RIGOROUS SPIN–SPIN CORRELATION FUNCTION OF ISING MODEL ON A SPECIAL KIND OF SIERPINSKI CARPETS

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ABSTRACT

We have exactly calculated the rigorous spin–spin correlation function of Ising model on a special kind of Sierpinski Carpets (SC's) by means of graph expansion and a combinatorial approach and investigated the asymptotic behaviour in the limit of long distance. The result shows there is no long range correlation between spins at any finite temperature which indicates no existence of phase transition and thus finally confirms the conclusion produced by the renormalisation group method and other physical arguments.

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1 Introduction

The existence of long range correlation usually is a fundamental feature of phase transition, therefore calculating rigorous correlation function and investigating its asymptotic behaviour in the limit of long distance becomes a basic and main way to test the existence of phase transition [?,?]. Usually this topic for high dimension translationally symmetric lattices is very difficult to reach, because the high connectivity of high dimensions strengthens the correlations of system.

In the 1980s a kind of geometric object with dilation symmetry—fractal has attracted great attention, many research works related to fractal have been done. In the field of phase transition and critical phenomena Gefen et al. [3-5] have finished some pioneering works, they have investigated the phase transitions of the Ising and Potts model on some regular (mathematical) fractals in terms of exact (decimation) and approximate (Migdal-Kadanoff bond-moving) renormalising group methods. Since then a lot of research works associated with the same topic have been reported [?]. An essential result is that only zero temperature transition takes place for the fractals with finite ramification, which is in consistence with previous physical argument.

Recently, we have noted that though the above conclusion is credible in physics, a rigorous calculation based on the fundamental principle of statistical physics still is necessary and valuable because it will finally confirm the validity of the previous conclusion mathematically and physically.

As we know, calculation of partition function and correlation function in the field of phase transitions and critical phenomena is a basic and difficult task, only a few results have been obtained so far. For fractal lattices only those with finite ramification and without loop structure can be exactly solved. In our previous article [?] we presented an exact calculation of partition function of Ising model on a special kind of Sierpinski Carpets. In this paper we will exactly calculate the spin—spin correlation function on the same fractal lattices in terms of graph expansion and combinatorial approach and investigate the asymptotic behaviour in the limit of long distance.

2 Graph expansion of spin—spin correlation function

First of all let us construct a special kind of Sierpinski Carpets via an iteration process. We start from a square and divide it into $b \times b$ sub-squares, then remove $\ell \times \ell$ sub-squares out with a special way like shown in Fig. 1. The process was repeated. In Fig.1 we choose $b = 3 \cdot \ell = 1$. Obviously, such a special kind of SC is of finite ramification in contrast with the usual Sierpinski Carpets.

We now put the Ising spin at each lattice site. The reduced Ising hamiltonian is

$$-\beta H = k \sum_{\text{n.n.}} \sigma_i \sigma_j \quad (k > 0, \beta = \frac{1}{kT})$$

(1)

where $\sigma_i$ denotes the Ising spin at site $i$ and $\sigma_i = \pm 1$. $k$ represents interaction parameter between nearest neighbour spins, the summation runs over all nearest neighbour spins. Correlation function between spins $\sigma_k$ and $\sigma_{k'}$ is, by definition,

$$\langle \sigma_k \sigma_{k'} \rangle = \frac{1}{Z_n} \sum_{\{\sigma\}} (\sigma_k \sigma_{k'}) \exp \left[ k \sum_{\text{n.n.}} \sigma_i \sigma_j \right]$$

(2)
in which \(Z_n\) is the partition function and \(\sum_{\{\sigma\}}\) is over all possible spin configurations of the system. Employing the following expression

\[
\exp(k\sigma_i\sigma_j) = (\cosh k)(1 + \sigma_i\sigma_j \tanh k)
\]

the expression of correlation function can be rewritten as

\[
\langle \sigma_k\sigma_{k'} \rangle = \frac{1}{Z_n} \sum_{\{\sigma\}} (\sigma_k\sigma_{k'}) \prod_{\text{n.n.}} \exp(k\sigma_i\sigma_j)
\]

\[
= \frac{1}{Z_n} (\cosh^M k) \sum_{\{\sigma\}} (\sigma_k\sigma_{k'}) \left\{ 1 + \sum_{\text{n.n.}} \sigma_i\sigma_j \tanh k + \right.
\]

\[
+ \sum_{\text{n.n.}} (\sigma_i\sigma_j_1)(\sigma_i\sigma_j_2) \tanh^2 k + \right.
\]

\[
+ \ldots + \sum_{\text{n.n.}} (\sigma_i\sigma_j_1)\ldots(\sigma_iM\sigma_jM) \tanh^M k \right\} \tag{4}
\]

where \(M\) is the total number of edges of the Sierpinski Carpet. As we see, (4) is an expansion in the power of \((\tanh k)\). To calculate the result of (4), we map each term in the right-hand side of the expression (4) onto a bond configuration. The rule of mapping is: join a bond between spins \(\sigma_i\) and \(\sigma_j\) \((\ell = 1, 2, \ldots, M)\) and assign a value \((\tanh k)\) on each bond.

Let us consider the typical term with coefficient \(\tanh^k\)

\[
\sum_{\{\sigma\}} (\sigma_k\sigma_{k'})[(\sigma_i\sigma_j_1)(\sigma_i\sigma_j_2)\ldots(\sigma_iM\sigma_jM)]. \tag{5}
\]

For the moment we regard \(\sigma_k\) and \(\sigma_{k'}\) as external spins and the spins within the bracket \([\ldots]\) as internal spins. The bond configuration might include three kinds of diagrams in general: isolated external points, linear diagrams (path) and ring diagrams. For the isolated external point it does not connect with other spins and must come from external spin. The linear diagram is an un-closed chain connecting spin \([\sigma_1\sigma_2\ldots\sigma_n, \sigma_1]\) while the ring diagram is a closed sub-square which results from the products of four internal spins like \([\sigma_1\sigma_2, \sigma_3\sigma_4, \sigma_5\sigma_6]\). Naturally, we are interested in those bond configurations which give non-zero contribution to expression (4). In the following we will investigate some typical cases in expression (5):

(a) The internal spins do not include \(\sigma_k\) and/or \(\sigma_{k'}\) and thus \(\sigma_k\) and/or \(\sigma_{k'}\) are isolated. Obviously the contribution of this kind of term will be zero. A sketch is shown in Fig.2.

(b) The term \(\sum_{\{\sigma\}} (\sigma_k\sigma_{k'})[(\sigma_i\sigma_n)\ldots(\sigma_i\sigma_k)]\) forms a linear path connecting \(\sigma_k\) and \(\sigma_{k'}\). The path looks like a teeth-of-saw type (see Fig.3). We can see that each site on the path has an in–bond and an out–bond and each end of the path has an in–(out–) bond and an external point corresponding to spins \(\sigma_k\) or \(\sigma_{k'}\). Such a term will give non–zero contribution to (4).

(c) The term \(\Sigma_{\{\sigma\}} (\sigma_k\sigma_{k'})[*\ldots*]_{\ell_1}[*\ldots*]_{\ell_2}\) in which \((\sigma_k\sigma_{k'})[*\ldots*]\) forms a linear path and
both are ring diagrams attaching to the ends of the path, gives a non-zero contribution to (4). A sketch is shown in Fig.4.

(d) A term representing ring diagrams attaching to the linear path between \( \sigma_k \) and \( \sigma_{k'} \), can be typically written as

\[
\sum_{\{\sigma\}} (\sigma_k \sigma_{k'})[\ldots \ast ]_L[\ldots \ast ]_r_1[\ldots \ast ]_r_2[\ldots \ast ]_r_3 \ldots [\ldots \ast ]_r_n
\]

where \((\sigma_k \sigma_{k'}[\ldots \ast ]_L\) forms the linear path connecting \( \sigma_k \) and \( \sigma_{k'} \) and \([\ldots \ast ]_r_i\) \((i = 1, 2, \ldots n)\) ring diagrams attaching to the path, as shown in Fig.5. It contributes a non-zero value to (4).

(e) A more general term, which corresponds to the diagram as shown in Fig.6, \( \Sigma (\sigma_k \sigma_{k'})[\ast \ldots \ast ]_L \prod_{i} [\ast \ldots \ast ]_r_i\) is composed of the linear path and different ring diagrams attaching and un-attaching to the linear path. It also gives non-zero contribution.

So far we have given all possible kinds of graphs with non-zero contribution to (4). We would like to point out that any complex graph will be factorized into linear diagram and a number of ring diagrams, which is clearly displayed in Fig.6. This is determined by the geometric feature of structure of the Sierpinski Carpet under consideration.

### 3 Exact expression of correlation function

We now proceed to calculate the correlation function. For the sake of clearness we employ Fig.6 to explain our calculation. In Fig.6 we display the linear path connecting \( \sigma_k \) and \( \sigma_{k'} \) in terms of heavy bonds with arrows; the number of bonds on the path is determined by the number of sub-squares passed by the path; each such sub-square provides two or one (three) bonds; in general the latter occurs near the ends of the linear path.

First we calculate the contribution of the linear path joining \( \sigma_k \) and \( \sigma_{k'} \). As we see at the site, say, \( \alpha \) there are two ways to cross the sub-square (\( \alpha \)-type sub-square), each way includes two bonds; at the site, say, \( \beta \) there are also two ways to cross the sub-square (\( \beta \)-type sub-square), but one way involves one bond and another way three bonds. Let us suppose that we have \( L \) \( \alpha \)-type and \( L' \) \( \beta \)-type sub-square along the linear path, thus they give the following contribution to (4):

\[
2^L(\tanh^2 k)^L(\tanh k + \tanh^3 k)^{L'}.
\] (6)

In other respect, the ring diagrams provide the contribution

\[
1 + (m - L - L') \tanh^4 k + \frac{(m - L - L')!}{2!(m - L - L' - 2)!} (\tanh^4 k)^2 + \ldots + (\tanh^4 k)^{m-L-L'}
\]

\[
= (1 + \tanh^4 k)^{m-L-L'}. \quad (7)
\]

Therefore from (4) to (7) we obtain
\[
\langle \sigma_k \sigma_{k'} \rangle = \frac{1}{Z_n} (\cosh k)^{M} 2^{N(n)} 2^L (\tanh^2 k)^L (\tanh k + \tanh^3 k)^{L'} (1 + \tanh^4 k)^{m-L-L'}
\]
\[
= \frac{(2 \tanh^2 k)^L (\tanh k + \tanh^3 k)^{L'}}{(1 + \tanh^4 k)^{L+L'}} = \left( \frac{2 \tanh^2 k}{1 + \tanh^4 k} \right)^L \left( \frac{\tanh k + \tanh^3 k}{1 + \tanh^4 k} \right)^{L'}
\]

where the partition function \( Z_n \) is given by [7]
\[
Z_n = (\cosh k)^M 2^{N(n)} (1 + \tanh^4 k)^m
\]

and \( m \) is the total number of sub-squares.

It is easy to check that
\[
\frac{2 \tanh^2 k}{1 + \tanh^4 k} < 1, \quad \frac{\tanh k + \tanh^3 k}{1 + \tanh^4 k} < 1.
\]

4 Asymptotic behaviour of correlation function

The expression (8) gives the exact spin-spin correlation function. We immediately see from (10) that \( \langle \sigma_k \sigma_{k'} \rangle \) does tend exponentially to zero as \( L + L' \) become large. Roughly speaking, \( L + L' \) represents the distance between spins \( \sigma_k \) and \( \sigma_{k'} \). The above result indicates that the correlation between spins will decrease rapidly with the increase of the distance \( L + L' \). In terms of the asymptotic formula
\[
\langle \sigma_k \sigma_{k'} \rangle \sim \exp[-(L + L')/\xi].
\]

The correlation length \( \xi \) would be
\[
\xi \sim \left[ \ln \left( \frac{1 + \tanh^4 k}{2 \tanh^2 k} \right) \right]^{-1}
\]

where we have ignored the factor \( \frac{\tanh k + \tanh^3 k}{1 + \tanh^4 k} \) since \( L' \) is always much less than \( L \).

When \( k \) approaches to infinity, equivalently, the temperature approaches to zero, the correlation length \( \xi \to \infty \). It thus indicates the existence of zero temperature phase transition. The result rigorously confirms the validity of the following conclusion [3-5]: no finite temperature phase transition occurs for fractals with finite ramification.

We have noted that our method is difficult to apply to fractal with loop structure, even if they are of finite ramification.

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References


Figure Captions

Fig. 1 The process of construction of Sierpinski Carpet embedded in two-dimensional Euclidean space. The shadow parts are removed out. The first stage of construction is the generator. The fractal dimension $d_f = 1.465$.

Fig. 2 The internal spins of (5) do not include $\sigma_k$ and $\sigma_{k'}$, therefore $\sigma_k$ and $\sigma_{k'}$ are isolated. The shadow region represents the internal spins.

Fig. 3 A sketch of a linear path formed by the term

$$(\sigma_k \sigma_{k'})[\sigma_k \sigma_{a_2}](\sigma_{a_1} \sigma_{a_2}) \ldots (\sigma_{a_4} \sigma_{k'}) .$$

Fig. 4 A sketch of a path, of which at the ends having two ring diagrams.

Fig. 5 A sketch of a path corresponding to $(\sigma_k \sigma_{k'})[*\ldots*]_L[*\ldots*]_{r_1} \ldots [*\ldots*]_{r_5}$ in which there are five ring diagrams attaching to the linear path.

Fig. 6 The $n = 3$ stage of construction of SC. The shadow parts are removed out. A sketch of a path connecting $\sigma_k$ and $\sigma_{k'}$ is shown. There are two ways like $\alpha \rightarrow 1 \rightarrow \alpha_1$ and $\alpha \rightarrow 2 \rightarrow \alpha_1$ to get site $\alpha_1$ from site $\alpha$, and there are also two ways like $\beta \rightarrow k$ and $\beta \rightarrow a \rightarrow b \rightarrow k$ to get site $k$ from site $\beta$. The heavy bonds indicate linear diagrams and ring diagrams.
Fig. 1