ is a d-sequence for all $n_1,...,n_d \geq 1$, from the exact sequence

$$0 \longrightarrow H_i(x_1^{n_1},...,x_{j-1}^{n_{j-1}};M)/x_j^{n_j}H_i(x_1^{n_1},...,x_{j-1}^{n_{j-1}};M)$$
$$\longrightarrow H_i(x_1^{n_1},...,x_j^{n_j};M) \longrightarrow \Big(0\ :\ x_j^{n_j}\Big)_{H_{i-1}(x_1^{n_1},...,x_{j-1}^{n_{j-1}};M)} \longrightarrow 0,$$

we get the following exact sequence

$$0 \to H_i(x_1^{n_1},...,x_{j-1}^{n_{j-1}};M) \to H_i(x_1^{n_1},...,x_j^{n_j};M) \to H_{i-1}(x_1^{n_1},...,x_{j-1}^{n_{j-1}};M) \to 0$$

for $i,\ j \geq 1$. Now, we will prove the formula (*) by induction on j. For $j = 1$, $H_i(x_1^{n_1};M) = 0$ for all $i \geq 2$, and $H_1(x_1^{n_1};M) = (0 : x_1^{n_1})_M = H^0_m(M)$. Therefore, the formula (*) holds for $j = 1$. Let $j > 1$, and assume that the formula (*) holds for $j-1$, we need to prove it for j. In the case $i = 1$, from the above exact sequence, we get the following exact sequence

$$0 \longrightarrow H_1(x_1^{n_1},...,x_{j-1}^{n_{j-1}};\ M) \longrightarrow H_1(x_1^{n_1},...,x_j^{n_j};\ M) \longrightarrow$$
$$\longrightarrow H^0_m(M/(x_1^{n_1},...,x_{j-1}^{n_{j-1}})M) \longrightarrow 0.$$

Therefore, by the induction hypothesis, we have

$$\ell(H_1(x_1^{n_1},...,x_j^{n_j};\ M)) = \ell(H_1(x_1^{n_1},...,x_{j-1}^{n_{j-1}};\ M)) + \ell(H^0_m(M/(x_1^{n_1},...,x_{j-1}^{n_{j-1}})M))$$
$$= \sum_{s=0}^{j-1} \ell(H^0_m(\ M\ /(x_1^{n_1},...,x_s^{n_s})M)).$$

Hence formula (*) is true for $j > 1, i = 1$. For $i > 1, j > 1$, we get by the induction hypothesis

$$\ell\big(H_i(x_1^{n_1},...,x_j^{n_j};M)\big) = \ell\big(H_i(x_1^{n_1},...,x_{j-1}^{n_{j-1}};M)\big) + \ell\big(H_{i-1}(x_1^{n_1},...,x_j^{n_j};M)\big)$$
$$= \sum_{s=0}^{j-1-i}\binom{j-s-2}{i-1}\ell(H^0_m(M\ /(x_1^{n_1},...,x_s^{n_s})M)) + \sum_{s=0}^{j-2-i}\binom{j-s-2}{i-2}\ell(H^0_m(M\ /(x_1^{n_1},...,x_s^{n_s})M))$$
$$= \sum_{s=0}^{j-i}\binom{j-s-1}{i-1}\ell\Big(H^0_m(M\ /(x_1^{n_1},...,x_s^{n_s})M)\Big).$$

The lemma is proved.

The main result of this section is following theorem.

Theorem 3.2. *Let M be A-module with $\dim A/a(M) \leq 1$, and $x_1,...,x_d$ a s.o.p for M such that $(x_2,...,x_d)A \subseteq \mathrm{Rad}\ a(M)$. Then, for all $n >> 0$, $d \geq j \geq i \geq 1$,*

$\ell(H_i(x_1^{n_1},...,x_j^{n_j};\ M))$ *is a constant if* $i > 1, j \neq d$ *and* $\ell(H_1(x_1^{n_1},...,x_d^{n_d};\ M))$ *is a polynomial of degree* $p(M)$. *Moreover it holds*

$$\ell(H_i(x_1^{n_1},...,x_j^{n_j};M)) = \sum_{s=0}^{j-i}\binom{j-s-1}{i-1}\sum_{t=0}^{s-1}\binom{s-1}{t}\ell(H_m^t(M/x_1^{n_1}M)).$$

Proof. Since $x_1^{n_1},...,x_d^{n_d}$ is a d-sequence for $n >> 0$ by Lemma 1.2, it follows from Lemma 3.1 that

$$\ell(H_i(x_1^{n_1},...,x_j^{n_j}\ ;\ M)) = \begin{cases} 0 & \text{if } j < i \\ \sum_{s=0}^{j-i}\binom{j-s-1}{i-1}l(H_m^0(M/(x_1^{n_1},...,x_s^{n_s})M)) & \text{if } j \geq i \end{cases}$$

for $i,j = 1,...,d$ and $n_1,....,n_d >> 0$. But, by Theorem 2.3, $x_2^{n_2},...,x_d^{n_d}$ is a standard s.o.p for $M/x_1^{n_1}M$, using [T, Lemma 1.7], we can easily show by induction on s that

$$\ell(H_m^0(M/(x_1^{n_1},...,x_s^{n_s})M)) = \sum_{t=0}^{s-1}\binom{s-1}{t}\ell(H_m^t(M/x_1^{n_1}M)).$$

So we get the formula of the theorem. Furthermore, it implies from Lemma 2.2 and (iii) of Theorem 2.3 that $\ell(H_m^0(M/x_1^{n_1},...,x_s^{n_s})M))$ is a constant for $s < d-1$ and it is a polynomial in n_1 degree $p(M)$ for $s = d-1$. Hence, $\ell(H_i(x_1^{n_1},...,x_j^{n_j};\ M))$ is a constant if $i > 1, j \neq d$ and it is a polynomial in n_1 of degree $p(M)$ if $i = 1,\ j = d$ (for $n >> 0$) as required.

Theorem 3.2 deduces the following consequence.

Corollary 3.3. *Under same assumptions as in Theorem 3.2. Then, for* $n >> 0$, *the higher Euler - Poincare characteristics*

$$X_k(x_1^{n_1},...,x_j^{n_j};M) = \sum_{i\geq k}(-1)^{i-k}\ell(H_i(x_1^{n_1},...,x_j^{n_j};M))$$

is a polynomial of degree $p(M)$ *for* $j = d,\ k = 1$ *and constant for the other cases.*

Proof. Since

$$X_k(x_1^{n_1},...,x_j^{n_j};M) + X_{k+1}(x_1^{n_1},...,x_j^{n_j};M) = \ell(H_k(x_1^{n_1},...,x_j^{n_j};M)),$$

the corollary follows immediately from Theorem 3.2.

4. Lengths of modules of generalized fractions

First of all, we recall some definitions about modules of generalized fractions which have bee introduced first in [S-Z1] by R.Y. Sharp and H. Zakeri.

Let k be a positive integer. We denote by $D_k(A)$ the set of all $k \times k$ lower triangular matrices with entries in A; for $H \in D_k(A)$, the determinant of H is denoted by $|H|$; and we use ^{T} to denote matrix transpose.

A ***triangular subset*** of A^k is a non-empty subset U in A^k such that (i) whenever $(u_1,...,u_k) \in U$, then $(u_1^{n_1},...,u_k^{n_k}) \in U$ for all choices of positive integers $n_1,...,n_k$, and (ii) whenever $(u_1,...,u_k)$ and $(v_1,...,v_k) \in U$, then there exist $(w_1,...,w_k) \in U$ and $H,\ K\ \in D_k(A)$ such that

$$H[u_1,...,u_k]^T = [w_1,...,w_k]^T = K[v_1,...,v_k]^T.$$

Given such a U and an A-module M, R.Y. Sharp and H. Zakeri have constructed the module of generalized fractions $U^{-k}M$ of M with respect to U as follows.

Let $\alpha = ((u_1,...,u_k),x),\ \beta = ((v_1,...,v_k),y) \in U \times M$. Then we write $\alpha \sim \beta$ when there exist $(w_1,...,w_k) \in U$ and $H,\ K\ \in D_k(A)$ such that

$$H[u_1,...,u_k]^T = [w_1,...,w_k]^T = K[v_1,...,v_k]^T$$

and

$$|H|x - |K|y \in (\sum_{i=1}^{k-1}Aw_i)M.$$

This relation is a equivalent relation on the set $U \times M$. For $x \in M$ and $(u_1,...,u_k) \in U$, we define the formal symbol $x/(u_1,...,u_k)$ to be the equivalence class of $((u_1,...,u_k),x)$ and let $U^{-k}M$ denote the set of all these equivalence classes. Then $U^{-k}M$ is an A-module described as follows. If $x,y \in M$ and $(u_1,...,u_k),(v_1,...,v_k) \in U$, then

$$x/(u_1,...,u_k) + y/(v_1,...,v_k) = (|H|x + |K|y)/(w_1,...,w_k)$$

for any choice of $(w_1,...,w_k) \in U$ and $H,\ K\ \in D_k(A)$ such that $H[u_1,...,u_k]^T = [w_1,...,w_k]^T = K[v_1,...,v_k]^T$. Also, with the the above notation, and for $a \in A$,

$$a(x/(u_1,...,u_k)) = ax/(u_1,...,u_k).$$

Consider the set

$U(M)_{d+1} = \{(x_1,...,x_d,1) \in A$: there exists j with $0 \leq j \leq d$ such that $x_1,...,x_j$ form a subset of a s.o.p of M and $x_{j+1} = ... = x_d = 1\}$.

Then $U(M)_{d+1}$ is a triangular subset of A^{d+1}. Let $x_1,...,x_d$ be a s.o.p for M. Denote by $M(1/(x_1^{n_1},...,x_d^{n_d},1))$ the submodule $\{m/(x_1^{n_1},...,x_d^{n_d},1); m \in M\}$ in $U(M)_{d=1}^{-d-1}M$, this cyclic submodule is annihilated by $\mathrm{Ann}_A M + (x_1^{n_1},...,x_d^{n_d})A$; so it has

$$\ell(M(1/(x_1^{n_1},...,x_d^{n_d},1))) < \infty.$$

Following [S-H] we also call this length the length of the generalized fraction

$$1/(x_1^{n_1},...,x_d^{n_d},1).$$

The aim of this section is to compute this length under the assumption that $\dim A/\mathfrak{a}(M) \leq 1$. To do this, we need some following definitions about *secondary representation* (see [S-H]).

Let L be an Artinian $A-$ module with a minimal secondary representation $L = \sum_{i=1}^{h} C_i$, C_i is P_i-secondary and set

$$\text{Att } L = \{P_1, ..., P_h\} \ , \ L_0 = \sum_{P_i \in \text{Att} L - \{m\}} {}_{i=1}^{h} C_i.$$

Then L_0 is a submodule of L which is independent of the choice of minimal secondary representation for L, will be called the residuum of L. Note that L/L_0 has finite length. We shall call this length the ***residual length*** of L and denote it by $R\ell(L)$.

An element $a \in m$ is called pseudo-L-coregular if a holds one of equivalent conditions:

(i) $a \notin \bigcup_{P \in \text{Att} L - \{m\}} P$,

(ii) $aL_0 = L_0$.

We define the stability index $s = s(L)$ of L to be the least integer $i \geq 0$ such that $m^i L = L_0$. Thus s is the least integer that $a^s L = L_0$ for each pseudo-L-coregular a.

Extending the proofs of [S-H, 2.1] and [S-Z3, 2.4 and 2.7] to modules we can easily prove the following two lemmas.

Lemma 4.1. *Put* $\overline{M} = M/H^0_m(M)$. *Let* $^- : M \to \overline{M}$ *be the natural module homomorphism. Then there is an isomorphism of A-modules*

$$f : U(M)^{-d-1}_{d+1}M \longrightarrow U(\overline{M})^{-d-1}_{d+1}(\overline{M})$$

which is such that, for $y \in M$ *and* $(u_1, ..., u_d, 1) \in U(M)_{d+1}$,

$$f(y/(u_1, ..., u_d, 1)) = \overline{y}/(u_1, ..., u_d, 1).$$

Lemma 4.2. *Suppose that* $d > 1$. *Let* $x_1 \in m$ *be a non-zero- divisor of* M. *Put* $\overline{M} = M/x_1M$ *and let* $^- : M \to \overline{M}$ *denote the natural homomorphism. Then there is a homomorphism*

$$\psi_{d+1} : U(\overline{M})^{-d}_{d}\overline{M} \longrightarrow U(M)^{-d-1}_{d+1}M$$

which is such that, for all $\overline{y} \in \overline{M}$ *and* $(u_2, ..., u_d, 1) \in U(\overline{M})_d$,

$$\psi_{d+1}(\overline{y}/(u_2, ..., u_d, 1) = y/(x_1, u_2, ..., u_d, 1).$$

Moreover, $\ker \psi_{d+1} \cong H^{d-1}_m(M)/x_1H^{d-1}_m(M)$.

The following lemma is analogous to Proposition 3.6 of [S-H].

Lemma 4.3. *Assume that* M *is a g.CM module and* $y_1, ..., y_d$ *is a standard s.o.p for* M. *Then*

$$(0 : (y_1, ..., y_d)^r A)_{U(M)^{-d-1}_{d+1}M} \subseteq M\big(1/(y_1^{r+d}, ..., y_d^{r+d}, 1)\big),$$

for all $r > 0$.

Proof. We use induction on d.

In the case $d = 1$, the lemma is proved by [S-H, Lemma 2.8]. For $d > 1$ and assume that the assertion is true for g.CM modules of dimension $< d$. Lemma 4.1 enables us to assume that depth $M > 0$. Then, by [C-S-T, Satz 3.3], every parameter element of M is a non-zero-divisor.

Let $\alpha \in U(M)^{-d-1}_{d+1}M$ such that $(y_1, ..., y_d)^r\alpha = 0$. By [S-Z2, Corollary 3.6], there exist $y \in M$ and positive integers $n_1, ..., n_d$ such that $\alpha = y/(y_1^{n_1}, ..., y_d^{n_d}, 1)$. Set $\overline{M} = M/y_1^{n_1}M$ and let $^- : M \to \overline{M}$ be natural module homomorphism. From Lemma 4.2 we get the following exact sequence

$$0 \longrightarrow H^{d-1}_m(M)/y_1^{n_1}H^{d-1}_m(M) \longrightarrow U(\overline{M})^{-d}_{d}\overline{M} \xrightarrow{\psi_{d+1}} U(M)^{-d-1}_{d+1}M$$

in which

$$\psi_{d+1}(\bar{c}/(u_2, ..., u_d, 1)) = c/(y_1^{n_1}, u_2, ..., u_d, 1)$$

for each s.o.p. $u_2, ..., u_d$ of $\overline{M}$ and $c \in \overline{M}$. Let

$$\beta = \bar{x}/(y_2^{n_2}, ..., y_d^{n_d}, 1) \in U(\overline{M})^{-d}_{d}\overline{M}.$$

Then $\alpha = \psi_{d+1}(\beta)$. Since $y_1, ..., y_d$ is a standard s.o.p of M, $(y_2, ..., y_d)H^{d-1}_m(M) = 0$. It follows by the hypothesis that

$$\beta \in (0 \ : \ (y_2, ..., y_d)^{r+1}A)_{U(\overline{M})^{-d}_{d}\overline{M}}.$$

Note that $\overline{M}$ is a g.CM module and $y_2, ..., y_d$ is a standard s.o.p of $\overline{M}$. Hence, by the inductive assumption,

$$\beta \in \overline{M}\big(1/(y_2^{r+d}, ..., y_d^{r+d}, 1)\big).$$

Thus $\alpha = z/(y_1^{n_1}, y_2^{r+d}, ..., y_d^{r+d}, 1)$ for some element z of M. Note that

$$\alpha = -z/(y_2^{r+d}, y_1^{n_1}, ..., y_d^{r+d}, 1)$$

(see [S-H, Corollary 2.5]) and every permutation of a standard s.o.p is also a standard s.o.p. Repeat the above argument we get

$$\alpha = v/(y_1^{r+d}, y_2^{r+d}, ..., y_d^{r+d}, 1)$$

for some element v of M. The induction is finished.

With the same method which has been used in the proof of Theorem 3.7 of [S-H] and applying Lemma 4.3 we get the following lemma.

Lemma 4.4. *Let M, $y_1,...,y_d$ be as in Lemma 4.3. Then, for all $n_1,...,n_d \geq d$, we have*

$$\ell(M(1/(y_1^{n_1},...,y_d^{n_d},1))) = n_1...n_d.e(y_1,...,y_d;\ M) - \sum_{i=1}^{d-1}\binom{d-1}{i-1}\ell(H_m^i(M)).$$

Lemma 4.5. *Suppose that $\dim A/a(M) \leq 1$. Let $x_1,...,x_d$ be a s.o.p of M such that $(x_2,...,x_d) \subseteq a(M)$. Then, for all $n \geq s(H_m^{d-1}(M))$, $x_1^n H_m^{d-1}(M)$ is the residuum of $H_m^{d-1}(M)$.*

Proof. Lemma 4.2 enable us to assume that depth $M > 0$. Then

$$(x_1,...,x_d) \not\subset \bigcup_{P\in Att(H_m^{d-1}M)-\{m\}} P.$$

It follows from [K, Theorem 124] that there exists $a \in (x_2,...,x_d)$ such that $y_1 := x_1 + a \notin \bigcup_{P\in Att(H_m^{d-1}M)-\{m\}} P$. Thus y_1 and all positive powers of it are pseudo- $H_m^{d-1}(M)$-coregular. Note that there is $b \in (x_2,...,x_d)$ such that $y_1^n = x_1^n + b$. Therefore the residuum of $H_m^{d-1}(M) = y_1^n H_m^{d-1}(M) = (x_1^n + b)H_m^{d-1}(M) = x_1^n H_m^{d-1}(M)$ as required.

Theorem 4.6. *Suppose that $\dim A/a(M) \leq 1$. Let $x = x_1,...,x_d$ be a s.o.p of M such that $(x_2,...,x_d) \subseteq \operatorname{Rad} a(M)$. Then the length of $M(1/(x_1^{n_1},...,x_d^{n_d},1))$ is a polynomial given by*

$$\ell\Big(M(1/(x_1^{n_1},...,x_d^{n_d},1))\Big) = n_1...n_d e(x_1,...,x_d;\ M)$$
$$- \sum_{i=1}^{d-2}\binom{d-2}{i-1}\ell(H_m^i(M/x_1^{n_1}M)) - R\ell(H_m^{d-1}(M))$$

for $n >> 0$.

Proof. Applying Lemma 4.1 we may assume that depth $M > 0$. Write $\overline{M} = M/x_1^{n_1}M$. Since x_1 is a non-zero-divisor of M by Lemma 1.2, it follows by the hypothesis that $\overline{M}$ is a g.CM module. let $^- : M \longrightarrow \overline{M}$ be the natural map. From Lemma 4.2 we have the exact sequence

$$0 \longrightarrow H_m^{d-1}(M)/x_1^{n_1}H_m^{d-1}(M) \longrightarrow U(\overline{M})_d^{-d}\overline{M} \xrightarrow{\psi_{d+1}} U(M)_{d+1}^{-d-1}M$$

in which

$$\psi_{d+1}(\overline{x}/(y_2,...,y_d,1)) = x/(x_1^{n_1},y_2,...,y_d,1)$$

for each s.o.p. $y_2,...,y_d$ of $\overline{M}$. Note that, there exist a positive integer n_0 enough large such that $(x_2^{n_2},...,x_d^{n_d})A \subseteq a(M)$ and by Theorem 2.3 such that $x_2^{n_2},...,x_d^{n_d}$ is a standard s.o.p. of $\overline{M}$ for all $n_2,...,n_d \geq n_0$. Since $(x_2^{n_0},...,n_d^{n_0})\ker\psi_{d+1} = 0$ it follows from Lemma 4.3 that

$$\ker\psi_{d+1} \subseteq \big(0\ :\ (x_2^{n_0},...,x_d^{n_0})\big)_{U(\overline{M})_d^{-d}\overline{M}} \subseteq \overline{M}(1/(x_2^{n_0+d},...,x_d^{n_0+d},1))$$
$$\subseteq \overline{M}(1/(x_2^{n_2},...,x_d^{n_d},1))$$

for all $n_2,...,n_d \geq n_0 + d$. So we get an exact sequence of A-modules of finite lengths

$$0 \longrightarrow H_m^{d-1}(M)/x_1^{n_1}H_m^{d-1}(M) \longrightarrow \overline{M}(1/(x_2^{n_2},...,x_d^{n_d},1)) \longrightarrow$$
$$\longrightarrow M\big(1/(x_1^{n_1},x_2^{n_2},...,x_d^{n_d},1)\big) \longrightarrow 0.$$

Therefore, by Lemma 4.5, there is a positive integer k such that

$$\ell(H_m^{d-1}(M)/x_1^{n_1}H_m^{d-1}(M)) = R\ell(H_m^{d-1}(M))$$

for all $n_1 \geq k$. Thus, applying Lemma 4.4 to the g.CM $\overline{M}$, we deduce for all $n_1 \geq k$ and $n_2,...,n_d \geq n_0 + d$ that

$$\ell\Big(M(1/(x_1^{n_1},...,x_d^{n_d},1))\Big) = \ell\Big(\overline{M}(1/(x_2^{n_2},...,x_d^{n_d},1))\Big) - R\ell(H_m^{d-1}(M))$$
$$= e(x_2^{n_2},...,x_d^{n_d};M/x_1^{n_1}M) - \sum_{i=1}^{d-2}\binom{d-2}{i-1}\ell(H_m^i(M/x_1^{n_1}M)) - R\ell(H_m^{d-1}(M)).$$

Since x_1 is a non-zero-divisor on $M/x_1^{n_1}M$,

$$e(x_2^{n_2},...,x_d^{n_d};\ M/x_1^{n_1}M) = e(x_1^{n_1},\ x_2^{n_2},...,x_d^{n_d}\ ;\ M).$$

Hence

$$\ell\Big(M(1/(x_1^{n_1},...,x_d^{n_d},1))\Big) = n_1...n_d e(x_1,...,x_d;\ M)$$
$$- \sum_{i=1}^{d-2}\binom{d-2}{i-1}\ell(H_m^i(M/x_1^{n_1}M)) - R\ell(H_m^{d-1}(M))$$

is a polynomial for all $n_1,....,n_d \geq \max\{n_0 + k\}$. The proof of Theorem is complete.

Theorem 4.6 and Theorem 2.3 lead to the following interesting result.

Corollary 4.7. *Suppose that $\dim A/a(M) \leq 1$. Let $x = x_1,...,x_d$ be a s.o.p. as in Theorem 4.6. Consider the following difference*

$$J_M(n;x) = n_1...n_d e(x;M) - \ell\Big(M(1/(x_1^{n_1},...,x_d^{n_d},1))\Big)$$

as a function in n. Then, for all $n >> 0$, $J_M(n;x)$ is a constant for $d \leq 2$ and it is a polynomial of degree $p(M)$ for $d \geq 3$.

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