CALCULATIONS ON SOME PROPERTIES OF THE PION IN THE FRAMEWORK OF SCHWINGER–DYSON AND BETHE–SALPETER EQUATIONS

Wan Shao-long

and

Wang Ke-lin

MIRAMARE-TRIESTE
I. Introduction

The calculation of the electromagnetic form factor, mass, radius and decay constant of the pion involves a variety of fundamental theoretical issues such as spontaneous and explicit chiral-symmetry breaking, the phenomenological potential, as well as the infrared and ultraviolet asymptotic behaviors of the effective quark-gluon coupling, and the propagator functions and mass function of the quark and the wave functions of the bound state of the pion which are obtained from the Schwinger-Dyson and Bethe-Salpeter equations. This applies not only to quantum chromodynamics but also to higher symmetry theories such as technicolor. Some of those quantities have been calculated in variety of approximation to quantum chromodynamics, which include potential models, bag models, QCD sum rules, and lattice QCD. Recently, Munczek and Jain 1 have assumed a phenomenological model and calculated the Bethe-Salpeter wave functions and decay constant of 0" mesons in the framework of the Schwinger-Dyson and Bethe-Salpeter equations, but not the electromagnetic form factor and radius of the 0" mesons. All of their results calculated are in accord with experimental values or those from other theories very well. In order to test further their phenomenological potential, we calculated the electromagnetic form factor and radius of the 0" mesons. All of their results calculated are in accord with experimental values or those from other theories very well. In order to test further their phenomenological potential, we calculated the electromagnetic form factor and radius of the pion in the same framework with their phenomenological model according to 1. We found that the electromagnetic form factor is in accord with experimental values only in a small $k^2$ region and the radius is much larger than experimental value in this phenomenological model. On the other hand, we have assumed a phenomenological potential model 2, the flat-bottom potential, and calculated some properties of the pion 2-3 and the kaon 4 in the framework of the Bethe-Salpeter equation with the phenomenological potential. All results we calculated are in conformity with the experimental values 5. However, in these studies, 1) the full propagator function was replaced with the propagator of the freedom quark in the Bethe-salpeter equation, and 2) no further physical consideration for the infrared and ultraviolet behaviors of the phenomenological potential was introduced. In order to test the phenomenolog-
ical potential model and consider effect of the infrared and ultraviolet behaviors of
the phenomenological potential, in present work, we combine our work with that of
Munczek and Jain, assume a flat-bottom phenomenological potential with considera-
tion for infrared and ultraviolet behaviors, the modified flat-bottom potential model,
and calculate the quark propagator function and mass function of the quark and the
Bethe-Salpeter wave functions of bound state, electromagnetic form factor, mass, ra-
dius and decay constant of the pion in the framework of the Schwinger-Dyson and
Bethe-Salpeter equations with this phenomenological potential. All of our results for
the asymptotic quark mass function and the bound state wave functions agree with
those derived from other theories 1,6 and for the electromagnetic form factor, mass,
radius and decay constant of the pion agree with experimental results 5, respectively.

In Sec.II, first, the expressions for the Schwinger-Dyson and Bethe-Salpeter equa-
tions are given and the expansion method of Lorentz invariant functions is introduced.
Then, the phenomenological potential model, the modified flat-bottom potential, is
assumed. Finally, the method of expression of SO(4) eigenfunctions is introduced. In
Sec.III, first, the matrix element of electromagnetic current in the Euclidean space
is reduced. Then, the formula for the physical spacelike electromagnetic form factor
of the pion is obtained. Finally, the relationship between the radius and the form
factor of the pion is given. In Sec.IV, the decay constant of the pion is obtained from
partial conservation of axial-vector current. In Sec.V, all numerical results for the
propagator functions and mass function of the quark, shown in Figs.[2-4], and for the
wave functions of bound state, shown in Figs.[5-7], mass, electromagnetic form factor,
shown in Fig.[8], radius and decay constant of the pion and calculating procedure are
given. Finally, conclusions and discussions are given in Sec.V.

II. Schwinger-Dyson and Bethe-Salpeter Equations
and Phenomenological Potential

A. Schwinger-Dyson Equation

In order to obtain the solutions to the Bethe-Salpeter equation for the wave func-
tions of the bound state, at first, it is necessary to find the solution to the Schwinger-
Dyson equation for the quark propagator function S(q). In the absence of mass-
breaking between u quark and d quark in the pion, the Schwinger-Dyson equation
with Landau-gauge takes the form

\[ S^{-1}(p) = \gamma \cdot p - m + \frac{i}{(2\pi)^2} \int d^4k \gamma_\mu G_\mu(k) G_\nu(k-p) \]  (1)

where

\[ S^{-1}(p) = \gamma \cdot p A(p^2) - B(p^2) \equiv A(p^2) \gamma \cdot p - m(p^2) \]  (2)

and \( G_\mu(k) \) is a phenomenological potential which can be written as the following
form

\[ G_\mu(k) = - \left[ g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \eta_\nu(k^2) \]  (3)

Based on Eqs.(1), (2) and (3), one can obtain two coupled integral equations
for the functions \( A(p^2) \) and \( B(p^2) \). We perform a rotation analytically continuing \( k \)
and \( p \) into the Euclidean region where we will denote them by \( \bar{k} \) and \( \bar{p} \), a procedure
which we will assume to be allowed for both the Schwinger-Dyson and Bethe-Salpeter
equations. Alternatively, one can derive the Schwinger-Dyson and Bethe-Salpeter
equations from the Euclidean path-integral formulation of the theory, thus avoiding
possible difficulties in performing the Wick rotation 7. From above, the Schwinger-
Dyson equation for the functions \( A(p^2) \) and \( B(p^2) \) can be written as

\[ A(p^2) = 1 + \frac{1}{(2\pi)^4} \int d^4k \frac{A(k^2) G((k-p)^2)}{k^2 A(k^2) + B(k^2)} \left[ \bar{p} \cdot \bar{k} + \frac{2 \bar{p} \cdot (k - \bar{k}) \bar{k} \cdot (k - \bar{k})}{(k-p)^2} \right] \]

\[ B(p^2) = m + \frac{3}{(2\pi)^4} \int d^4k \frac{B(k^2) G((k-p)^2)}{k^2 A(k^2) + B(k^2)} \]  (4)
Before we solve Eqs.(4), it is important to introduce a mass scale parameter $\Lambda_{QCD}$, and to change $B(p^2)$, $k$, and $p$ into dimensionless quantities by

$$B(p^2) = \frac{B(p^2)}{\Lambda_{QCD}} \quad \tilde{k} = \frac{k}{\Lambda_{QCD}} \quad \tilde{p} = \frac{p}{\Lambda_{QCD}} \quad \mu(L) = \frac{\mu}{\Lambda_{QCD}}$$

(5)

In terms of the symmetry for $k_0$, $k_1$, $k_2$, and $k_3$ in Eqs.(4) and the dimensionless changes (5), when a cutoff $L$ is introduced, one can simplify Eqs.(4) as the following form

$$A(p^2) = 1 + \frac{4}{3\pi^2} \int_0^L dk \frac{A(k^2)}{k^2} \frac{A(k^2)}{k^2} K_A(k^2, p^2)$$

$$B(p^2) = \mu(L) + \frac{4}{\pi} \int_0^L dk \frac{B(k^2)}{k^2} \frac{B(k^2)}{k^2} K_B(k^2, p^2)$$

(6)

where $k = |k|$, $K_A(k^2, p^2)$ and $K_B(k^2, p^2)$ are given in the appendix.

### B. Bethe-Salpeter Equation

When the mass of quark and antiquark is the same, the wave functions of the bound states of the quark-antiquark pairs should be the solutions to the Bethe-Salpeter equation which takes the form

$$S^{-1}(p + \frac{1}{2}Q)\chi(Q,p)S^{-1}(p - \frac{1}{2}Q) = \frac{-i}{(2\pi)^4} \int d^4k \gamma_\mu \gamma_\nu \gamma_\rho \Gamma_{\mu\nu\rho}(k - p)$$

(7)

where $\chi(Q,p)$ is the Bethe-Salpeter wave function of bound state. $S(p)$ is the quark propagator function. $\Gamma_{\mu\nu\rho}$ is the gluon propagator (the phenomenological potential). $Q$ is four-momentum of the center of mass of the pion. $p$ is relative four-momentum between quark and antiquark in the pion.

Considering the Bethe-Salpeter equation in the rest frame of the bound state, one can perform a rotation analytically continuing $k$ and $p$ into the Euclidean region where we will denote them by $\tilde{k}$ and $\tilde{p}$ as in the Schwinger-Dyson equation. In addition, we define

$$\overline{Q} = -iQ$$

then the Bethe-Salpeter equation can be written as

$$S^{-1}_\nu(Q, \overline{Q})\chi(Q, p)S^{-1}_\mu(Q, \overline{Q}) = \frac{1}{(2\pi)^4} \int d^4k \gamma_\mu \gamma_\nu \gamma_\rho \Gamma_{\mu\nu\rho}(\tilde{k} - \tilde{p})$$

(8)

where

$$S^{-1}_\nu(Q, \overline{Q}) = \gamma \cdot (p \pm iQ)A_{\nu}(p \pm iQ) - B_{\nu}(p \pm iQ)$$

In addition, the Bethe-Salpeter wave function $\chi(Q, p)$ of the bound state can be expanded as Lorentz invariant functions

$$\chi(Q, p) = \gamma_\nu \chi^{(1)}(p, p \cdot Q) + \gamma_\nu \gamma_\rho \chi^{(2)}(p, p \cdot Q)$$

(9)

In order to obtain the numerical solutions of the wave function to the Bethe-Salpeter equation, $\chi^{(1)}(p, p \cdot Q)$ is expanded in terms of $SO(4)$ eigenfunction, Tschebyshev function. In addition, as we know, when the masses of the quark and antiquark are equal, the bound state of the system satisfies the charge conjugation symmetry and the time reversal invariance and the Lorentz invariant wave functions, $\chi^{(1)}(p, p \cdot Q)$ can be expressed as

$$\chi^{(1)}(p, p \cdot Q) = \sum_{n=0}^\infty \chi^{(1)}(p, Q)(-1)^n p^n T_n^4(\cos \theta)$$

(10)

where, the expression of Tschebyshev function is

$$T_n^4(\cos \theta) = \cos(n\theta)$$

where, $\theta$ is the angle between $\overline{p}$ and $Q$. Before solving the Bethe-Salpeter equation, it is important to translates the wave functions $\chi^{(1)}(p)$ into the form of same dimension

$$F_n^{(1)} = \Lambda_{QCD} x_n^{(1)} \quad F_n^{(2)} = \Lambda_{QCD} x_n^{(2)} \quad F_n^{(3)} = \Lambda_{QCD} x_n^{(3)} \quad F_n^{(4)} = \Lambda_{QCD} x_n^{(4)}$$

(12)
Based on Eqs.(10-12) and Eq.(5), the Bethe-Salpeter equation (8) can be projected into the infinite set of coupled integral equations for $F^{(1)}_a, F^{(3)}_a, F^{(4)}_a$, and $F^{(5)}_a$. In fact, it is impossible to solve the infinite set of coupled integral equations for $F^{(1)}_a, F^{(3)}_a, F^{(4)}_a$, and $F^{(5)}_a$, therefore the first step in solving the Bethe-Salpeter equation is to truncate the infinite sum over $n$ in Eq.(11). Because it is possible to calculate many properties of the bound state with accuracy in the lowest order approximation, we only retain nonvanishing terms ($n=0$ or $n=1$) in Eq.(11). The results calculated show that this approximation works well. As a result, in the lowest order approximation, the Bethe-Salpeter wave function is expressed as follows

$$\chi(Q,\bar{P}) = \gamma_3\chi_0^{(1)} + \gamma_3\gamma_\mu\gamma_\nu\chi_0^{(2)} - \gamma_3\gamma_\mu\gamma_\nu\chi_0^{(3)}(\cos\theta) + \gamma_3[\gamma_\mu\gamma_\nu\gamma_\sigma]\chi_0^{(4)}$$

From the above, the resulting explicit expression, in the lowest order approximation, for the Bethe-Salpeter equation can be written as

$$H_{11}F^{(1)}_0 + H_{12}F^{(3)}_0 + H_{13}F^{(4)}_0 + H_{14}F^{(5)}_0 = \int_0^L dk K_{11}F^{(1)}_0$$

$$H_{21}F^{(1)}_0 + H_{22}F^{(3)}_0 + H_{23}F^{(4)}_0 + H_{24}F^{(5)}_0 = \int_0^L dk K_{22}F^{(2)}_0 + \int_0^L dk K_{23}F^{(3)}_0$$

$$H_{31}F^{(1)}_0 + H_{32}F^{(3)}_0 + H_{33}F^{(4)}_0 + H_{34}F^{(5)}_0 = \int_0^L dk K_{33}F^{(3)}_0 + \int_0^L dk K_{34}F^{(4)}_0$$

$$H_{41}F^{(1)}_0 + H_{42}F^{(3)}_0 + H_{43}F^{(4)}_0 + H_{44}F^{(5)}_0 = \int_0^L dk K_{44}F^{(5)}_0$$

where $H_{ij}$ and $K_{ij}$ are given in the appendix.

Since the phenomenological potential does not depend on the total momentum of the pion, the normalization of the wave function is given by

$$\frac{1}{(2\pi)^4} \int d^4Q \chi(Q, \bar{Q}) \left[ \frac{\partial}{\partial Q^\mu} S^{-1}\left(q + \frac{1}{2} Q\right) \chi(Q, \bar{Q}) S^{-1}\left(q - \frac{1}{2} Q\right) + S^{-1}\left(q + \frac{1}{2} Q\right) \chi(Q, \bar{Q}) \frac{\partial}{\partial Q^\mu} S^{-1}\left(q - \frac{1}{2} Q\right) \right] = 2Q^\mu$$

where $\chi(Q, \bar{Q}) = \gamma_3\chi_0^{(1)}(Q, \bar{Q})\gamma_0$.

C. Phenomenological Potential

Now we discuss the phenomenological potential in the Schwinger-Dyson and Bethe-Salpeter equations. First, based on renormalization-group analysis, we know that the ultraviolet asymptotic behavior of $G(k^2)$ for large $-k^2$ takes the form

$$G(k^2) = \frac{16\pi}{3} \frac{\alpha_s(k^2)}{k^2} = \frac{16\pi^2}{3k^2} \ln\left(k^2/\Lambda_{QCD}^2\right)$$

where $d = 12/(33 - 2\beta_0)$, $b = 2\beta_0^2$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{11}{12}$, and $f$ is the number of quark flavors. $\Lambda_{QCD}$ is the QCD scale parameter, and we have used the two-loop asymptotic expression for $\alpha_s(k^2)$. That the asymptotic form appears to be compatible with experimental for $\Lambda_{QCD} \approx 180$ (MeV) and for $p$ larger than a few GeV an assumption which we will make here. Second, as we know, very little is known, theoretically or experimentally, about the low-momentum, small $-k^2$ behavior of $G(k^2)$. An infrared behavior $G(k^2) \approx 1/k^4$, presumably leading to confinement, has been assumed 10 or obtained from approximation calculation in covariant and axial gauges 11. Such a behavior, however, makes the Schwinger-Dyson and Bethe-Salpeter equations highly singular. On the other hand, the $1/k^4$ infrared behavior has been contested by Lattice calculation in Landau gauge and by theoretical and lattice calculations in the axial gauge 12. These objections don’t necessarily apply to the form $G(k^2) \approx \delta^{(4)}(k)$ a regulated alternative to the behavior $1/k^4$, which may still be indicative of confinement 13. Finally, in the intermediate $-k^2$, considering that the Yukawa potential has the singular point when $r = 0$, we apply the phenomenological potential of Ref.[2], the flat-bottom potential, which is the sum of a set Yukawa potentials.

$$G(k^2) \approx \sum_{j=1}^N \frac{a_j}{k^2 + (N + j\rho)^2}$$

It is analogous to the exchange of a series of particles with different mass. where, $N$ is the minimum value of the mass of the exchange particle, $\rho$ is the difference of the exchange particles, and $\{a_j\}$ is the relative coupled constant of different particles. In
addition, the three dimensional form of the phenomenological potential corresponding to this potential to is
\[ V(r^2) \approx \sum_{J} e^{-\frac{(N+1)\rho}{r}} \]
in order to suppress the singular point at \( r=0 \), we assume that the phenomenological potential satisfies the following conditions:
\[ V(0) = \text{constant} \]
\[ \frac{dV(0)}{dr} = \frac{d^2V(0)}{dr^2} = \ldots = \frac{d^{n-1}V(0)}{dr^{n-1}} = 0 \]
so \( n+1 \) relative coupled constants \( a_j \) can be determined by the following equations [2], the flat-bottom condition
\[ \sum_{j=0}^{n} a_j (N+j\rho)^n = 0 \]
\[ \sum_{j=0}^{n} a_j (N+j\rho)^{n-1} = 0 \]
\[ \ldots \]
\[ \sum_{j=0}^{n} a_j (N+j\rho)^{n+1} = 0 \] (17)

In view of the above, we assume that a phenomenological potential takes the following form which displays a combination of the different features and models discussed above
\[ G(k^2) = \frac{16\pi^2}{3} \left[ \frac{d}{k^2 \ln(x_0 + k^2/\Lambda^2_{CD})} \right] + \left[ 1 + \frac{\ln[x_0 + k^2/\Lambda^2_{CD}]}{\ln(x_0 + k^2/\Lambda^2_{CD})} \right] \]
\[ + \sum_{j=0}^{n} \frac{a_j}{k^2 + (N+j\rho)^2} + (2\pi)^2 \langle \psi | \psi \rangle \]
(18)

According to this phenomenological potential model, the modified flat-bottom potential, we can obtain numerically the propagator functions and mass function, shown in Figs.[2-4], for the quark, and the wave functions, shown in Figs.[5-7], of bound state for the pion from the Schwinger-Dyson and Bethe-Salpeter equations, respectively. Using the propagator functions and mass function for the quark and the wave functions of bound state for the pion, we can study some properties of the quark and the pion as following.

III. Electromagnetic Form Factor and Radius

A. Matrix Element of Electromagnetic Current

Recently, many experiments [5] for the electromagnetic form factor and radius of the pion have been done, which not only give the information of the distribution of the charge of hadron but also give the information of the internal structure of hadron. When one discusses the phenomenological potential, it is effective test to see whether the phenomenological model can obtain the electromagnetic form factor, radius and decay constant of the pion which give a good fit to experimental results. For this reason, first, we calculate the electromagnetic form factor and radius of the pion with the phenomenological potential.

As we know, the relationship between the electromagnetic form factor and the matrix element of the electromagnetic current between two bound states is
\[ F(k^2)(P_1 + P_f) = \langle P_1 | J_\mu(0) | P_f \rangle \]
\[ 2k = P_f - P_1 \] (19)
where \( J_\mu \) is the operator of the electromagnetic current, \( | P_1 \rangle \) and \( | P_f \rangle \) are two bound states. Eq.(19) can be written as
\[ F(k^2) = \frac{1}{2(k^2 + M^2)} P_\mu (P_1 | J_\mu(0) | P_f) \] (20)
In the lowest order, the matrix element of electromagnetic current between two bound states takes the form
\[
(P_f | J_\mu(0) | P_i) = \frac{i}{(2\pi)^4} \int d^4q Tr \left[ \chi(P_i, q - k) i e_\mu \frac{\partial}{\partial P_\mu} S^{-1}(q + \frac{i}{2} P_i) \chi(P_f, q + k) S^{-1}(q + \frac{i}{2} P_f) \right] \]

(21)

The Feynman figures corresponding to the matrix element of the electromagnetic current in the lowest order are shown in Fig. [1]. Under the spacelike condition \( k^2 > 0 \), it is convenient to consider the following choice of kinematic variables:

\[
k = (k, 0, 0, 0)
\]
\[
P_i = (k, 0, 0, i\sqrt{k^2 + M^2})
\]
\[
P_f = (-k, 0, 0, i\sqrt{k^2 + M^2})
\]

(22)

where, \( M \) is mass of the pion.

B. Electromagnetic Form Factor and Radius

The physical quantities of Eq.(21) are in Minkowski space. In order to obtain the electromagnetic form factor from the wave functions which have been calculated in Euclidean region, we perform a rotation to Euclidean variables and obtain the expression of the electromagnetic form factor

\[
F(k^2) = \frac{e}{4(k^2 + M^2)(2\pi)^4} \int d^4q \frac{1}{2} \epsilon^\mu \chi_\mu(P_f, q - k)
\]

\[
\left[ -\gamma \cdot P_i A_+ + 2 \gamma \cdot (i\vec{q} - \frac{i}{2} P_i)(i\vec{q} - \frac{i}{2} P_i) \cdot P_i A_+ + 2 (i\vec{q} - \frac{i}{2} P_i) \cdot P_i B_+ \right]
\]

\[
\chi(P_i, q) \left[ i (i\vec{q} + \frac{i}{2} P_i) A_+ + B_+ \right]
\]

\[
-\frac{1}{2} \epsilon \chi(P_f, q + k) \left[ i (i\vec{q} - \frac{i}{2} P_i) A_+ + B_+ \right] \chi(P_i, q)
\]

\[
\left[ -\gamma \cdot P_i A_+ + 2 \gamma \cdot (i\vec{q} + \frac{i}{2} P_i)(i\vec{q} + \frac{i}{2} P_i) \cdot P_i A_+ + 2 (i\vec{q} + \frac{i}{2} P_i) \cdot P_i B_+ \right]
\]

(23)

where

\[
P_i = (-ik, 0, 0, \sqrt{k^2 + M^2})
\]
\[
P_f = (ik, 0, 0, \sqrt{k^2 + M^2})
\]
\[
A_+ = A \left[ \left( q_0 \mp \frac{1}{2} k_0 \right)^2 + q_i^2 + q_0^2 + \frac{1}{4}(k^2 + M^2) \right]
\]
\[
B_+ = B \left[ \left( q_0 \mp \frac{1}{2} k_0 \right)^2 + q_i^2 + q_0^2 + \frac{1}{4}(k^2 + M^2) \right]
\]

From the solution of the wave functions of bound state, Fig.[7], we know \( \bar{Q} P \chi_\nu \), \( \bar{Q} P x_\nu \). As a result, we can ignore the contribution from \( \bar{Q} P \chi_\nu \) and \( \bar{Q} x_\nu \). In this case, Eq.(23) can be reduced as

\[
F(k^2) = \frac{e}{4(k^2 + M^2)(2\pi)^4} \int d^4q \chi_\nu \left[ \left( q_0 \mp \frac{1}{2} M^2 \right) A_+ A_+ + 2 \left( q_0 \mp \frac{1}{2} M^2 \right) B_+ B_+ \right]
\]

\[
+2 \left( q_0 \mp \frac{1}{2} M^2 \right) \left( q_i^2 + q_0^2 + q_0^2 + \frac{1}{4}(k^2 + M^2) \right) A_+ B_+ + 2 \left( q_0 \mp \frac{1}{2} M^2 \right) B_+ B_+
\]

\[
+\chi_\nu \left[ M^2 A_+ B_+ + 2 \left( q_0 \mp \frac{1}{2} M^2 \right) q_i^2 \left( k^2 + M^2 \right) \right] A_+ B_+
\]

\[
+2 \left[ q_i^2 k^2 - \frac{1}{4} M^4 - q_0^2 \left( k^2 + M^2 \right) \right] B_+ A_+
\]

\[
+\chi_\nu \left( 2q_0 k^2 + M^2 \right) A_+ B_+ + 2 \left( q_0 \mp \frac{1}{2} M^2 \right) q_i^2 \left( k^2 + M^2 \right) A_+ B_+
\]

\[
+2 \left[ q_i^2 k^2 + \frac{1}{4} M^4 + q_0^2 \left( k^2 + M^2 \right) \right] B_+ A_+
\]

\[
+\chi_\nu \left( q_0 \mp \frac{1}{2} M^2 \right) A_+ A_+ + 2 \left( q_0 \mp \frac{1}{2} M^2 \right)
\]

\[
\left( q_0 \mp \frac{1}{2} M^2 \right)
\]

\[
\nu \]
\[
\left[ \left( q^2 + \frac{1}{2} M^2 \right) \left( q^2 + \frac{1}{2} (2k^2 + M^2) \right) - (2k^2 + M^2) \right] \left( q^2 + q_1^2 + q_2^2 + q_0^2 + \frac{1}{4} M^2 \right) - \left( q^2 + \frac{1}{2} M^2 \right) \left( q^2 + \frac{1}{2} (2k^2 + M^2) \right) A_4 A_-
\]
\[
-2 \left( 2k^2 + M^2 \right) \left( q^2 + \frac{1}{2} M^2 \right) B_4 B_-
\]  
\tag{24}
\]

where
\[
\chi_0^{(1)} = \chi_0^{(1)} \left[ (q_1 - k)^2 + q_1^2 + q_2^2 \right]
\]
\[
\chi_0^{(2)} = \chi_0^{(2)} \left[ q_1^2 + q_2^2 + q_0^2 \right]
\]

In terms of the quantities introduced in the translations (5) and (12) and considering that \( q_1, q_2, \) and \( q_0 \) are symmetric in Eq.(24), we can simplify the integral by the following change of integration variables
\[
Q_2 = q_1 + q_2 + q_0
\]

Then the form factor can be written as
\[
F(k^2) = \frac{4\pi}{N_F} e^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q^2 dQ \left[ F_0^{(1)} F_0^{(2)} \left( q^2 + \frac{1}{2} M^2 \right) A_4 A_- \right] + 2 \left( q^2 + \frac{1}{2} M^2 \right) \left( q^2 + \frac{1}{4} M^2 \right) A_4 A_- + 2 \left( q^2 + \frac{1}{2} M^2 \right) B_4 B_-
\]
\[
+ F_0^{(1)} F_0^{(2)} \left[ M^2 A_4 B_- - 2 \left( \left( q^2 + \frac{1}{2} M^2 \right)^2 - \frac{1}{2} Q^2 \left( k^2 + M^2 \right) \right) A_4 B_- \right] + 2 \left[ q^2 k^2 - \frac{1}{4} M^4 - \frac{1}{6} Q^4 \left( k^2 + M^2 \right) \right] B_4 A_-
\]
\[
+ F_0^{(2)} \left[ (2k^2 + M^2) A_4 B_- + 2 \left( q^2 + \frac{1}{2} M^2 \right)^2 - \frac{1}{3} Q^2 \left( k^2 + M^2 \right) \right] A_4 B_- 
\]
\[
- 2 \left[ q^2 k^2 + \frac{1}{4} M^2 \left( 2k^2 + M^2 \right) + qk \left( k^2 + M^2 \right) + \frac{1}{3} Q^2 \left( k^2 + M^2 \right) \right] B_4 A_-
\]
\[
+ F_0^{(1)} \left[ -qk + \frac{1}{2} (2k^2 + M^2) A_4 A_- + 2 \left( q^2 + \frac{1}{2} M^2 \right) \right]
\]  
\tag{25}
\]

where
\[
F_0^{(1)} \equiv F_0^{(1)} \left[ (q - k)^2 + Q^2 \right]
\]
\[
F_0^{(2)} \equiv F_0^{(2)} \left[ q^2 + Q^2 \right]
\]
\[
A_k = A \left[ q \pm \frac{1}{2} k \right] + Q^2 \pm \frac{1}{3} (k^2 + M^2)
\]
\[
B_k = B \left[ q \pm \frac{1}{2} k \right] + Q^2 \pm \frac{1}{3} (k^2 + M^2)
\]
\[
N_F \equiv \frac{(2\pi)^4}{4\lambda Q_0^2} F(0)
\]

From above, the electromagnetic form factor of the pion is obtained from Eq.(25) by using the computer VAX8700, which is shown and compared with the experimental value of Ref.[5] in Fig.[8]. The calculated result for the electromagnetic form factor is in conformity with the experimental values in a large \( k^2 \) region very well.

C. Radius

As we know, Eq.(25) can be used to obtain the radius of the pion
\[
\left( r_s^2 \right) = -\frac{1}{6} \frac{dF}{dQ^2} \bigg|_{Q^2=0}
\]  
\tag{26}
\]
the pion radius calculated from Eq.(26) by using the computer VAX8700, which, \( (r_s^2) = 0.671 fm \), is in conformity with the experimental data of the pion radius [5], \( (r_s^2) = 0.663 \pm 0.023 fm \).
IV. Decay Constant for the Pion

In the absence of isospin breaking, the decay constant $f_0$ for the pion of momentum $Q$, can be obtained from PCAC (Partial Conservation of Axial-vector Current) which is expressed through the diagram in Fig.[9]

$$iQ_0f_0 = \langle 0| A_\mu^a(0) | \pi^a(Q) \rangle$$

(27)

where

$$A_\mu^a(x) = \xi(x)\gamma_\mu \gamma_5 q^a(x)$$

is the axial-vector current. A summation over color indices is understood.

It follows that

$$iQ_0f_0 = \left[ \frac{N_c}{2} \right]^{1/2} \int \frac{d^4 q}{(2\pi)^4} \left[ \gamma_\mu \gamma_5 \int \lambda(Q,q) \left( \frac{d^4 q}{(2\pi)^4} \right) \chi(Q,q) \right]$$

(28)

where, $N_c$ is the number of colors, $\chi(Q,q)$ is the bound state wave function of the pion, the normalized solution to the Bethe-Salpeter equation. Then, this equation can be written as

$$iQ_0f_0 = -i \left[ \frac{N_c}{2} \right]^{1/2} \int \left[ \gamma_\mu \gamma_5 \left( \frac{d^4 q}{(2\pi)^4} \right) \right]$$

(29)

In the lowest approximation, the equation is reduces as

$$f_0 = -i \left[ \frac{N_c}{2} \right]^{1/2} \int \left[ \gamma_\mu \gamma_5 \left( \frac{d^4 q}{(2\pi)^4} \right) \right]$$

(30)

Performing a rotation to Euclidean region and angular integration, we obtain the following form

$$f_0 = \left[ \frac{N_c}{2} \right]^{1/2} \int \left[ \gamma_\mu \gamma_5 \left( \frac{d^4 q}{(2\pi)^4} \right) \right]$$

(31)

The numerical calculation of Eq.(31) is completed by using the computer VAX8700. The result calculated, $f_{\text{num}} = 120 (\text{MeV})$, is compatible with the experimental value.

V. Numerical Results

A. Input Parameters

The propagator functions and mass function for the quark and the wave functions of bound state, mass, electromagnetic form factor, radius and decay constant for the pion discussed in this section were fitted or predicted with parameter values in Eq.(18) as follows: $x_0 = 10$, $\Lambda_{\text{QCD}} = 200 (\text{MeV})$, $\gamma = 280 (\text{MeV})$, $V(0)/N = 10$, and $N = S(\Lambda_{\text{QCD}})$. In addition, the number of flavor $f$ is chosen as five and $n'$ is predicted as nine in Eq.(18). Assuming $u$ and $d$-quark degeneracy, the cutoff-dependent chiral-symmetry-breaking mass were $\rho_0(L) = 3.25 (\text{MeV})$. The cutoff in both the Schwinger-Dyson and Bethe-Salpeter equation was $L = 600 (\Lambda_{\text{QCD}})$.

B. Quark Propagator and Mass Function

The coupled integral Eqs.(6) for the propagator functions $A(k^2)$ and $B(k^2)$ were solved numerical by simultaneous iterations on VAX8700 computer. The iterations converged rapidly to a unique solution and independently of the choice of the initial guess functions for $A$ and $B$. The propagator functions calculated for $A(k^2)$ and $B(k^2)$ are shown in Figs.[2-3]. And the mass function $m(k^2) = B(k^2)/A(k^2)$ is shown in Fig.[4]. From Fig.[2], we find that, at small $-k^2$, $A(k^2)$ differs from the value 1 appreciably, while it tent to 1 for large $-k^2$. In addition, for large $-k^2$, the asymptotic behavior of the mass function, $k^2 > 2000 (\text{MeV})$, is in very good numerical agreement with the behavior of the irregular solution found in renormalization-group analysis of QCD, which, to two loop, is given by

$$m(q^2) \rightarrow \frac{\rho_0(L)}{0.5m(q^2)} \left[ 1 - \frac{2\gamma_1 \beta_1 m(q^2)}{\beta_1^2} \right] \ln(q^2)$$

(32)
where $\gamma_1 = 2$, $\gamma_2 = (\frac{19}{14} - 5f/18)$, $\beta_1$, and $\beta_2$ as above given. From Eq.(32), we find the mass $m$ have the value $m = 6.56(MeV)$. Frequently quoted in the literatures is the value $\overline{m}(1GeV)$, which we find to be $\overline{m}(1GeV) = 6.32(MeV)$. The values $m$ and $\overline{m}(1GeV)$ agree with the range of values obtained from current-algebra and sum-rule considerations [6]. Neither $m$ nor $\overline{m}(1GeV)$ are close to the constituent-quark mass. A closer quantity appears to be $m(0) = 800(MeV)$.

C. Wave Functions, Electromagnetic Form Factor, Radius and Decay Constant

First, the Bethe-Salpeter equation (14) are solved numerically by iteration starting with the initial guess wave functions. The iteration converged rapidly to a unique solutions of wave function of bound state and independently of the choice of the initial guess wave functions. The accuracy required at any point was less than $10^{-10}$.

Figs.[5-6] show graphs for the wave function $\chi_0^{(1)}(\beta^2)$ and $\chi_0^{(2)}(\beta^2)$, respectively. Fig.[7] show graphs for $F1(\beta^2)$, $F2(\beta^2)$, $F3(\beta^2)$, and $F4(\beta^2)$. In addition, the mass of the pion is given which is $M = 133(MeV)$. After the solutions for the quark propagator functions to the Schwinger-Dyson equation and for the wave functions of bound state to the Bethe-Salpeter equation are obtained, the electromagnetic form factor of the pion can be obtained according to Eq.(25) and shown and compared with the experimental values [5] in Fig.[8]. The result for the electromagnetic form factor gives a good fit to the experimental values in a large $k^2$ region. Then the radius of the pion is obtained from Eq.(26), which, $\langle r^2_{\text{tot}} \rangle^{1/2} = 0.671(fm)$, is conformity with the experimental data of the pion radius [5], $\langle r^2_{\text{exp}} \rangle^{1/2} = 0.663 \pm 0.023(fm)$. Finally, the decay constant of the pion is calculated from Eq.(31), which, $f_\pi = 120(MeV)$, is compatible with the experimental value. All results are calculated numerically by using the computer VAX8700 and with same variable parameters as above given. All results obtained show that the phenomenological potential is good and the lowest order approximation works well.

V. Conclusion and Discussion

In the present work, first, the expressions for the Schwinger-Dyson and Bethe-Salpeter equations in Euclidean space are reduced. Second, the flat-bottom potential model 2 is combined with the same physical considerations for the infrared and ultraviolet asymptotic behavior of the effective quark-gluon coupling 1, and the modified flat-bottom potential is established. Third, the expression of the electromagnetic form factor for the pion in this framework is reduced. Fourth, the radius for the pion is given according to the electromagnetic form factor. Then, the expression of the decay constant is given. Finally, the propagator functions and mass function for the quark, and the wave functions of bound state, mass, electromagnetic form factor, radius and decay constant for the pion are obtained numerically in the framework of the Schwinger-Dyson and Bethe-Salpeter equations with the modified flat-bottom potential, in the Euclidean space, by using the computer VAX8700. All results calculated are in agreement with other theoretical results or experimental data. All the above mentioned show that the phenomenological potential is reasonable. In addition, the fact that all of the wave functions, mass, electromagnetic form factor, radius and decay constant for the pion are in conformity with those given in the flat-bottom potential model [2,3] shows that all of results obtained in the flat-bottom potential model are credible, too. Further tests of the modified flat-bottom potential and the approach described in this work, as well as of parameterization of the running coupling $\alpha_s(\bar{q}^2)$, will be done by extending it to other $q\bar{q}$ system, $qQ$ system, vector meson, and $qqq$ system or by applying the results to the calculation of quantities and processes requiring detailed knowledge of the quark propagators and Bethe-Salpeter wave functions.
Acknowledgments

One of the authors (Wan Shao-long) would like to thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, Italy.

Appendix:

A. The kernels functions, $K_A(p^2, k^2)$ and $K_B(p^2, k^2)$, appearing in the Schwinger-Dyson equation:

$$K_A(p^2, k^2) = \frac{3k^2}{16\pi^2} \int_0^\infty \left[ p\bar{k}\cos\theta + \frac{2(p\bar{k}\cos\theta - \bar{k})(p - \bar{p}\cos\theta)}{(p^2 + k^2 - 2pk\cos\theta)} \right] G(p, \bar{k}, \cos\theta) \sin^2\theta d\theta$$

$$K_B(p^2, k^2) = \frac{3k^2}{16\pi^2} \int_0^\infty G(p, \bar{k}, \cos\theta) \sin^2\theta d\theta$$

where

$$G(p, \bar{k}, \cos\theta) = \Lambda_{QCD}^2 G([p - \bar{k}]^2)$$

where $\theta$ is the angle between $p$ and $\bar{k}$.

B. The matrices, $H_{ij}$, appearing in the Bethe-Salpeter equation:

$$H_{11} = \bar{k}^2 + \frac{1}{4} M^2 + m_+ m_-$$

$$H_{12} = \frac{1}{2} M^2 [m_+ + m_- - \frac{1}{2} (m'_+ + m'_-) \bar{k}^2]$$

$$H_{13} = \frac{1}{8} M^2 k^2 [m_+ + m_- - 2 (m'_+ + m'_-)]$$

$$H_{14} = -\frac{3}{2} M^2 k^2$$

$$H_{17} = \frac{1}{2} (m_+ + m_-)$$

$$H_{22} = -\bar{k}^2 + \frac{1}{4} M^2 + m_+ m_-$$

$$H_{23} = -\frac{1}{8} M^2 k^2$$

$$H_{24} = -2k^2 [m_+ + m_- - \frac{1}{8} \bar{k}^2 (m'_+ + m'_-)]$$

$$H_{31} = m'_+ + m'_-$$
\[ H_{93} = -2 \]
\[ H_{39} = k^2 - \frac{1}{4} M^2 + m_+ m_- \]
\[ H_{34} = -2(m_+ + m_- - \frac{1}{2} M^2(m'_+ + m'_-) \]
\[ H_{41} = 2 \]
\[ H_{42} = 2(m_+ + m_-) \]
\[ H_{43} = \frac{1}{4} M^2 k^2 (m'_+ + m'_-) \]
\[ H_{44} = -k^2 - \frac{1}{4} M^2 + m_+ m_- \]

where
\[ m_k \equiv m \left[ q^2 + Q^2 + \frac{1}{4} M^2 \right] \]

C. The kernel functions, \( K_\nu \), appearing in the Bethe-Salpeter equation:

\[ K_{11} = \frac{3k^3}{4\pi^3} \frac{1}{A \omega A \omega} \int_0^\infty \left[ G(\bar{p}, \bar{k}, \cos \theta) \sin \theta \right] \delta \theta \]

\[ K_{22} = -\frac{k^3}{12\pi^3} \frac{1}{A \omega A \omega} \int_0^\infty \left[ G(\bar{p}, \bar{k}, \cos \theta) \sin \theta \right] \left[ 3 + \frac{2k^3 \sin^2 \theta}{p^2 + k^2 - 2pk \cos \theta} \right] \delta \theta \]

\[ K_{23} = \frac{k^3}{12\pi^3} \frac{1}{A \omega A \omega} \int_0^\infty \left[ G(\bar{p}, \bar{k}, \cos \theta) \sin \theta \right] \left[ \frac{4k^3 \sin^2 \theta}{p^2 + k^2 - 2pk \cos \theta} - 3 \right] \delta \theta \]

\[ K_{24} = -\frac{k^3}{12\pi^3} \frac{1}{A \omega A \omega} \int_0^\infty \left[ G(\bar{p}, \bar{k}, \cos \theta) \sin \theta \right] \left[ 9 - \frac{10p^2 + 2k^2 - 6pk \cos \theta}{p^2 + k^2 - 2pk \cos \theta \sin^2 \theta} \right] \delta \theta \]

\[ K_{44} = \frac{k^3}{\pi^3} \frac{1}{A \omega A \omega} \int_0^\infty \left[ \frac{2}{3} \left( p^2 + k^2 \right) \cos \theta - 2k^2 + \frac{2}{3} k^2 \sin^2 \theta \right] \]

\[ A_k = A \left[ q^2 + Q^2 + \frac{1}{4} M^2 \right] \]

References:


Figure Captions:

Fig.1 Feynman diagrams for the pion electromagnetic current element.

Fig.2 Function $A(p^2)$ shown as function of $p$ in which $p$ is in units of $\Lambda_{QCD}$.

Fig.3 Function $B(p^2)$ in units of $\Lambda_{QCD}$ shown as function of $\overline{p}$ in which $\overline{p}$ is in units of $\Lambda_{QCD}$.

Fig.4 Mass function $m(p)$ in units of $\Lambda_{QCD}$ shown as function of $p$ in which $p$ is in units of $\Lambda_{QCD}$.

Fig.5 Wave function $\chi_{0}^{(1)}(p_{2})$ shown as function of $\overline{p}$ in which $\overline{p}$ is in units of $\Lambda_{QCD}$.

Fig.6 Wave function $-\chi_{0}^{(2)}(p_{2})$ in units of $\Lambda_{QCD}$ shown as function of $\overline{p}$ in which $\overline{p}$ is in units of $\Lambda_{QCD}$.

Fig.7 Plot for $F_{1}$, $F_{2}$, $F_{3}$, and $F_{4}$ in which $F_{1} = \chi_{0}^{(1)}$, $F_{2} = \overline{Q}p\chi_{0}^{(1)}$, $F_{3} = \overline{Q}p^{2}\chi_{0}^{(3)}$, and $F_{4} = \overline{Q}p\chi_{0}^{(4)}$ in which $\overline{p}$ is in units of $\Lambda_{QCD}$.

Fig.8 The plot calculated for the electromagnetic form factor of the pion is compared with the experimental values [5].

Fig.9 The pion decay constant.