A GENERAL SOLUTION
OF THE BV-MASTER EQUATION
AND BRST FIELD THEORIES

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1 Introduction

The Batalin-Vilkovisky (BV) method [1] which is a systematic way of quantizing reducible gauge theories is based on finding out the proper solution of the (BV-) master equation. Unfortunately, obtaining this solution is usually a tedious task. In ref. [2] (which will be denoted as 1) we presented a procedure of discovering the desired solution straightforwardly for some first order gauge theories, by utilizing the general grading concept which was revealed in the study of topological quantum field theories [3]-[7] (for a review see ref. [8]). In 1 it was only shown that the method is suitable for a very restricted class of actions.

Here it is demonstrated that, for an action whose kinetic term is bilinear in fields and linear in derivatives, to know if the gauge generators are related to the action in a certain way is sufficient to conclude that the method of 1 is applicable. Thus, one avoids to check explicitly (i.e. by using the original fields, ghosts, etc.), if the action found in terms of the generalized fields satisfies the master equation. By inspection of the action and the generators which are the main ingredients of the gauge theories, one can easily determine if the generalized field method of 1 is relevant to find out the proper solution of the master equation. In Section 3 as an example we study the application of the general method to the generalization of the Chern-Simons theory to any odd dimension introduced in ref. [9].

In Section 4 it is shown that, by replacing the role of the exterior derivative with the first quantization BRST charge, the general formalism is suitable to quantize BRST field theories. This permits to obtain readily the proper solution of the master equation for the string field theories.

2 General Formalism

The original and the ghost fields of a gauge theory can be treated on the same footing, by generalizing the exterior derivative, $d$, as

$$d \rightarrow d + \delta_B,$$

where $\delta_B$ denotes the BRST transformation [3]. Extension of the ordinary grading to include also the ghost number is revealed to be useful in writing the solution of the master equation in a compact form for the topological quantum field theories [3]-[7].

In order to utilize the generalized grading to obtain the solution of the master equation we act according to the following procedure.

If the original gauge theory is not already first order in $d$, and the terms containing $d$ are not bilinear in fields, one should find an equivalent formulation of it possessing these properties. Thus we deal with the Lagrangian (action)

$$L(A, B) = BdA + V(A, B),$$

where

$$d = d + 6_B,$$

(1)

(2)
which is supposed to be invariant under the gauge transformations

$$\delta^{(0)}(A, B) = R^{(0)}(A, B)\lambda.$$  \hspace{1cm} (3)

The minimal ghost content of the theory can be figured out by analysing the reducibility of the gauge transformations (3). Then, generalize the original fields to include also the ghosts and antifields which possess the same grading with the original ones in terms of $d$, and substitute the original fields $A, B, \tilde{A}, \tilde{B}$, in the Lagrangian, (2). The resulting action

$$S = L(\tilde{A}, \tilde{B}) = \tilde{B} d\tilde{A} + V(\tilde{A}, \tilde{B}),$$  \hspace{1cm} (4)

is invariant under the transformations generated by

$$\tilde{R}(\tilde{A}, \tilde{B}) = R^{(0)}(\tilde{A}, \tilde{B}).$$  \hspace{1cm} (5)

Of course, one should also generalize the gauge parameter, $\lambda$, as $\tilde{\lambda}$ whose generalized grading is the same with the grading of $\lambda$. To discover the conditions which $\tilde{R}$ should fulfill to guarantee that $S$ satisfies the master equation, let us first examine the invariances of $S$ which would yield that it is a solution of the master equation.

We choose the signs of the field and antifield contents of $\tilde{A}$ and $\tilde{B}$ as

$$\tilde{A}_k = (\phi_k, \phi_k^*), \quad \tilde{B}_k = (-\phi_k^*, \phi_k),$$

so that under the transformations

$$\delta_B \tilde{A}_k = -\frac{\partial S}{\partial B_k} \lambda^A_k,$$

$$\delta_B \tilde{B}_k = -\frac{\partial S}{\partial A_k} \lambda^B_k,$$

variation of $S$ will be

$$\tilde{\epsilon}_B S = (S, S) \equiv 2 \frac{\partial S}{\partial B_k} \frac{\partial S}{\partial A_k}.$$  \hspace{1cm} (6)

If $S$ satisfies the master equation

$$(S, S) = 0,$$  \hspace{1cm} (7)

(6) will define the BRST transformations in accordance with the BV formalism.

By taking the derivatives of (6), one can define the transformations

$$\tilde{\epsilon}_A \tilde{A}_k = -\frac{\partial \delta S}{\partial B_k} \lambda^A_k - \frac{\partial \delta S}{\partial A_k} \lambda^B_k,$$$$

\tilde{\epsilon}_B \tilde{B}_k = \frac{\partial \delta S}{\partial \tilde{A}_k} \lambda^A_k + \frac{\partial \delta S}{\partial \tilde{B}_k} \lambda^B_k,$$  \hspace{1cm} (8)

where $\lambda_{1,2}$ are some parameters. Variation of $S$ under (8) can be shown to yield

$$\tilde{\epsilon} S = \frac{\partial (S, S)}{\partial \tilde{A}_k} \lambda^A_k + \frac{\partial (S, S)}{\partial \tilde{B}_k} \lambda^B_k.$$  \hspace{1cm} (9)

Thus if $S$ is invariant under the transformations (8) with $\lambda_1 \neq 0, \lambda_2 \neq 0$, one can conclude that it satisfies the master equation, (7).

Fortunately, in some circumstances to show that $S$ satisfies the master equation, it is sufficient to know whether $S$ is invariant under (8) even if one of the parameters $\lambda_{1,2}$ is vanishing: suppose that $S$ is invariant under the transformations (8) with the parameters $\lambda_1 \neq 0, \lambda_2 = 0$. Thus, we may only observe that $(S, S)$ is independent of $A$. $(S, S)$ possesses $(0, 1)$ grading (the first number indicates the usual grading and the other one denotes the ghost number). If the gradings of the components of $\tilde{B}$ are such that it is impossible to construct a function possessing $(0, 1)$ grading only in terms of $\tilde{B}$, we can conclude that $(S, S)$ vanishes. The other case, $\lambda_1 = 0, \lambda_2 \neq 0$, can be examined similarly.

Therefore, $S$ which is obtained by substituting the original fields with the generalized ones in the original action so that invariant under the transformations generated by (5), is a solution of the master equation, (7), if $\tilde{R}$ generates the transformations (8) where both of the parameters are non-vanishing or one of them vanishes but $(S, S)$ cannot depend on the field related to the vanishing parameter due to its grading.

By construction $S(\tilde{A}, \tilde{B})$ possesses the correct classical limit. Moreover, $\tilde{A}$ and $\tilde{B}$ include all the fields of the minimal sector and because of the form of $S$, (4),

$$\text{rank} \left[ \frac{\partial S}{\partial (\tilde{A}, \tilde{B})} \right] = N,$$

where $N$ is the number of the components of $\tilde{A}$ or $\tilde{B}$. Hence, we conclude that under the above mentioned conditions $S(\tilde{A}, \tilde{B})$ is the proper solution of the master equation.

3 Chern-Simons theory in $d = 2n + 1$

The theories studied before in refs. [2]-[7], can be shown to satisfy the above conditions. Nevertheless to illustrate the method we would like to study the Chern-Simons theory in any $d = 2n + 1$, which is given with the action (we suppress Tr)

$$L_d = \frac{1}{2} \int_{M_d} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$  \hspace{1cm} (10)

When we define

$$A = \phi + \psi \equiv \sum_{i=1}^{v_1} \phi_{2i+1} + \sum_{i=0}^{v_2} \psi_{2i},$$

where $\phi$ and $\psi$ are Lie-algebra valued, respectively, bosonic $2i + 1$-form and fermionic $2i$-form. It reads

$$L_d = \frac{1}{2} \int_{M_d} \left( \phi \wedge d\phi + \frac{2}{3} \phi \wedge \phi \wedge \phi + D\psi \psi \right),$$  \hspace{1cm} (11)
where $D_k = d + \{ \phi, \}$. (11) is followed from the fact that in the integral only the terms possessing odd grading survive. In (11) one recognizes the theory introduced in ref. [9] (see [10] for a supersymmetric formalism of the BF theory which possesses some common features).

The action (11) is invariant under

$$\delta A = d \Sigma + [A, \Sigma],$$

where

$$\Sigma = \Lambda + \Xi = \sum_{m=0}^{n-1} \Lambda_m + \sum_{m=1}^{n-1} \Xi_{m+1}. \quad (13)$$

$\Lambda$ and $\Xi$ are bosonic and fermionic, respectively. (12) are reducible on mass shell. Indeed, when the equations of motion

$$F^\alpha = 0, \quad D^\alpha \psi = 0,$$

are satisfied, one can see that

$$J_m J_{m+1} = 0, \quad m = 0, \cdots, 2n - 2,$$

where $J_m$ is the gauge generator and

$$J_{2m} = \left( \begin{array}{cc} D_\phi & \psi \\ \psi & D_\phi \end{array} \right), \quad J_{2m+1} = \left( \begin{array}{cc} D_\phi & -\psi \\ -\psi & D_\phi \end{array} \right).$$

By examining the reducibility properties one introduces ghosts and antighosts of ghost fields, the related antifields and obtains

$$\delta \phi = \sum_{i=0}^{n-1} \left[ \phi_{2i+1,0} + \sum_{j=1}^{2i+1} \eta_{2i+1,2i+1-j} + \sum_{j=0}^{2i+1} \eta_{2i+1,2i+1+j} \right],$$

$$\delta \psi = \sum_{i=0}^{n-1} \psi_{2i,0} + \sum_{j=0}^{2i+1} \xi_{2i,2i-j} + \sum_{j=0}^{2i+1} \xi_{2i,2i+j} + \sum_{j=0}^{2i+1} \xi_{2i,2i+1+j}.$$  

(14)

The antifield of the field $\phi_{2i,0}$ is defined as $\bar{\phi}_{2i+1, -2i, 0}$. Observe that $\delta \phi$ and $\delta \psi$ are, respectively, collection of $2i + 1$-forms and $2i$-forms. Now, in terms of $A = \phi + \psi$, we can write

$$S_A = -\frac{1}{2} \int_{M_d} \left( \hat{\delta} A \tilde{A} + \frac{2}{3} \hat{\delta}^2 \right).$$

(15)

$S_A$ is a proper solution of the master equation, because it is invariant under the transformations generated by

$$\hat{R} = d + [\hat{A},] = \frac{\partial S_A}{\partial \tilde{A}},$$

following as the generalization of (12). Because of the sign assignments in (14) the transformations (8) are given as

$$\delta A = \omega_{AB} \frac{\partial \delta S}{\partial A_B} - S_B; \quad \omega_{AB} = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right),$$

where the generalized gauge parameter

$$\hat{\Sigma} = \Lambda + \Xi,$$

$$\hat{\Lambda} = \sum_{i=0}^{n-1} \Lambda_{2i,0} + \sum_{i=1}^{n-1} \Lambda_{2i-1,2i} + \sum_{i=0}^{n-1} \Lambda_{2i,2i-1} + \sum_{i=0}^{n-1} \frac{1}{4} \sum_{j=m-2}^{2i+1} \frac{1}{2} \xi_{2i+1-j}.$$  

This example is somewhat different from the general case, because the general gradings of the components of $A$ are not the same. But the integral selects only the terms with the correct grading. One could write the solution of the master equation by using the generalized forms each of which possessing only one grading, and then gather them to obtain (15). In terms of $\delta$ and $\psi$ components (15) is given as

$$S_A = \int_{M_d} \left( \hat{\delta} \phi + \frac{1}{3} \hat{\delta}^2 + \psi \hat{d} + \hat{\delta} \tilde{\psi} \right).$$

(16)

by observing: i) $f_{M_d} \hat{\delta} \phi = 0$, integral is defined to possess zero ghost number so that one of the two factors of the integrand should be a field and the other one an antifield, which belong to the same reducibility level, but then is an even form, ii) $f_{M_d} \hat{\delta} \tilde{\psi} = 0$, integral will be a $M$-form, where $M = 2i + 1 - j + 2m + 1 - k + 2p + 1 - q$, but also due to condition on the ghost number we have $j + k + q = 0$, therefore $M = \text{even}$, iii) $f_{M_d} \hat{\delta} \phi = 0$, integrand will be a $K$-form, where $K = 2i - j + 2m - k + 2p - q$, and again due to fact that its ghost number should vanish, we have $j + k + q = 0$, therefore $K = \text{even}$. Hence their integrals on $d = 2n + 1$-manifold vanish.

4 BRST Field Theory

When we deal with a constrained hamiltonian system (gauge theory) its first quantization will yield the BRST charge $Q$, which is fermionic and nilpotent, $Q^2 = 0$. To define this charge one should introduce first quantization ghosts, and associate them the "algebraic ghost number", $N$. By construction $Q$ possesses $N_4(Q) = 1$. The BRST charge, $Q$, acts on the fields which are
valued in the Hilbert space of the first quantized theory, and the physical states are defined as

\[ Q\chi = 0, \quad \chi \neq Q(\ldots). \]

The fields can be written as functionals or in terms of the normal modes. In general we can attribute algebraic ghost numbers to the fields and define an inner product to write a free action whose equations of motion are the physical state conditions\[^{[12]}\]. Inner product can be allowed to carry an algebraic ghost number or not. We deal with an inner product which does not carry algebraic ghost number. Then the free action is

\[ L_0(\chi, \bar{\chi}) = \bar{\chi}Q\chi, \tag{17} \]

where (\(\epsilon\) denotes the Grassmann parity)

\[ N_0(\chi) = -1, \quad N_0(\bar{\chi}) = 0, \]

\[ \epsilon(\chi) = 1, \quad \epsilon(\bar{\chi}) = 0. \]

Because of the nilpotency of \(Q\), (17) is invariant under the gauge transformations

\[ \delta \chi = Q\Lambda, \quad \delta \bar{\chi} = \bar{\Lambda}Q, \tag{18} \]

where the gauge parameters possess

\[ N_0(\Lambda) = -1, \quad N_0(\bar{\Lambda}) = -2. \]

This algebraic ghost number assignment follows from the fact that we would like to interpret the first quantization BRST charge, \(Q\), as the exterior derivative, though in this case there are "negative forms" (\(\equiv\) negative \(N_0\) states). For the sake of generality, we suppose that any integer algebraic ghost number is available. Obviously, the gauge invariance (18) is infinitely reducible:

\[ \delta_n = Q\Lambda_{n+1}, \quad \delta_{\bar{n}} = \bar{\Lambda}_{n+1}Q, \]

\[ N_n(\Lambda_n) = -n, \quad N_n(\bar{\Lambda}_n) = -(n + 1), \]

\[ \epsilon(\Lambda_n) = n, \quad \epsilon(\bar{\Lambda}_n) = n + 1. \tag{19} \]

To perform the BV quantization we need to introduce the ghost and the ghost of ghost fields, \(n, \bar{n}\), which possess the following "gauge ghost number", \(N_0\) (of course, gauge ghost number of \(\chi\) and \(\bar{\chi}\) is zero),

\[ N_0(n) = N_0(\bar{n}) = n, \quad \epsilon(n) = \epsilon(n + 1), \quad \epsilon(\bar{n}) = \epsilon(\bar{n}) + 1. \]

They also carry algebraic ghost number according to (19).

Now, we can use the formalism given in Section 1, by the replacement

\[ d \rightarrow Q, \quad \delta_B \rightarrow \Delta_B, \]

where \(\Delta_B\) denotes the BRST transformation resulting from the second quantization. We can attribute a generalized grading to the fields introduced above: \(\chi, \eta, \bar{\chi}, \bar{\eta}\) possess 0-generalized grading and \(\xi, \bar{\xi}\) possess -1-generalized grading. In the BV quantization scheme we should also introduce the related antifields. Antifields are defined such that the sum of the total ghost numbers of a field and its antifield should be -1. Thus they are given as

\[ \chi_{(0,-1)} = \bar{\chi}_{(1,-1)} = \eta_{(n,-n+1)} = \bar{\eta}_{(n+1,-n-1)} = \]

where the first number in the parenthesis indicates the algebraic ghost number and the second denotes the gauge ghost number.

According to the rules given in Section 1 we can group the fields which possess the same generalized grading:

\[ \bar{\chi} = \chi_{(0,0)} + \eta_{(-n,n)} + \bar{\chi}_{(1,1)} + \bar{\eta}_{(-n-1,n+1)}, \]

\[ \bar{\xi} = -\chi_{(0,-1)} - \eta_{(n,-n-1)} + \hat{\chi}_{(-1,0)} + \hat{\eta}_{(-1,-n)}. \]

Then we generalize the original action, (17), as

\[ S_0 = L_0(\chi, \bar{\chi}) = \bar{\chi}Q\chi, \tag{20} \]

where the multiplication is defined such that \(\epsilon(S_0) = 0\), and \(N_0(S_0) = N_0(\hat{S}_0) = 0\). One can easily check that \(S_0\) satisfies the master equation by observing that it is invariant under

\[ \delta \bar{\chi} = \frac{\partial S_0}{\partial \bar{\chi}}, \quad \delta \chi = \frac{\partial S_0}{\partial \chi}, \]

and it is also proper:

\[ \text{rank} \left[ \frac{\partial S_0}{\partial \chi, \partial \bar{\chi}} \right] = \# \text{ of } \chi = \# \text{ of } \bar{\chi}. \]

To expose the resemblance with the general formalism introduced in Section 1 and for future use, observe that the original gauge invariance (18), can be written as

\[ \left( \begin{array}{c} \delta \chi \\ \delta \bar{\chi} \end{array} \right) = \hat{R}^{(0)}(L_0, \chi, \bar{\chi}) \left( \begin{array}{c} \Lambda_1 \\ \bar{\Lambda}_1 \end{array} \right) \equiv \left( \begin{array}{cc} \partial_\Lambda_{\chi} & \partial_{\bar{\Lambda}}_{\bar{\chi}} \\ \partial_{\chi}^{\Lambda} & \partial_{\bar{\chi}}^{\bar{\Lambda}} \end{array} \right) \left( \begin{array}{c} \Lambda_1 \\ \bar{\Lambda}_1 \end{array} \right). \tag{21} \]

Thus the generalized action \(S_0\) will be invariant under the transformations generated by

\[ \hat{R}(S_0) \equiv \hat{R}^{(0)}(S_0, \chi, \bar{\chi}). \tag{22} \]

This is equivalent to the transformation (8), used in Section 1 to show that \(S_0\) satisfies the master equation.

Interactions can be introduced in terms of some vertex operators \(V^{(N)}\), as

\[ L_I(\chi, \bar{\chi}) = \sum_{N} \sum_{k=1}^{N} \chi(1) \cdots \chi(N-k)V^{(N)}_{N-k+1} \chi(N-k+1) \cdots \chi(N) \tag{23} \]

where \(L_I\) is defined to be bosonic and to possess zero ghost numbers. The ranges of \(N\) and \(k\) depend on the requirements.
There are mainly two ways of specifying the interaction terms: i) keep the gauge invariance as it was for the non-interacting case (18), by demanding $Q V_n = 0$ [13], or ii) require that the gauge invariance is generated by the generators $R(0)(L, \chi, \dot{\chi})$, where $R(0)$ is defined in (21), and

$$L(\chi, \dot{\chi}) = L_0 + L_1.$$ 

We deal with the latter case which allows us to obtain the proper solution of the master equation as before:

$$S = L(\chi, \dot{\chi}); \{ S, S \} = 0.$$ 

BRST field theory is suitable to construct string field theories, and most of the string theories studied earlier can be seen to obey the above conditions (e.g. see refs. [14]-[15]). Recently this is observed in ref. [16] for the closed bosonic string field theory. There the inner product carries algebraic ghost number, so that the original action depends only on one field $\chi$. Nevertheless, one can see that all of the conditions of the general procedure introduced above are satisfied, hence one can directly obtain the proper solution of the master equation by replacing the original field with the one including all of the ghost fields and the antifields (compare with the effort spent in ref. [16] to show that indeed this generalization yields the proper solution of the master equation).

To perform gauge fixing one introduces some new fields and Lagrange multipliers, which can be grouped as $C$ and $\tilde{C}$ with 0- and 1-generalized grading, respectively. Moreover in terms of the gauge fixing fermion $\Psi$ which possesses $-1$-generalized grading one fixes the gauge freedom as $\phi^e = \partial \Psi / \partial \phi$.

In the enlarged space the proper solution of the master equation is

$$S_\gamma = S + C^* \tilde{C}.$$ 

Let us deal with $k = 1$ in (23) and choose the gauge fixing fermion as

$$\tilde{\Psi} = \tilde{C}(\tilde{\chi} - \tilde{\chi} O),$$ 

where $O$ is any operator possessing $N_a(O) = -1$, depending on the first quantization variables. First of all, this gauge fixing fermion renames the antifields which are present in $\tilde{\chi}$ as antighosts. In the related path integral after integrating over the Lagrange multipliers, $\tilde{\Psi}$, the gauge fixed action will be

$$S_\gamma = \tilde{\chi} O Q \tilde{\chi} + \sum_{N} O V_N \tilde{\chi}(1) \cdots \tilde{\chi}(N).$$ 

(24)

This procedure suggests that one can proceed in the reverse direction: suppose that in the superspace given in terms of the first quantization variables there is an action in the form (24). It may be possible to construct a gauge invariant action which leads to it after gauge fixing, by finding the appropriate $Q$ and $O$ operators.

To elucidate the above procedure we briefly discuss its application to the relativistic particle (see also ref. [17]), which is defined in terms of the BRST charge

$$Q = c(p^2 + m^2),$$ 

where $c^2 = 0$ and $p^\mu = -i \partial / \partial x^\mu$. In this case the available algebraic ghost numbers are $(-1, 0, 1)$, so that the linear gauge invariance is

$$\delta \chi = Q \lambda, \; \delta \tilde{\chi} = 0,$$

and moreover, it is irreducible. Hence the generalized fields are

$$\chi = \chi (0,0) + \eta (1,0) + \chi (1,-1),$$

$$\tilde{\chi} = - \chi (0,-1) - \eta (0,-1) + \chi (1,-1).$$

In terms of the vertex operator

$$V_N(1, \cdots, N) = c\phi(x_1 \cdots x_2) \cdots \delta(x_{N-1} - x_N),$$

one can write the action

$$L_\gamma(x, \tilde{\chi}) = - \int d^4 x Q \chi + \int \sum_{N=3} d^4 z N \sum_{N=3} \delta(x(1) V_N x(2) \cdots x(N)), \; (25)$$

which can easily be seen to be invariant under the transformations generated by $R(0)(L, \chi, \tilde{\chi})$ (see (21)). Because of the algebraic ghost number restrictions $\partial^2 S_\gamma / \partial \Phi^2$ should vanish, so that the action given in (25) is the general one.

When one generalizes the action $L_\gamma$ as

$$S_\gamma = L_\gamma(\tilde{\chi}, \tilde{\chi}),$$

its invariance under the transformations generated by $\tilde{R}(S_\gamma)$ yield that $(S_\gamma, S_\gamma)$ is independent of $x$ only, because $\lambda_1 = 0$. Obviously $(S_\gamma, S_\gamma)$ cannot depend only on $\tilde{\chi}$, so that it vanishes.

Let us choose the following gauge fixing fermion after enlarging the space of fields as described above,

$$\tilde{\Psi} = \tilde{\chi} (\tilde{\chi} - \chi p),$$ 

where $\{ p, c \} = 1$. The effective gauge fixed action will be

$$S_\gamma = \int d^4 z \left( \tilde{\chi} [\partial^2 - m^2] \tilde{\chi} + \sum_{N=3} \chi^N \right),$$

where $\chi^N$ should be named as antighost.

Acknowledgments

I am grateful to G. Thompson for helpful discussions.

I thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the ICTP.

This work is partially supported by the Turkish Scientific and Technological Research Council (TUBITAK).
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