THE LAMB SHIFT
IN THE CHARGE-MAGNETIC MONOPOLE SYSTEM

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1 Introduction

It is well-known that the magnetic monopole hypothesis leads, already on the quantum-mechanical level, i.e. before second quantization, to a number of non-trivial consequences including, e.g., the appearance of half-integral angular momenta in the boson-boson bound system [1], violation of parity $P$ in the symmetric electrodynamics if magnetic charge is a scalar [2], and fermions from bosons [3], and other effects. These features have been demonstrated on the simple example of the model system of “dyonium”, i.e. the dyon ($v_1, j_1$) - dyon ($v_2, j_2$) system which can be solved exactly for relativistic and nonrelativistic case [4]. The special case of charge-dyon system has also been studied separately [5]. The investigation of the charge-dyon atom is a simpler problem because in this case the strong $g_1 g_2$ coupling disappears and we can suppose that an interaction with external fields in the first order of perturbation theory take place only with the charge (electron) $e$ but not with the dyon. The study of this system will make it possible to describe the various interaction processes of slow monopoles with a hydrogen-like atom which are of interest in connection with the experimental search for the magnetic charges (see, e.g., Ref. 6).

From the point of view of quantum electrodynamics the existence of the magnetic monopoles will also lead to effects having no analog in the standard theory [7]. However, there are some problems in the QED description of the $e - g$ interaction. One of them is the impossibility of using perturbation theory because the interaction constant between electric and magnetic charges is not small. The other problem is the absence of a consistent local Lagrangian or Hamiltonian formulation of the theory with two kinds of charges (unless the Dirac string is made into a dynamical variable; see, e.g., [8]). Therefore a possible approach consists in calculating the amplitudes of quantum electrodynamical processes in the monopole (dyon) background field using the exact solutions of the equations for the electrically charged particles in this background field. The examples of this kind of calculations of the radiative corrections in the field of a monopole or a dyon [9] are based on the Wichmann-Kroll method [10] of calculating vacuum polarization contribution.

On the other hand there is the new approach by Barut applied to the calculation of the QED effects. General formulation of this “self-field quantum electrodynamics” is also based on using the exact solutions of relativistic wave equations in external fields [11,12]. This formalism has been applied to spontaneous emission [13], vacuum polarization in a Coulomb field [14] and Lamb shift [15].

In the present note the radiative corrections to the spectrum of the “charge-dyon” atom are considered in the framework of the self-field QED as well as in the Pauli approximation.

2 Selffield Quantum Electrodynamics

The general energy shift $\Delta E_n$ of a level $n$ of a quantum system due to radiative self energy effects is given by the expression [11]:

$$\Delta E_n = \frac{e^2}{2} \int dx \bar{\psi}_n(x) \gamma_0 \gamma_0 \psi_n(x) P \int \frac{dk}{(2\pi)^3} \int \frac{dy}{(2\pi)^3} \frac{e^{ik(x-y)}}{k^2} \int \sum^\infty_{s} \bar{\psi}_s(y) \gamma^s \psi_s(y)$$

$$= \frac{e^2}{2} \int \sum^\infty_{s} dx dy \bar{\psi}_n(x) \gamma_0 \gamma_0 \psi_n(x) \int \frac{dk}{(2\pi)^3} \frac{e^{ik(x-y)}}{k^2} \bar{\psi}_s(y) \gamma^s \psi_s(y) \frac{\pi}{2k} (E_n - E_s - k)$$
Here $P$ stands for the principal value integral and $\int \sum$ implies a sum over the discrete part and an integration over the continuum part of the system's spectrum and $\psi_i$ is a Dirac spinor corresponding to a fixed level. The first term of Eq.(1) is the contribution of the vacuum polarization, the second that of spontaneous emission and the third corresponds to the Lamb shift. For evaluation of it let us consider the relativistic Dirac wave functions of the bound system charge-dyon (see e.g. [16]). The Hamiltonian is

$$H = \mathbf{\Pi} \cdot \mathbf{\Pi} + M \beta - \frac{Ze^2}{r}$$

where $\mathbf{\Pi} = \mathbf{p} - eA^\mu$, $A^\mu = g^\mu_{\alpha\beta}^\rho \phi^\rho$ is the Dirac potential, $M$ is the charge (electron)'s mass and $\alpha, \beta$ is the standard set of Dirac matrices. The conserved angular momentum operator corresponding to the above Hamiltonian is

$$\mathcal{J} = [\mathbf{\tau} \times \mathbf{\Pi}] - \frac{1}{2} \sigma_0 \mathbf{\tau} = \mathcal{J} + \frac{1}{2} \mathcal{K}$$

where $\mathcal{J} = [\mathbf{\tau} \times \mathbf{\Pi}] - \frac{1}{2} \sigma_0 \mathbf{\tau}$ is the non-relativistic angular momentum operator and $\mathcal{K} = e\gamma_1$. According to the analysis made by Kazama et al. [17], the bound states of a 1/2-spin particle in the dyon field have three types of solutions:

$$\phi_1 = \frac{1}{r} \left( F(r)\xi_{nm}^{(1)}(\theta, \phi) ; \right) ; \quad \phi_2 = \frac{1}{r} \left( F(r)\xi_{nm}^{(2)}(\theta, \phi) ; \right)$$

$$\phi_3 = \frac{1}{r} \left( F(r)\xi_{nm}^{(3)}(\theta, \phi) ; \right) ;$$

where $\xi_{nm}^{(1)}(\theta, \phi)$, $\xi_{nm}^{(2)}(\theta, \phi)$ and $\xi_{nm}^{(3)}(\theta, \phi)$ are two component spinors

$$\xi_{nm}^{(1)}(\theta, \phi) = A\varphi_1(\theta, \phi) + B\varphi_2(\theta, \phi), \quad \text{if } j \leq |\xi| + \frac{1}{2};$$

$$\xi_{nm}^{(2)}(\theta, \phi) = -B\varphi_1(\theta, \phi) + A\varphi_2(\theta, \phi), \quad \text{if } j \leq |\xi| + \frac{1}{2};$$

$$\xi_{nm}^{(3)}(\theta, \phi) = \varphi_1(\theta, \phi), \quad \text{if } j = |\xi| + \frac{1}{2};$$

and

$$\varphi_1(\theta, \phi) = \left( \frac{\sqrt{2\xi + 1}}{\sqrt{2j + 1}} Y_{j+1/2, \pm j+1/2} \right)(\theta, \phi);$$

$$\varphi_2(\theta, \phi) = \left( \frac{\sqrt{2\xi + 1}}{\sqrt{2j + 1}} Y_{j+1/2, \pm j+1/2} \right)(\theta, \phi).$$

The common eigenfunctions $Y_{nm}(\theta, \phi)$ of the operators $\mathcal{J}^2$ and $\mathcal{J}_3$

$$\mathcal{J}^2 Y_{nm}(\theta, \phi) = (j + 1)Y_{nm}(\theta, \phi),$$

$$\mathcal{J}_3 Y_{nm}(\theta, \phi) = n Y_{nm}(\theta, \phi)$$

are the generalized spherical harmonics [18]:

$$Y_{nm}(\theta, \phi) = N(1 - x) P^m_n(\cos \theta) e^{i(m+n+1)\phi},$$

where

$$N = 2^n \left( \frac{2j + 1}{4\pi} \right)^{1/2} \frac{(2j + 1)(j + m)!}{(j + m + 1)!};$$

$$x = \cos \theta,$$

and $P^m_n(\cos \theta)$ are the standard Jacobi polynomials:

$$P^m_n(\cos \theta) = \left( \frac{1 + x}{2} \right)^{m+n} \left( \frac{1 - x}{2} \right)^m \frac{d^{m+n}}{dx^{m+n}} \left( (1 - x)^{m+n} (1 + x)^{-m-n} \right).$$

The coefficients $A$ and $B$ in Eq.(5) are defined as:

$$A = B = \sqrt{2j + 2q + 1} \frac{1}{\sqrt{2j + 1}}.$$

It is worth to note that the radial equation corresponding to the third type of solutions does not lead to a mathematically well defined problem [17]. For this reason we shall not discuss this case in this section. For the types 1 and 2, after setting the corresponding wave functions (4) into (2), we get the radial equations [16]:

$$\frac{d^2 F}{dr^2} + \left( \frac{2\xi + 1}{2j + 1} - \frac{j^2}{r^2} \right) F = \left( -M + E + \frac{q^2}{2j + 1} \right) F;$$

$$\frac{d^2 G}{dr^2} + \left( \frac{2\xi + 1}{2j + 1} - \frac{j^2}{r^2} \right) G = \left( -M + E - \frac{q^2}{2j + 1} \right) G;$$

where $\ell = \sqrt{(j + 1/2)^2 - q^2}$, i.e. the radial equations for type 2 coincide with the corresponding equations for the first type by replacing $\ell$ with $-\ell$. The equations (9) are precisely those radial equations describing a Dirac particle in the Coulomb field except for the replacement $\ell \to \epsilon = \pm(j + 1/2)$ [19]. So, the bound states radial functions are [16]:

$$F(r) = C_{N\ell}(r) \sqrt{1 + \frac{E}{M}} \left( -R_{\ell}(r) + R_\ell(r) \right);$$

$$G(r) = C_{N\ell}(r) \sqrt{1 - \frac{E}{M}} \left( R_{\ell}(r) + R_\ell(r) \right)$$

where

$$C_{N\ell}(r) = M \frac{1}{2\pi} \frac{1}{\sqrt{2\ell + 1}} \frac{(2\ell + 1)!}{(2\ell + 2)!} e^{-ir};$$

and

$$F(r) = n_1 F_1(1 - n, 2\gamma + 1, 2kr), \quad G(r) = (N + \ell) F_1(n, 2\gamma + 1, 2kr).$$
with
\[ k = \sqrt{M^2 - E^2} = \frac{eQM}{N}, \quad \gamma = \sqrt{\left(j + \frac{1}{2}\right)^2 - e^2Q^2 - q^2} = \sqrt{\gamma^2 - e^2Q^2}, \]
\[ N = \sqrt{e^2Q^2 + (n + \gamma)^2}, \quad n = 0, 1, 2 \ldots \]
is the radial quantum number and \( F_1 \) is the confluent hypergeometric function. Thus we obtain the energy spectrum [4]:
\[ E_k = M \sqrt{1 - \frac{e^2Q^2}{2(n + j + \frac{1}{2})^2} - \frac{e^2Q^2}{2(n + j + \frac{1}{2})(j + \frac{1}{2})}} = E_2 - \delta_2, \quad (12) \]
where
\[ \delta_2 = \frac{e^2Q^2}{2(n + j + \frac{1}{2})(j + \frac{1}{2})} \]
is the correction to the hydrogen-like atom spectrum due to the presence of a magnetic charge.

We see that the most essential difference between charge-dyon atom and dyonium is in the energy spectrum. Indeed, in the dyonium case we have just the same expression (11) but for the replacement \( eQ \rightarrow e_3e_2 + g_3g_2 \). However, since \( g_3g_2 \gg 1 \), practically for any values of the quantum numbers the expression under the square root will be negative. The self-adjointness of the Hamiltonian breaks down and a new method is necessary. The detailed solution of this problem has been given in [4] and [20].

Using the solutions (4)-(10) along the lines of Ref [15] we introduce the energy dependent Green function \( G(x,y;i) \) of the relativistic charge-dyon system as
\[ G(x,y;i) = \sum_{\gamma} \psi_\gamma(x) \psi_\gamma^*(y;i - E_k) \]
The sum over all intermediate states can be written as a contour integral
\[ \int \sum_{\gamma} \psi_\gamma(x) \psi_\gamma^*(y,i) = \int_{C_{2\pi i}} \frac{dz}{2\pi i} G(x,y;z) \]
and the expression for the Lamb shift of the kth level is (with \( \alpha = \frac{e_3e_2}{4\pi} \))
\[ \Delta E_k = -4\alpha \int_{C_{2\pi i}} \frac{dz}{2\pi i} \frac{2}{(z - E_k)^2 - k^2 + i\epsilon} b_j(kz) b_j(ky), \quad (14) \]
Here the standard expansion into spherical harmonics
\[ \int dk e^{i(kx-y)} = (4\pi)^2 \sum_{\gamma} Y_{\gamma}(\hat{x}) Y_{\gamma}(\hat{y}) \]
was used. Then after integration over \( k \) we have
\[ \Delta E_k = -4\alpha \int_{C_{2\pi i}} \frac{dz}{2\pi i} \frac{dxdy \psi_\gamma(x) \psi_\gamma^*(y) \sum_{i,m} Y_{\gamma}(\hat{x}) Y_{\gamma}(\hat{y})}{(z - E_k)^2 - k^2 + i\epsilon} b_j(kz) b_j(ky), \]
where
\[ r_1 \text{ is the smaller one of } (|x|, |y|) \text{ and } r_2 \text{ is the larger one of } (|x|, |y|) \]
and
\[ \omega^2 = (z - E_k)^2 + i\epsilon \text{ or } \omega = \sqrt{(z - E_k)^2 + i\epsilon}. \]
Let us note that the only difference from the hydrogen atom case consists in the shift of the Green function poles corresponding to the bound states energy on the value \( \delta_k(12) \). As it was shown in [15], we can deform the contour \( C \) in the \( z \) plane so that the expression (15) has two contributions which correspond to the so-called high-energy and low-energy parts of the Lamb shift:
\[ \Delta E_k^{(s-1)} = \alpha \int_{\infty}^{\infty} \frac{dz}{2\pi i} \frac{dxdy \psi_\gamma(x) \gamma G(x,y,z) \psi_\gamma^*(y) e^{-i(kz-ky)}}{|x - y|} \]
\[ \Delta E_k^{(l-1)} = \alpha \int_{0}^{\infty} \frac{dz}{\pi} dxdy \psi_\gamma(x) \gamma G(x,y,z) \psi_\gamma^*(y) \sin(E_k - z) \frac{(x - y)}{|x - y|} \]
that agree exactly with the expansions of Mohr [21].

It is worth to note that the expression (15) is very convenient for the numerical evaluation of the radiative corrections to the spectrum. The examples of this kind of calculations have been done in [22,23]. On the other hand it was shown [21] that the high-energy part of the Lamb shift (16) is exactly what one would obtain using standard perturbation theory to calculate the energy shift due to the interaction of the electron with the external electromagnetic field (see e.g. [24]).

3 Effective Potential Approach
Using the expression for the effective potential energy of the electron in the external electromagnetic field to first order in \( \alpha \) [25]; we have
\[ \delta U_e = -\alpha \frac{e_3e_2}{2\pi M\lambda^2} \left( \frac{M}{3} - \frac{3}{8} + \frac{1}{5} \right) \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda} + \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda} F_{\Phi}^{\omega}, \]
After calculating the diagonal matrix elements between unperturbed wave functions of the Dirac particle in the dyon field \( \Psi_\gamma \) (here \( k \) is the total set of quantum numbers) we obtain the quantum correction to the energy of the k-th energy level stipulated by the interaction with the short-wave photons:
\[ \delta E_k^{(s-1)} = \delta E_k^{(1)} + \delta E_k^{(2)}, \]
where
\[ \delta E_k^{(1)} = \frac{\alpha e}{4\pi M} \left( \frac{M}{3} - \frac{3}{8} + \frac{1}{5} \right) \left( \frac{\partial}{\partial \rho} - g(\Phi - gD^\Phi) \right) \Psi_\gamma \]
is the correction to the potential energy of the electron due to its interaction with zero energy fluctuations of the electromagnetic field (the vacuum polarization term \( z = \frac{1}{2} \) is discussed in [14]), and
\[ \delta E_k^{(1)} = \frac{4\alpha e}{3\pi M^2} \left( \ln \frac{M}{\lambda} - \frac{3}{8} - \frac{1}{5} \right) |p_k| |p_k| \]  

(19)

is the correction connected with the interaction between anomalous momentum and external electric \((E_e = e_0 \beta)\) and magnetic \((H = \gamma_5 H)\) fields of dyon [25, 26]. Here \(\Phi = -e_0 \beta\) is the scalar potential of the electric field of dyon and \(\lambda\) is the cut-off "mass" of the photon.

Unfortunately there is a serious problem if we use Eq. (18) directly for the calculation of the radiative corrections caused by the monopole field. This difficulty is connected with the indefiniteness of the expression \(Q/t\) in this case. As is known a consistent potential description of the Abelian monopole field had been achieved for instance in the Wu-Yang approach [27]. In this formalism one may obtain \(\Delta A^D = \Delta A^D = 0\) everywhere in the \(R^3 = R^3 - \{0\}\). However the definition of this expression at the point \(\{0\}\) cannot be made in the usual way because the problem of the distribution lifting in the fiber bundle is still unsolved [28]. Keeping it in mind we will assume ad hoc that the magnetic charge is not generated by electric currents i.e. \(\Delta A^D = 0\).

Next let us use the first relativistic approximation i.e. write the wave function as an expansion in the "big" and "small" components:

\[ \Psi_k = \begin{pmatrix} \psi_k \phi_k \end{pmatrix}; \quad \chi = \frac{1}{2M} \langle \phi, \bar{\sigma} \Phi \phi \rangle - \frac{e}{2M} \langle \bar{\sigma} \Phi \phi \rangle \]

(20)

where \(\phi\) is the Pauli spinors:

\[ \phi_1(\chi) = \langle \eta^1 | \chi \rangle \]  

(21)

corresponding to above mentioned types of solutions. The corresponding radial wavefunctions are [3]:

\[ R_{N_1}^{(1)}(r) = C_1 r^{-l} e^{-Br} F_l(-N, 2l, 2l); \quad R_{N_2}^{(2)}(r) = C_2 r^{-l} e^{-Br} F_l(-N + 2, 2l, 2l); \]

(22)

where

\[ C_1 = \frac{2/(N + 1)}{(2l)! \sqrt{N(N + 1)}}, \quad C_2 = \frac{2/(N + 1)}{(2l + 1)! \sqrt{N(N + 1)}} \]

and \(k = \sqrt{2M E}, \quad l = \frac{1}{2} + \sqrt{j(j + 1) - \frac{q^2}{r^2}}\)

Note, that the states of types 1 and 2 have degenerate energy spectra:

\[ E_{1,2} = \frac{M(\epsilon Q)^2}{2(N + 1)^2}; \quad E_3 = \frac{M(\epsilon Q)^2}{2(N + 1)^2} \]

(23)

Then, we have for the correction \(\delta E_k^{(1)}\) up to higher relativistic corrections:

\[ \delta E_k^{(1)} = \frac{4\alpha e}{3\pi M^2} \left( \ln \frac{M}{\lambda} - \frac{3}{8} - \frac{1}{5} \right) |p_k| |p_k| \]

(24)

Using Eq. (20) and the well-known properties of the Pauli matrices the second term can be put into the form:

\[ \delta E_k^{(2)} = \frac{e^2}{15M^2} |p_k| \Delta \Phi |p_k| + \delta E_k^{(1)} \]

(25)

where

\[ \delta E_k^{(2)} = \frac{e^2}{4\pi M^2} |p_k| \Delta \Phi |p_k| \]

and \(J = [x \times \Phi] - q \Phi\) is the non-relativistic angular momentum operator whose action on the wave functions \(\phi_k\) is defined by Eq. (6). The low-frequency component of the total shift of a level \(E_k\) in the "dyonium" spectrum can be computed as usual from the second order perturbation theory:

\[ \delta E_k^{(1)} = \sum_{n} V_{nk} V_{kn} (E_n - E_k) - \omega \]

where \(V_{nk}\) is the matrix element \(\langle \phi_n | V | \phi_k \rangle\) of the electron - photon interaction and the summation must be performed over all quantum numbers of the electron and photons including polarizations. As it was shown in [15] this expression can be obtained from general selffield formula [11] in the dipole approximation, i.e. the operator \(V\) in the non-relativistic case is proportional to the dipole momentum operator and after integration over frequencies and addition of the mass renormalization term [25] we have:

\[ \delta E_k^{(1)} = \frac{2\epsilon^2}{3M} \sum_{\nu} |\langle \phi_n | \nu | \phi_k \rangle|^2 (E_n - E_k) \ln \frac{1}{2 |E_n - E_k|} \]

(26)

Let us note that

\[ \sum_{\nu} |\langle \phi_n | \nu | \phi_k \rangle|^2 (E_n - E_k) = - \frac{\epsilon^2}{M^2} \sum_{\nu} |\langle \phi_n | \nu | \phi_k \rangle|^2 (E_n - E_k) = \]

\[ = \frac{\epsilon^2}{M^2} \sum_{\nu} |\langle \phi_n | \nu | \phi_k \rangle|^2 (E_n - E_k) = - \frac{\epsilon^2}{M^2} \delta_{nk} \]

(27)

where \(H\) is the Pauli Hamiltonian

\[ H = \frac{1}{2M} (|x|)^2 - \frac{Zq^2}{r} \]

Using the property of the Dirac potential \(\nabla \Delta^D = \gamma \Delta^D = 0\) one can write

\[ |H, \|, \|, \| | \phi_k \]

(28)

Thus the low-frequency component of the radiative shift of the \(k\)-th "dyonium" level is:

\[ \delta E_k^{(2)} = \frac{2\epsilon^2}{3M} \sum_{\nu} |\langle \phi_n | \nu | \phi_k \rangle|^2 (E_n - E_k) \ln \frac{1}{2 |E_n - E_k|} + \]

\[ + \frac{4\alpha e}{3\pi M^2} |p_k| \Delta \Phi |p_k| \ln \frac{\lambda}{M} \]

(28)
Adding the terms (24), (25) and (28) we obtain the expression for the radiative correction to the $k$-th energy level:

$$
\delta E_k = \frac{2e^2}{3\pi} \sum_{\ell} \left| \langle \phi_\ell | \mathcal{E} | \phi_k \rangle \right|^2 \left( E_k - E_\ell \right)^2 \ln \frac{M}{2 | E_k - E_\ell |} + \frac{e^4}{3\pi M^2} \frac{19}{30} \left( \phi_k | \Delta \mathcal{E} | \phi_k \right) + \frac{e^4}{4\pi M^2} \left( \phi_k | (\mathcal{D} L)^{-3} | \phi_k \right).
$$

This is the generalization of the well known formulae for the Lamb shift of the hydrogen atom [19, 25].

Let us consider the corrections to the states $\phi_{1,2}$ and $\phi_3$ separately. Owing to field equations $\Delta \Phi = 4\pi Z e \delta(x)$ it is clear from the behavior of the radial function (22) as $r \to 0$ that only the wave functions of the third type are not vanishing at $r = 0$. So the second term in (29) equals to zero for the wavefunctions $\phi_{1,2}$.

Next it follows from the definition of the total angular momentum operator $\mathcal{J}$ (3) that

$$
(\mathcal{J} \sigma) = \mathcal{J}^2 - \frac{\mathcal{J}^2}{2} - \frac{3}{4}
$$

and so

$$
(\mathcal{J} \sigma) \phi = \left( \kappa(j + 1) - j(j + 1) - \frac{3}{4} \right) \phi,
$$

where $\kappa = j \pm 1/2$ is the eigenvalue of the operator $\mathcal{J}$ i.e.

$$
(\mathcal{J} \sigma) \phi = \lambda \phi, \text{ where } \lambda = \begin{cases} j & \text{if } \kappa = j + \frac{1}{2}; \\ -(j + 1) & \text{if } \kappa = j - \frac{1}{2}. \end{cases}
$$

The formal difference of this formula from the analogous expressions for the hydrogen atom is the half-integral values of the quantum numbers $j$ in Eqs.(30), (31) instead of integral ones [19, 25].

Using the expression (30) one can write

$$
\delta E_k^{(1)} = \frac{e^2}{4\pi M^2} \left( \phi_k | (\mathcal{D} L)^{-3} | \phi_\ell \right) =
\frac{e^2}{4\pi M^2} \left( \kappa(j + 1) - j(j + 1) - \frac{3}{4} \right) \int_0^\infty dr r^{-1} \left( R^{(2)}(r) \right)^2
$$

where $R^{(2)}(r)$ is the radial functions (22). Consequently calculating the integral in Eq. (32) by means of the well known formulae (see, e.g. [28]) we have for the corrections to the first and second type of wave functions:

$$
\delta E_k^{(1)} = \frac{2e^2}{4\pi M^2} \left( \frac{\left( \mathcal{J} \sigma \right)^2}{(N + 1)^3} \right) \left( \kappa(j + 1) - j(j + 1) - \frac{3}{4} \right),
$$

$$
\delta E_k^{(2)} = \frac{2e^2}{4\pi M^2} \left( \frac{\left( \mathcal{J} \sigma \right)^2}{(N + 1)^3} \right) \left( \kappa(j + 1) - j(j + 1) - \frac{3}{4} \right).
$$

Thus the degeneracy between first and second type of wave function describing the "dyonium" atom bound states is now lifted. Introducing the value

$$
L_{nj} = \frac{(N + 1)^3}{2M(\mathcal{Z} e)^2} \sum_{\ell} \left( \frac{\left( \mathcal{J} \sigma \right)^2}{(N + 1)^3} \right) \left( \frac{\left( \mathcal{J} \sigma \right)^2}{(N + 1)^3} \right) \ln \frac{M(\mathcal{Z} e)^2}{2 | E_\nu - E_k |}
$$

which can be found numerically (here the summing must be performed over all intermediate quantum numbers including continuum) one can write the expression for the radiative shift of the first and second type levels

$$
\delta E_n^{(1)} = \frac{4M \mathcal{Z} e^2}{\pi(N + 1)^3} \left( L_{nj} + \frac{3}{8} \frac{\kappa(j + 1) - j(j + 1) - \frac{3}{4}}{(j + 1)^2 + \frac{1}{4}} \right),
$$

$$
\delta E_n^{(2)} = \frac{4M \mathcal{Z} e^2}{\pi(N + 1)^3} \left( L_{nj} + \frac{3}{8} \frac{\kappa(j + 1) - j(j + 1) - \frac{3}{4}}{(j + 1)^2 + \frac{1}{4}} \right)
$$

In order to obtain the corrections to the $S$ wave states with $j$ equal to zero we take into account that for them $\delta E_n^{(0)} = 0$ and

$$
\left| \phi_{\ell}(0) \right|^2 = \left( \frac{\mathcal{Z} e^2}{\pi(N + 1)^3} \right).
$$

It is easy to see that in this case ($S=0$)

$$
\delta E_{n=0} = \frac{4M \mathcal{Z} e^2}{3\pi(N + 1)^3} \left( \ln(\mathcal{Z} e)^2 + \frac{19}{30} + L_{n=0} \right)
$$

which coincides exactly with the standard expression for the $S$ wave Lamb shift in the hydrogen atom case [20, 25].

In conclusion, we have shown that the methods of calculating radiative corrections to the bound electron in Coulomb field can easily be generalized to the electron bound to a dyon. But the dyon-dyon system is much more complicated and remains to be investigated.

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