UNBOUNDED COMPONENTS
IN PARAMETER SPACE OF RATIONAL MAPS

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ABSTRACT

We apply the pinching construction to the study of boundaries of space of quasi-
conformal deformations of rational maps.

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§0. INTRODUCTION

Let \( d > 1 \) be an integer, then space of all rational maps of degree at most \( d \) may be
identified with projective space \( \mathbb{CP}^{2d+1} \) by coefficient correspondence. That means that all
limit points of any sequence of rational maps of degree at most \( d \) are rational maps of
degree at most \( d \). But the question of convergence of dynamics of these maps is more
complicated. The well-known results of Mane, Sad, Sullivan see [MSS] and Lyubich see [L]
show that there is an open everywhere dense set \( W \subset \mathbb{CP}^{2d+1} \), where convergences of maps
and dynamics are equivalent. By these results the question reduces to investigation of the
boundary of the open set.

In this paper we give sufficient conditions on dynamic of a rational map \( R \) guaranteeing
unboundedness of the \( J \)-stable component containing the map (see theorem A below). This
theorem is connected with more easy parts of the next two related problems.

1) C. McMullen see [MM]. If \( R \) is a rational map of degree \( d \) with connected Julia
set and no common periodic point on the boundary of two components of Fatou set, is the
closure of the \( J \)-stable component of \( R \) compact in \( \text{Rat}_d = \mathbb{CP}^{2d+1}/\text{PSL}_2(\mathbb{C}) \) ?

2) J. Milnor see [M]. Let \( d = 2 \). How can one decide when some given hyperbolic
component has compact closure in \( \text{Rat}_2 \) or when it is unbounded?

Definition 0.1. The following set is called \( A \)-domain of a given rational map \( R \) and
denoted by \( A(R) \).

\[
A(R) = \{ F \in \mathbb{CP}^{2d+1}; \text{there are neighborhoods } U_R \text{ and } U_F \text{ of } J(R) \text{ and } J(F), \text{respectively and a quasiconformal homeomorphism } h_F : U_R \to U_F \text{ such that } F = h_F \circ R \circ h_F^{-1} \}/\text{Möb}.
\]

If \( R \in W \), then \( A(R) \) coincides with component of \( J \)-stability of the map \( R \).

If \( R \) is hyperbolic, then \( A(R) \) is called hyperbolic component.

Definition 0.2. The following set is called space of quasiconformal deformations of a given
rational map \( R \) and denoted by \( \text{qc}(R) \).

\[
\text{qc}(R) = \{ F \in \mathbb{CP}^{2d+1}; \text{there is a quasiconformal automorphism } h_F \text{ of the Riemann } \text{ sphere } \mathbb{C} \text{ such that } F = h_F \circ R \circ h_F^{-1} \}/\text{Möb}.
\]

If \( R \in W \), then \( \text{qc}(R) \) is everywhere dense subset of \( A(R) \).

By results due to Sullivan see [S] we know that there are only 5 types of periodic domains
in Fatou set of the given rational map \( R \) and each type has an associated Riemann surface.

Namely.

The cases with fundamental domain.

(1) Let \( D \) be an attractive periodic domain. Then associated Riemann surface \( T_D \)
is a torus with marked points \( a_1, ..., a_t \), where \( a_t \) correspond to orbits of critical points
from the full orbit of \( D \). Denote by \( S_D \) the surface \( T_D \backslash \{ a_1, ..., a_t \} \) and by \( D_R \)
the set \( \{ \text{the full orbit of } D \} \cup \{ \text{the full orbit of the attractive point} \} \cup \{ \text{set of full } \text{orbits of the critical points} \} \). Then there is an unbranched covering

\[
P_D : D_R \to S_D.
\]
(2) Let \( D \) be a parabolic periodic domain. Then associated Riemann surface \( S_{PD} \) is a twice punctured sphere with marked points \( a_1, \ldots, a_k \) where \( a_2 \) again corresponds to orbits of critical points from the full orbit of \( D \). Denote by \( S_D \) the surface \( S_{PD} \setminus \{a_1, \ldots, a_k\} \) and by \( D_R \) the set \( \{\text{the full orbit of } D\} \setminus \{\text{set of full orbits of critical points}\} \). Then there is an unbranched covering
\[
P_D : D_R \rightarrow S_D.
\]

Foliated cases.

(3) Let \( D \) be a periodic Siegel disk of period \( k \). Then associated Riemann surface \( D_D \) is a disk conformally equivalent to \( D \) with marked points \( a_1, \ldots, a_{k+2} \) and cyclic group \( G_D \) generated by rotation on an angle \( 2\pi \alpha \), where \( \alpha \) is irrational. Here the point \( a_1 \) is in \( 0D \) but points \( a_2, \ldots, a_{k+1} \) are corresponding to the first hits of forward orbits of the critical points and \( G_D \) corresponds to the action \( R^{\alpha} : D \rightarrow D \). Denote by \( S_D \) the surface \( D_D \setminus \{a_1, a_{k+2}\} \), then there is an unbranched covering
\[
P_D : \{\text{full orbit of } D \} \setminus \{\text{full orbits of critical points}\} \rightarrow S_D
\]

(4) Let \( D \) be a periodic Herman ring of period \( k \). Then associated Riemann surface \( H_D \) is a circular ring conformally equivalent to \( D \) with marked points \( a_1, \ldots, a_{k+2} \) and cyclic group \( G_D \) generated by rotation on an angle \( 2\pi \alpha \), where \( \alpha \) is irrational. Here the points \( a_1 \) and \( a_{k+2} \) belong to different components of \( 0D \) and points \( a_2, \ldots, a_{k+1} \) again correspond to the first hits of forward orbits of critical points. The group \( G_D \) corresponds to the action \( R^{\alpha} : D \rightarrow D \). Denote by \( S_D \) the surface \( H_D \setminus \{a_1, a_{k+2}\} \), then there is an unbranched covering
\[
P_D : \{\text{full orbit of } D \} \setminus \{\text{full orbits of critical points}\} \rightarrow S_D
\]

(5) Let \( D \) be a superattractive periodic domain of the period \( k \). Then associated Riemann surface \( S_D \) is a circular ring with marked points \( a_1, \ldots, a_{k+2} \) and group \( G_D \) generated by rotations on angles \( 2\pi \alpha, \alpha = \frac{1}{D} \) for all \( n \geq 1 \). Here the points \( a_1 \) and \( a_{k+2} \) belong to different components of \( 0D \) and all points \( a_2, \ldots, a_{k+1} \) correspond to forward orbits of critical points and \( d \) is the local order of the superattractive points for \( R^d : D \rightarrow D \). There is no a map from the full orbit of \( D \) onto \( S_D \).

Circles define an invariant foliation for the cases (3) - (5). Denote by \( S_{RD} \) the set \( U_{RD} \) and by \( P_R \) the unbranched covering from \( F(R) \setminus \{\text{full orbit of all superattractive periodic domains}\} \cup \{\text{full orbit of all critical points}\} \cup \{\text{the full orbit of all attractive periodic points}\} \) into \( S_R \) by setting \( P_R = P_D \) on the full orbit of \( D \).

Consider the following sets of rational maps.

\[W_1 = \{\text{rational maps } R \text{ with disconnected Julia set}\}\]
\[W_2 = \{\text{rational maps } R \text{ with connected Julia set such that there is an periodic component } D \text{ of the Fatou set and an periodic point } x \in 0D \text{ having more than one access from } D\}\]
\[W_3 = \{\text{rational maps } R \text{ with connected Julia set with two periodic components } D_1 \text{ and } D_2 \text{ of the Fatou set such that the intersection } 0D_1 \cap 0D_2 \text{ contains two periodic points accessible from } D_1 \text{ and } D_2\}\]

**Definition 0.3.** Let \( D \subset F(R) \) be an attractive (parabolic) periodic domain for the given rational map \( R \). Let \( \gamma \subset S_D \) be a closed simple geodesic. Then component \( \beta \in P_{R}^{-1}(\gamma) \cap D \) is called the main component of the lifting of \( \gamma \) or main component of \( \gamma \) iff \( \beta \) lands in the attractive (parabolic) periodic point.

Let \( x \in 0D \) be an accessible from \( D \) periodic point, for a given rational map \( R \). An access \( (x, [w]) \) is called geodesic access iff there is a geodesic \( \gamma \subset S_D \) and a main component \( \beta \) of \( \gamma \) such that \( \beta \) lands in the point \( x \) and defines the same access.

We call a periodic point \( x \in 0D \) geodesic accessible iff \( x \) has geodesic-access from \( D \).

We call two geodesic accesses \( (x, [\gamma]) \) and \( (x, [\gamma']) \) independent iff there are two geodesics \( \gamma' \) and \( \gamma'' \subset S_D \) defining the same accesses, respectively and either \( \gamma' = \gamma'' \) or \( \gamma' \cap \gamma'' = \emptyset \).

The main result of this paper is the theorem.

**THEOREM A.**

1. Let \( R \in W_1 \), then A-domain of \( R \) does not have compact closure in the space \( \text{Rat}_4 \).
2. Let \( R \in W_2 \cup W_3 \). Assume there is a map \( R_1 \in A(R) \) such that accesses in definitions of \( W_2 \) and \( W_3 \) are independent for \( R_1 \). Then A-domain of \( R \) does not have compact closure in the space \( \text{Rat}_4 \), where \( d \) is degree of \( R \).

**COROLLARY A.** Let \( R \in W_2 \cup W_3 \). Then there is an integer \( N(R) \) such that A-domain of \( R^n \) does not have compact closure in the space \( \text{Rat}_4 \) for all \( n \) divided by \( N(R) \).

Arisimg question about non-triviality of the number \( N(R) \) is satisfied by the following example.

**EXAMPLE A.** Let \( R \) be a rational map with completely invariant parabolic domain \( D \). Assume that \( J(R) \) is connected and \( S_{PD} \) is twice punctured sphere with only one marked point. Then closure of the space \( qC(R) \) is compact in the space \( \text{Rat}_4 \) and
1. If \( \deg(R) > 2 \), then \( A(R) \) may be unbounded;
2. If \( \deg(R) = 2 \), then \( A(R) \) has compact closure in the space \( \text{Rat}_4 \).
The test for checking maps from $W_j \cup W_3$ given by theorem A is connected with existence of geodesic accessibility of respective periodic points. It is clear that (already for degree two) there are non-geodesic accessible periodic points even when Julia set is a Jordan curve. We show two cases of existence of independent accesses for the map from $W_j$ and $W_3$ (see theorems 3.1 and 4.1 below) and prove the following theorems.

Denote by $L(A)$ the length (period) of a periodic set $A$ for the given rational map $R$.

**THEOREM 1.** Let $R \in W_3$ be a rational map and domains $D_1$ and $D_2$ be non-parabolic. Let $d_i$ be degree of the restriction map $R|_{D_i}$, where $D_i = R^i(D)$, $j = 1, 2$. Assume that $L(x), L(y) \leq 2L(D)$ and $\frac{d_1^2d_2^2}{2} < \sum_{i=1}^{L(D)} (d_i - 1)$, for $j = 1, 2$.

Then $A$ - domain of $R$ does not have compact closure in space $\text{Rat}_A$.

**COROLLARY 1.** Let $R \in W_3$ and let domains $D_1$ and $D_2$ be non-parabolic and invariant. Assume that $x$ and $y$ are fixed. Then $A$ - domain of $R$ does not have compact closure in the space $\text{Rat}_A$.

**THEOREM 2.** Let $R \in W_2$ and domain $D$ and point $x$ are as in definition of $W_2$.

Assume that

1. The domain $D$ is non-parabolic and $L(z) \leq L(D)$;
2. Let $d_i$ be degrees of the restriction maps $R|_{D_i}$, where $D_i = R^i(D)$, then $d_i = 1$ for all $i$ except one $i_0$.

Then $A$ - domain of $R$ does not have compact closure in the space $\text{Rat}_A$.

**COROLLARY 2.** Let $R$ be a rational map with completely invariant domain $D$. Assume that fixed points of $R$ are neither parabolic nor Siegel nor Cremer. Then $A$ - domain of $R$ does not have compact closure in the space $\text{Rat}_A$.

The main tool in proving these theorems is application of the "pinching" construction to Riemann surfaces associated with rational maps.

Let $S_R$ be a Riemann surface associated with a rational map $R$. Let $S$ be a component of $S_R$ with a hyperbolic (for example Poincare) metric and let $\gamma \subset S$ be a closed, simple (without self intersection) geodesic (or a leaf of foliation in case of foliated Riemann surface see remark 1 below). Then there are (see [Str]) small ring neighborhoods $A_i \subset S$ of $\gamma$ and a conformal map $h$ from the standard ring $C = \{z : 1/r < |z| < r\}$ onto $A$ such that $h^{-1}(\gamma) = \{z : |z| = 1\}$. Then modulus of $A$ in terms of $r$ is $\ln(r)/r$.

Let $F(z) = z/|z|$ be the quasiconformal map and $C_n = F^{-n}(C)$ be rings, then $F^n(C_n) = C$.

Denote by $\nu_n$ the Beltrami differential on $C$ as follows

$$
\nu_n = \begin{cases} 
0 & \text{on } C \setminus C_n; \\
\frac{d^2}{dz^2} & \text{on } C_n.
\end{cases}
$$

Define the Beltrami differentials $\nu_n$ on $S$ by

$$
\nu_n = \begin{cases} 
0 & \text{if } z \in S \setminus A; \\
\nu_n(h^{-1}(z)(h^{-1})'(h^{-1})^{-1}) & \text{if } z \in A.
\end{cases}
$$

and the Beltrami differentials $\omega_n$ on $\bar{C}$ by

$$
\omega_n = \begin{cases} 
0 & \text{if } z \in \bar{C} \setminus P_{R}^{-1}(A); \\
\omega_n(P_{R}^{-1}(z)(P_{R})^{-1} \cdot (P_{R})^{-1}) & \text{if } z \in P_{R}^{-1}(A).
\end{cases}
$$

Then we have $\omega_n(R(z))(\bar{C}) = \omega_n(z)(\bar{R}(z))$. The application of the Riemann measurable theorem gives a sequence $H_n$ of quasiconformal automorphisms of the Riemann sphere such that $R_n = H_n \circ R \circ H_n^{-1}$ are rational maps. If a limit map has the same degree as $R$, then we can call this map a pinching deformations of $R$ near geodesic $\gamma$ and denote by $R_\gamma$. Let $R_n = \lim_{n \to \infty} R_i$ for some $\{n_i\} \subset \{n\}$. Then we can call $R_{n_i}$ and $H_{n_i}$ the pinching sequences of rational maps and quasiconformal homeomorphisms, respectively.

**REMARK 1.** It is clear that a leaf of foliation (in foliated cases) which is not passing through a critical point is a simple closed geodesic. In this case a geodesic always means a leaf of foliation. So, if a ring neighborhood of the geodesic is invariant under $G_R$, then taking into account the fact that $F(z)$ commutes with a rotation of the ring $C$, the pinching deformation for this case is well-defined.

**REMARK 2.** Assume that on $S$ exists a collection of simple closed geodesics $\gamma_1, \ldots, \gamma_n$ such that $\gamma_i \cap \gamma_j = \emptyset$ for $i \neq j$. Then it is clear that pinching deformations may be defined for all geodesics at once. If pinching deformation of $R$ near $\gamma_1, \ldots, \gamma_n$ exists we denote it by $R_{\gamma_1, \ldots, \gamma_n}$.

**REMARK 3.** Let $A_n = h^{-1}(C_n)$ and $\Psi_n$ maps from $S$ into $S_{R_n}$ with dilatations equal to $\nu_n$, respectively. Then it easily to calculate that moduli of the rings $B_i$ and $B_n' = \Psi_n(A_n)$ are coincided and equal to $\ln(r)/2\pi$. Moreover maps $\Psi_n \circ \Psi_n^{-1}$ are conformal on both components $K_n$ and $K_n'$ of $\Psi_n(A_n)$ for all $i \geq n + 1$ and

$$m(K_n), m(K_n') \geq \sum_{i=n+1}^{n} m(B_i) = \frac{n\ln(r)}{2\pi}.$$

As a result we conclude that maps $\Psi_n \circ \Psi_n^{-1}$ are conformal on $\{\Psi_n(S) \setminus A\} \cup \{K_n \cup K_n'\}$ for all $i \geq n + 1$.

Finally we give the following necessary fact.

**LEMMA 1.** Let $R$ be a rational map and $D \subset F(R)$ is an attractive or parabolic periodic domain. Denote by $d_i$ the degree of the restriction $R|_{D_i}$, where $D_i = R^i(D)$. Then there is a rational map $R_i \subset A(R)$ and a homeomorphism $h : J(R) \to J(R_i)$ conjugating $R$ with $R_i$ such that

1. The domain $D_i \subset F(R_i)$ bounded by $h(\partial D)$ is attractive or parabolic, respectively;
2. for any $i$ the domain $D_i = R(D_i)$ contains $d_i - 1$ critical points of $R_i$ and its forward orbits are infinite;
3. There are no critical points in $\cup_{i=1}^{n} \{R_i(D_i)\}$ having intersecting forward orbits.

**PROOF.** By the quasiconformal surgery. \boxed{\text{END}}
PROOFS OF THE THEOREM A AND COROLLARY A

We divide proof of the theorem on 3 cases, whenever $R$ belongs to either $W_1$ or $W_2$. Or $W_3$.

The case $R \in W_1$.

For begin assume that $F(R)$ contains a cycle of Herman rings of a period $k$. Let $D$ be a component of this cycle and $I \subset D$ invariant leaf does not intersecting orbits of critical points of $R$. Let $A \subset D$ be a small invariant ring neighborhood of $I$ and $R_n$, $h_n$ be pinching sequences of rational maps and quasiconformal homeomorphisms, respectively. Assume that there is a limit map $R_{\infty}$ for $R_n$, of the same degree as $R_n$, Then maps $h_n$, on $C_1 \cup C_2 = D$ form a normal family of conformal maps. Now we show that all limit maps of both families $F^\infty$ are constants. Otherwise, assume that $F^\infty$ has non-constant limit map $H_R$, then

$$H_R \circ R_i = H_R \circ h_i.$$  \[(1)\]

Moreover, by induction the family $\{h_n \mid V\}$ is normal for any component $V$ from $R^{-1}(C_1)$ and therefore if $H_1$ is a limit map for $(h_n)_{\mid V}$ then

$$H_1 \circ R_i = H_1 \circ h_i.$$  \[(2)\]

Because there are $1 < k > 1$ and components $V_1$ and $V_2$ from $R^{-1}(C_1)$ separated by $C_1$ we conclude that sets $w_1 = \lim_{n \to \infty} h_n(V_1)$ and $w_2 = \lim_{n \to \infty} h_n(V_2)$ belong to different components of $H_1(C_1)$ and

$$R_1^\infty(w_1) = R_1^\infty(w_2) = H_1(C_1).$$  \[(3)\]

Therefore, if $O$ is a component of $F(R_{\infty})$ containing $H_1(C_1)$, then $O$ is an invariant Herman ring for $R^\infty$. But by definition of pinching there is an integer $N$ such that $h_n \circ h_{N+1}$ are conformal on a ring $K \subset h_1(D)$ with modulus $m(K) > m(O)$, for all $n > N$. Such as $H_{\infty} \circ h_{N+1}$ may be continued to a conformal map from $K$ into $O$ we obtain a contradiction. For further we need the following lemma.

LEMMA 1.1. Let $R$ be a rational map of degree $d$. Let $\text{Per}_n(R)$ be the set of periodic points of the period $l$ with $l \leq n$. Then there is an integer $N_R$ such that

\begin{enumerate}
  \item The set $\text{Per}_n(R)$ consists of repulsive points for any $n \geq k > N_R$.
  \item For any $N_1 \geq N_2 \geq N_R$, there is a neighborhood $U_{N_1,N_2} \subset \mathbb{CP}^{d+1}$ of $R$ such that
    
    $\text{card}(\text{Per}_n(F)) = \text{card}(\text{Per}_n(R))$
    
    for all $N \leq n \leq N_1$ and $F \in U_{N_1,N_2}$;
  \item For any fixed $N_1 \geq N_2 \geq N_R$ if $F, G \in U_{N_1,N_2}$ and $F_i \to R$ then
    
    $\text{Per}_n(F_i) \to \text{Per}_n(R)$
    
    uniformly. Moreover for all big $i$ there are homeomorphisms $f_i$ mapping the sets $\text{Per}_n(F_i)$ onto $\text{Per}_n(R)$ and conjugating the action $R$ with the action $F_i$.
\end{enumerate}

Homeomorphisms $f_i$ depends continuously on $F$ and $f_i \to \text{id}$ for $i \to \infty$.

PROOF. The number of non-repulsive points of $R$ are bounded and this gives (1). The cases (2) and (3) are immediate corollary of investigation of the solutions of the following equation

$$F^m(z) - z = 0, \text{ for } n \leq k \leq m \leq n \leq N_1,$$

where $F_k(z) \in U_{N_1,N_2}$ and $U_{N_1,N_2}$ is such that all solutions of the equation above with initial conditions

$$F^m(z_0) = R^m(z_0) = z_0, \text{ for } z_0 \in \text{Per}_n(R)$$

are well defined on $U_{N_1,N_2}$.

Now return back to the theorem A. By above we have that all limit functions of the both families $F^\infty$ are constants. That means that the spherical diameter of the both components $B^t$ and $B^{t'}$ of $h_{\infty}(D/A)$ tends to $0$ as well as $n_j \to \infty$. Let $x$ and $y$ be different points from $\text{Per}_n(R) \cap B^t_i$ with $k \geq N_R$. Then by the construction we have $x_j = h_{n_j}(x)$ and $y_j = h_{n_j}(y) \in \{\text{Per}_n(R_n)\} \cap \{B^t_i\}$. Therefore $\lim_{n \to \infty} x_j = \lim_{n \to \infty} y_j$ for some $j \in \{j\}$. This is contradiction with the lemma 1.1.

Now examine general case. Let $D$ be a non-simply connected Fatou component and $O$ is the periodic component in the forward orbit of $D$. Then $O$ is either attractive (superattractive) or parabolic or Siegel. Let $R^\infty = R^\infty(D)$. Assume there is a Jordan curve $\gamma \subset O$ (in Siegel case $\gamma$ is a leaf of foliation) satisfying

\begin{enumerate}
  \item there exists a small ring neighborhood $A \subset D$ of $\gamma$ such that $R^\infty|_A$ is invariant for all $n \geq 1$;
  \item there is a component $O$ of the set $D \cap (\text{the full orbit of } A)$ separating components of $\partial D$;
  \item in parabolic and attractive cases $R^\infty(A) \cap R^\infty(A) = \emptyset$ for any $n \neq m$.
\end{enumerate}

Then we claim that the space $\text{qc}(R)$ does not have compact closure in the space $\text{Rat}_d$. Proof of the claim. Condition (1), (2), (3) imply that dropping of $\gamma$ into $\text{Rat}_d$ is a closed Jordan curve. Redefine $\gamma$ to be the geodesic in freely homotopical class of the dropping.

Under assumption (2) there exists a component $O_1 \subset D$ of $F^m(R) \cap (\text{Rat}_d \setminus \text{Rat}_d)$ separating components of $\partial D$. Let $h_i$ and $R_i$ be pinching sequences of quasiconformal homeomorphisms and rational maps constructing by $\gamma$. Then there is a subsequence $(h_i) \subset \{h_i\}$ such that the family $\{h_i\} \subset \{h_i\}$ for some $i \subset \{i\}$ is normal. By the lemma 1.1 we conclude that all limit maps of $\gamma$ cannot be constants. Let $H_{\infty}$ be a limit map and $O$ component of $F(\infty)$ containing the set $O_1 = H_{\infty}(O_1)$, then $O_1$ separates components of $\partial D$.

Again by pinching construction we know that for any $N$ there is $N_1$ such that maps

$$\psi_j = h_i \circ h_j^{-1}$$

are conformal on $O_1 \subset F(R_i)$ containing the set $h_1(O_1)$ for any $s \geq N_1$. Moreover the complementary set $O_1 \setminus h_1(O_1)$ consists of rings $B_i, i = 1, \ldots, k \geq 2$ with moduli $m(B_i) \leq N$. Therefore, if $\Psi_{\infty}$ is a limit map for $\psi_j$, then

$$\Psi_{\infty}|_{h_1(O_1)} = H_{\infty} \circ h^{-1}_{\infty} \{h_1(O_1)\}.$$  \[(4)\]

Moreover any component of $\overline{\text{C}}(\text{H}(O_1))$ contains a ring $C_l = \Psi_{\infty}(B_l)$ for some $l$ and $\{\partial C_l \cap H_{\infty}(O_1)\} \subset H_{\infty}(O_1)$. Because moduli of $C_l$ may be arbitrary big we conclude that the
set $E(Q)$ consists of a finite number of points. This is contradiction with lemma 1.1. The claim is proved.

Now we show existence of a rational map $R_t \in A(R)$ and a curve $\gamma \subset F(R_t)$ satisfying to conditions (1), (2), (3).

Let $D$ be a Siegel disk, then define $\gamma$ to be an invariant leaf interior of which (up to Möbius changing coordinates) contains all first hits of forward orbits of critical points from full orbit of $D$. Note that the full orbit of $D$ contains at least two critical points. By consideration of the branched covering $R^k : O \rightarrow D$ conditions (1), (2), (3) are satisfied.

Let $D$ be an attractive periodic domain of the period $k$. Then we define $\gamma$ to be boundary of a small disk neighborhood $U$ such that $R(U) \subset U$ and $R(U)$ is univalent. Then (1) and (3) are satisfied. Show (2). If $D$ is not simply connected, there is an iterated preimage of $\gamma$ separating components of $\partial D$. Otherwise all components of $R^\infty(U) \cap D$ are simply connected for any $n$. Let $U_n$ be component of $R^\infty(U) \cap D$ containing all first hits of forward orbits of critical points from the full orbit of $D$. Consider branched covering $R^m : O \rightarrow D$. Then $U_n$ is homotopical to $\partial D$, thus $R^m(\partial U_n)$ is homotopic to $\partial D$ and (2) is satisfied.

The superattractive case is reduced to the attractive one by the lemma 1.

Let $D$ be a parabolic periodic domain. Then by using the lemma 1 we have a map $R_t \in A(R)$ and homeomorphism $h : J(R) \rightarrow J(R_t)$ such that

1. the component $D_1$ bounded by $h(\partial D)$ is parabolic and
2. the sets of forward orbits of critical values belonging to the forward orbit of $D_1$ are mutually disjoint.

Then by [Mak, theorem 1] there is a horoneighborhood $U \subset D_1$ of the parabolic point such that

1. $R(U) \subset U$ and $R(U)$ is univalent,
2. $U(R(U))$ contains all first hits of critical values from the full orbit of $D_1$.

Let $\gamma \subset U \cap R(U)$ be a closed Jordan curve interior of which contains all these first hits from (3). Again consider branched covering $R^k : O_1 \rightarrow D_1$, where $O_1$ is bounded by $h(\partial D)$. Then the interior of $\gamma$ contains all critical values of this covering and $m(\gamma) \neq 1$ we conclude that $\gamma$ satisfies to (1), (2), (3).

The case $R \in W_1$ is proved.

The case $R \in W_2$.

We begin with Maskit inequalities.

Lemma 1.2. The Maskit inequalities. Let $S$ be a hyperbolic surface of finite topological type (that is $\pi_1(S)$ is finitely generated) and $\gamma$ is a simple closed geodesic on $S$. If $A$ is a ring neighborhood of $\gamma$ in $S$ and $m(A)$ is the extremal length of the family of all rectifiable curves from $A$ freely homotopic to $\gamma$. Then

1. $L_1 \leq 2m(A) \leq \pi L_2 \exp(L_1) / 2$, where $L_1$ is the length of $\gamma$.
2. Let $T$ be a torus and $p : C(0) \rightarrow T$ the unbranched covering with the covering group $G = \langle g = \lambda, \rho \rangle$, for some $|\lambda| > 1$. Let $c$ be a generator of $\pi_1(C(0))$ which may be represented by a curve $c$ with counter clockwise orientation. Let $\Sigma, \Gamma$ be the generator of the fundamental group of the torus $T$ such that $p^*(g) = \Gamma, p^*(c) = \Sigma$.

Let $\gamma$ is homologous to a loop $\Sigma^p \Gamma^q$, then the lifting $p^{-1}(\gamma)$ consists of $q$ curves $\gamma_1, ..., \gamma_q$. If $A \subset T$ is a small ring neighborhoods of $\gamma$, then

$$\ln^2 |\lambda| + (\arg(\lambda) - \frac{x + y}{2})^2 \leq m(\lambda),$$

the number $p/q$ is called the combinatorial rotation number of the curves $\gamma_1, ..., \gamma_q$ with respect to the point $0$.

If $|\lambda| < 1$, then by the above notions we have

$$\ln^2 |\lambda| + (\arg(\lambda) + \frac{x + y}{2})^2 \leq m(\lambda),$$

and the combinatorial rotation number of curves $\gamma_1, ..., \gamma_q$ with respect to the point $0$ is $-p/q$. In other words, the combinatorial rotation numbers $a_0, a_{\infty}$ of the curves $\gamma_1, ..., \gamma_q$ with respect to endoe, respectively, satisfy to the equation $a_0 + a_{\infty} = 0$.

3. Let $B$ be a topological ring in $C(0)$ separating $0$ and $\infty$ and $g^m(B) \cap B = \emptyset$ for some $g(z) = \lambda z, |\lambda| > 1$ and any $n$. If $B$ is conformally equivalent to $A$, then

$$\frac{2\pi}{\ln |\lambda|} \leq m(A).$$

Moreover all these inequalities are quasiconformal invariants.

PROOF. See [Mas1], [Mas2]. The case (1) is the corollary of the well known collar lemma. The cases (2), (3) are results of comparison of $m(A)$ with the Euclidean metric on $B$.

THEOREM 1.1. Let $R$ be a rational map and $D$ be an attractive (parabolic) periodic domain. Let $S_D \subset S_R$ be the Riemann surface associated with $D$ and $\gamma_1, ..., \gamma_n$ closed geodesics on $S_D$ with non-empty sets of main components and $\bar{\gamma}, \gamma_1 \neq 0$ for $j \neq i$. Assume that pinching deformation $R_j \rightarrow \gamma_j$ exists. Then there is a continuous map $h$ from $D$ into $S$ such that $h$ is a quasiconform map on $D \setminus \{\gamma_1, ..., \gamma_n\}$ and $h \circ R = R_1 \circ \cdots \circ h$, where $k$ is period of $D$.

PROOF. Without loss of generality we can analyze case with only one geodesic $\gamma \subset S_D$. Fix some ring neighborhood $A \subset S_D$ of $\gamma$. Let $D_j = j = 1, ..., m$ be components of $D \setminus h^{-1}(A)$ touching the attractive (parabolic) point $z_0 \in D(z_0 \in \partial D)$. Then there is an integer $m = kl, l \geq 1$ such that $R^m(D_j) = D_j$ and each $D_j$ contains at least one critical point of $R^k$. Consider pinching sequences $h_i$ of quasiconformal maps, then there is a subsequence $\{\bar{h}_i\} \subset \{h_i\}$ such that families $\bar{h}_i$ are normal. Note that we can think that sequence $\{h_i\}$ is convergent for any $j$. Denote by $H_j$ the limit maps for these sequences, respectively, then it easily to check that for all $j$ these maps satisfy

(i) $H_j$ is conformal;
(ii) $H_j = R^k \circ h_i$.

Denote by $O_j$ component of $F(R_j)$ containing the set $H_j(D_j)$, then all these components touch the point $x = H_j(z_0) = ... = H_m(z_0)$. By Maskit inequalities $x$ is parabolic point for $R$, and $O_j$ belong to the immediate basin of attraction of $x$. 


Now we claim that $O_i \backslash H_i(D_i)$ consists of the full orbit of two invariant under $R_{\infty}$, topological disks $w_1$ and $w_2$, both touching the point $x$. These disks are called petals of the point $x$ in $O_i$.

Proof of claim. For begin we show that $O_i \backslash H_i(D_i)$ does not contain critical points. Otherwise $h_i$ induces a conformal map $H_i'$ from the component $B \in \mathcal{S}_i(D_i)$ into $S_0$ (or from components $B_1$ and $B_2 \in \mathcal{S}_i(D_i)$ in parabolic case) such that there is a component of $O_i \backslash H_i'(B)$ containing a marked point $\alpha$. By definition of the pinching deformation we know that maps $\psi_j = h_j \circ h_j^{-1}$ induced by maps $\phi_j = \phi_j \circ \psi_j^{-1}$ (see remark 3) which are conformal on the rings $K_1 \cup K_2 \subset \psi_j(A)$ and moduli of these rings satisfy

$$m(K_1), m(K_2) \leq \frac{\log r}{2\pi}.$$ 

Therefore map $H_i' \circ h_j^{-1}$ may be extended to a conformal embedding $H_i$, from $K_1 \cup K_2$ into $S_0$. Thus for all big $s$ the set $H_i(K_1 \cup K_2)$ contains the point $\alpha$. This is contradiction with number of marked points of the component $S_i(D_i)$.

So, $S_i \backslash H_i(D_i)$ consists of two punctured disks $W_1$ and $W_2$ without marked points. Therefore we conclude that the set $W_1 = P_i^{-1}(W_1 \cup W_2)$ is the full orbit of two components $w_1$ and $w_2 \subset W_1 \cap O_i$, touching the point $x$. The map $R_i$ restricted on any component of $W_1$ is one-to-one. The claim is proved.

Then by claim we have that the map $H_i$ defines a continuous map on any component $B \in \partial D_i \cap D_i \cap \{A \} \cup \{A \}$ and $H_i(\gamma_1) = H_i(\gamma_2)$, where $\gamma_1 \cup \gamma_2 = \partial D_i$. On $B \partial \partial D_i$ the map $H_i$ is a homeomorphism.

Denote by $B^*$ the set $S_0 \backslash \gamma_1$, then $B^* \supset B \supset \{B_1 \cup B_2\}$ in parabolic case. Let $\Psi_j$ be a diffeomorphism from $B^*$ onto $S_0$, coinciding with $H_i$ on $B$. This diffeomorphism is unique up to isotopy on $B^*$. Then $\psi_j$ may be lifted on the component $D_i \cup P_i^{-1}(\gamma_1)$ containing the set $D_i$ to a diffeomorphism $\Psi_j$ from $D_i$ onto conjugating the action $R_i^\infty$ with action $R_i$. Let $C$ be a component of $D_i \backslash \{B\}$ and $D = D_i \cap \partial D_i \cap \{A \} \cup \{A \}$, then $\Psi_j(C \cup \cap D_i) = C$ for some $\omega \in W_1$. Define $\phi_j$ on $\beta$, then $\phi_j(\partial D)$, then it is easy to see that by using actions of $R_i^\infty$ and $R_i$ we obtain continuous map from $D_i \cup \{ P_i^{-1}(\gamma_1) \}$ onto $O_i$. If $\beta \in \{ D_i \cup \{ P_i^{-1}(\gamma_1) \}$, then $\Psi_j(\beta) = \psi_j(\beta)$ by construction. Therefore $\Psi_j$ defines a continuous map $\Psi_j$ on $\{ D_i \cup \{ P_i^{-1}(\gamma_1) \}$ onto $O_i$, where conjugates the action $R_i^\infty$ with the action $R_i$. By induction we complete the proof of the theorem.

By the theorem we see that if pinching deformation $R_i$ near a geodesic $\gamma$ exists for a rational map $R$, then the set $P_i^{-1}(\gamma)$ does not contain components the closure of an union of which divides the Riemann sphere. This is important conclusion for further proof.

So, let $R \in \mathcal{W}_2$ and $R_1 \in \mathcal{A}(R)$. Let $D$ be a component of $\mathcal{F}(R_1)$ and $x \in \partial D$ is an accessible periodic under point with at least two independent accesses.

Assume that $\gamma \in \mathcal{F}(R_1)$ and $\gamma \neq \gamma_1$ and $\gamma_2$ with main components $\beta_1$ and $\beta_2$ of the lifting of $\gamma$ defining the different accesses to $x$ or there are two geodesics $\gamma_1$ and $\gamma_2$ with main components $\beta_1$ and $\beta_2$ defining two accesses to $x$, respectively. Again, consider pinching near either geodesic $\gamma_1$ or $\gamma_2$. If either $R_i \in \mathcal{M}_1$, then by theorem 1.1 we have a continuous map $h$ from $D$ into $C$ semi-conjugating $R_i$ with $R_i^\infty$ and restriction of $h$ on closure of a component of $P_i^{-1}(\gamma)$ is a constant, where $\gamma$ is period of the periodic domain $D$. Under assumption in both cases the set $\overline{\beta_1} \cup \overline{\beta_2}$ defines a closed Jordan curve separating $\overline{\mathbb{C}}$ such that the both components $B_1$ and $B_2$ of $\mathbb{C} \backslash \{ \overline{\beta_1} \cup \overline{\beta_2} \}$ contain points of $\mathcal{F}(R_1)$. Consequently we have $h(\overline{\beta_1} \cup \overline{\beta_2}) = \text{const}$. Now, let $h_i$ be a sequence of the pinching sequence of quasiconformal maps used for construction of $h$. Then either $\lim_{i \to \infty} \text{diam}(h_i(B_i \cap \mathcal{F}(R_1))) = \text{const}$ or $\lim_{i \to \infty} \text{diam}(h_i(B_i \cap \mathcal{F}(R_1))) = \text{const}$. This is contradiction with lemma 1.1. The claim and the case $R \in \mathcal{W}_2$ are proved.

The case $R \in \mathcal{W}_3$. Consider $R_1$ and consider geodesics generating accesses from $D_1$ and $D_2$ to periodic points $x$ and $y$, respectively. Again pinching near these geodesics gives desired.

The case $R \in \mathcal{W}_3$ is proved.

Proof of the corollary 1.

For begin we consider Blaschke maps with connected Julia sets.

PROPOSITION 1.2. Let $B$ be a Blaschke map of a degree $d$ with connected Julia set. (we mean that $\mathcal{J}(B)$ is unit circle) and $\mathcal{F}(B)$ is the set of fixed repulsive points of $B$. Let the component $S_\Delta \subset S_\Delta$ associated with unit disk $\Delta$ be either torus with $d - 1$ punctured points or sphere with $d - 1$ punctured points. Then for any point $x \in \mathcal{F}(B)$ there exists a geodesic $y \subset S_\Delta$ such that the main lifting $\gamma_x$ of $\gamma$ lands in the point $x$. Moreover $\gamma_x \cap \gamma_y = \emptyset$ for $x \neq y \in \mathcal{F}(B)$.

Proof. Let $S$ and $T$ be natural generators of $\pi_1(T_3)$ (parabolic case $T$ is natural generator of $\pi_1(S_\Delta)$) such that $P(B) = T$.

Let $\gamma_1, \ldots, \gamma_{d-1}$ be different geodesics on $S_\Delta$ freely homotopic to $T$, $T_3$ such that $S \Delta \backslash \{ \gamma_1, \ldots, \gamma_{d-1} \}$ consists of non-degenerated rings with only one puncture. In parabolic case let $\gamma_1, \ldots, \gamma_{d-1}$ be different geodesics on $S_\Delta$ freely homotopic to $T$ on $S_\Delta$, such that $S \Delta \backslash \{ \gamma_1, \ldots, \gamma_{d-1} \}$ consists of two twice punctured disks (each disk contains exactly one puncture of $S_\Delta$ and non-degenerated rings with only one puncture.

Now, let $\beta_1$ be main component of the lifting of $\gamma_1$, respectively. Then we have

1. $\beta_1$ is unique main component of $P_i^{-1}(\gamma_1)$;
2. $\beta_1$ lands in some point $x \in \mathcal{F}(B)$;
3. $\beta_i(\beta_1) = \beta_1$ and $\beta_i(\beta_1)$ is one-to-one.

Assume that $\beta_1 \neq \beta_1$ land in common point $x$. Then the interior $L \subset \Delta$ of the loop $\overline{\beta_1} \cup \overline{\beta_1} \subset \Delta$ contains at least one critical point of $B$. Besides $L$ can not contains any component from $B^{-n}(\overline{\beta_1} \cup \overline{\beta_1})$ for any $n \geq 0$, because $L \cap \Delta = x$. Then this one together with conditions (1) - (3) imply that $B(L) = L$ and $B(L)$ is one-to-one. This is contradiction. Proposition is proved.

Now return back to our case. Let $R \in S_2 \cup S_3$ and $N(R)$ be minimal integer such that for map $R_1 = R^{N(R)}$ in definition of $S_2 \cup S_3$ the domains $D_1, D_2$ are invariant, the periodic points $x$ and $y$ are fixed and all accesses to these points from $D_1, D_2$ are invariant also. By quasiconformal surgery we may find a rational map $R_{\infty} \in \mathcal{A}(R_1)$ such that the Fatou set of $R_{\infty}$ does not contain a superattractor periodic domains. If $O$ is a periodic domain of $\mathcal{F}(R_2)$ and $x \in O$ is a critical point, then $R_{\infty}^{-n}(x) \neq 0$ and the forward orbit of $x$ does not interacting with a forward orbit of another critical point (or a periodic point, when $O$ is attractive periodic domain).
Then in case $\mathbf{R}_2 \in \mathbf{W}_2$ the component $\mathbf{S}_D \subseteq \mathbf{S}_R$ is either torus with $d - 1$ punctures or sphere with $d - 3$ punctures, where $d = \text{deg}(\mathbf{R}_2)$. For the case $\mathbf{R}_2 \in \mathbf{W}_3$ we have similar conclusion for $\mathbf{S}_D$ and $\mathbf{S}_R$

Therefore, by consideration of Blaschke models for the either actions $\mathbf{R}_2 : D \rightarrow D$ or the actions $\mathbf{R}_2 : D_i \rightarrow D_i$, $i = 1, 2$, by the proposition 1.2 and theorem A we complete proof of the corollary A.

\section{Example 1}
Let $\mathbf{R}$ be a rational map with completely invariant domain $D \subseteq \mathbf{F}(\mathbf{R})$ and $\mathbf{T}(\mathbf{S}_D) = \mathbf{T}(\mathbf{S}_R) \times \mathbf{T}(\mathbf{S}_1) \times \ldots \times \mathbf{T}(\mathbf{S}_m)$ be the Teichmüller space of the Riemann surface $\mathbf{S}_D$. Then by results of [S] we know that there is a branched covering $\pi$ from $\mathbf{T}(\mathbf{S}_D)$ into $\mathbf{R}(\mathbf{D})$. In case of absence of measurable fields on $\mathbf{R}(\mathbf{D})$ compatible with $\mathbf{R}$ the image $\pi(\mathbf{T}(\mathbf{S}_D))$ coincides with $\mathbf{R}(\mathbf{D})$.

Let $\mathbf{S}_D \subseteq \mathbf{S}_R$ be component associated with $D$ and $\mu \in \mathbf{T}(\mathbf{S}_D)$ is a conformal structure. Denote by $\mathbf{D}_\mu$ the set $\{(\omega, \omega_1, \ldots, \omega_m) \in \mathbf{T}(\mathbf{S}_D) \times \mathbf{T}(\mathbf{S}_1) \times \ldots \times \mathbf{T}(\mathbf{S}_m); \mu \equiv \mu\}$. Then we have the following simple conclusion.

\textbf{PROPOSITION 2.1.} Let $\mathbf{R}$ be a rational map with completely invariant domain $D$ and $\mathbf{D}_\mu \subseteq \mathbf{T}(\mathbf{S}_D)$ is a conformal structure. Then closure of $\pi(\mathbf{D}_\mu)$ in $\mathbf{R}(\mathbf{D})$ is compact.

\textbf{PROOF.} Let $\mathbf{R}_1$ be any sequence from $\pi(\mathbf{D}_\mu)$ and $\mathbf{R}_2$ is sequence of quasiconformal maps the Riemann sphere onto itself such that $\mathbf{R}_1 = \mathbf{R}_2 \circ \mathbf{R}_1 \circ \mathbf{R}_2^{-1}$ for $i \geq 2$. Then under assumption our can think, that

1. maps $\mathbf{R}_i$ are conformal on completely invariant domain $D_i$ of $\mathbf{R}_i$;
2. the point $\infty$ belong to $D_i$ and $\mathbf{R}_i(\infty) = \infty + 1/2 i z$ for $i = \infty$.

In other words the maps $\mathbf{R}_i(\infty)$ define the normal family. Let $\mathbf{R}_\infty$ be a limit map for a subsequence $\{\mathbf{R}_i\}$ of $\mathbf{R}_i$. Then $\mathbf{R}_\infty$ is rational map. If $\mathbf{R}_\infty$ is limit map for $\{\mathbf{R}_i\}$, then by (2) above we conclude $\mathbf{R}_\infty(\infty) = \infty + 1/2 i z$. Therefore by setting $\mathbf{R}_\infty(\infty) = \infty + 1/2 i z$.

Now show the case (2) of the example.

Let $\mathbf{R}_1$ and $\mathbf{R}_2 \in \mathbf{A}(\mathbf{R}_1)$ be parabolic degree two maps with connected Julia sets. Let $h_1$ and $h_2$ be conformal maps from completely invariant domains $D_1$ and $D_2$ respectively onto upper half plane mapping parabolic fixed points onto $\infty$, respectively. Then $h_1 \circ \mathbf{R}_1 \circ h_1^{-1} = h_2 \circ \mathbf{R}_2 \circ h_2^{-1}$ is a continued onto real line and is identical at there (because $h$ preserves orientation, commutes with $z \mapsto z + \frac{1}{2}$ and fixes the point $\infty$). Therefore by setting $h = \{h_2^{-1} \circ h_1 \}$ on $D_1$;

$$h = \begin{cases} h_2^{-1} \circ h_1 & \text{on } D_1; \\ h & \text{on } U_1 \setminus D_1 \end{cases}$$

we obtain extension of $h$ on $D_1$. Further by arguments of the proposition 2.1 we complete the case (2).

The case (1). Let $\mathbf{S}_D$ be an invariant union of periodic Fatou components for the given rational map $\mathbf{R}$. Suppose that $\mathbf{S}_D$ contains a periodic point $x$ of a period $n$. Consider accesses from interior of $\mathbf{S}_D$ to $x$. An access $[x, \gamma]$ is periodic if there is an integer $k > 1$, such that, the access $[\mathbf{R}^k(x), [\mathbf{R}^k(\gamma)]]$ defines the same access $[x, \gamma]$. Then follow [GM] we say that the periodic point $x$ has a \textit{combinatorial rotation number} $\frac{\ell}{q}$ if there is a periodic access $[x, \gamma]$, such that $\gamma$ performs a $\frac{\ell}{q}$ rotation around $x$ under iterations of $\mathbf{R}^n$.

\section{Proofs of the Theorem 1 and the Corollary 1}

\textbf{DEFINITION 3.1.} Let $\mathbf{D}$ be an invariant attractive domain for a given rational map $\mathbf{R}$. An annulus $\mathbf{A} \subset \mathbf{D}$ is called a fundamental domain iff $\mathbf{P}_\mathbf{D}(\mathbf{A}) = \mathbf{A}_1$, the restriction of $\mathbf{P}_\mathbf{D}$ onto interior of $\mathbf{C}$ is one-to-one and onto $\mathbf{C}$ is two-to-one.

A fundamental domain $\mathbf{C}$ is called tame fundamental domain iff the interior of $\mathbf{C}$ contains all critical values of $\mathbf{R}$ from $D$ and first hits of forward orbits of critical points from $\{\text{the back orbit of } \mathbf{D}\}$. The proof of the fact see [Mak, theorem 1].

\textbf{PROPOSITION 3.1.} Let $\mathbf{B}$ be a hyperbolic Blaschke map of a degree $d \geq 2$ with connected Julia set. Assume, there are a tame fundamental domain $\mathbf{C} \subset \mathbf{D}$ and number critical values of $\mathbf{B}$ in $\mathbf{D}$ is bigger than $\frac{d - 2}{2}$. Then any two fixed point $x$ and $y \in \partial \mathbf{D}$ are independent accessible from $\mathbf{C}$.

\textbf{PROOF.} Denote by $\mathbf{U}_\mathbf{B}$ the set of all repulsive fixed points of $\mathbf{B}$ and fix a tame fundamental domain $\mathbf{C}$, the torsus $\mathbf{T}_\mathbf{D}$. $\mathbf{U}_\mathbf{B}(\mathbf{T}_\mathbf{D})$ be generated by $\mathbf{S} = \mathbf{P}_\mathbf{B}(\mathbf{A})$ and $\mathbf{T} = \mathbf{P}_\mathbf{B}(\mathbf{B})$. Further, as in proposition 1.2, let $\gamma_1, \ldots, \gamma_m$ be different geodesics on $\mathbf{S}$ freely homotopic to $\mathbf{D}$ and such that $\mathbf{S}_\mathbf{A}(\mathbf{U}_\mathbf{B})$ consists of non - degenerated rings with one puncture. Then, it is clear that main components $\gamma_i$ of the $\gamma_i$ are independent accessible from $\mathbf{C}$.

Now, let $\mathbf{A} \subset \mathbf{S}_\mathbf{A} \subset \mathbf{T}_\mathbf{D}$ be the geodesic freely homotopic to $\mathbf{A}$. $\mathbf{S}_\mathbf{A} \subset \mathbf{A} \subset \mathbf{S}_\mathbf{A}$ is a small ring neighborhood of $\mathbf{S}$. Consider the twist map $\mathbf{B} : \mathbf{A} \rightarrow \mathbf{A}$, in ring coordinates $< t, \theta >$, $1 \leq t < \theta < 2 \leq 2 \pi$ on $\mathbf{A}$ by $\mathbf{B}(t, \theta) = (t, \theta + 2\pi - t)$.

Then by using of the tameness of the fundamental domain $\mathbf{C}$ we conclude, that

1. $\mathbf{B}$ is continued to a homeomorphism $\mathbf{B} : \mathbf{D} \rightarrow \mathbf{D}$ commuting with $\mathbf{B}$ and $\mathbf{B}$ generates a map of order $d - 1$. In other words $\mathbf{B}$ acts transitively on the set of fixed points of $\mathbf{B}$.

Now our goal is showing of the existence of the an integer $k < d - 1$ such that $\mathbf{B}^k$ contains both points $x$ and $y$.

Without loss of generality we may think that $x \in \mathbf{A}$, $y \in \mathbf{A}$, $n_1 = 0 < n_1 \leq \ldots \leq n_{m - 1} < d - 1$ be the full collection of non-negative integers such that $\mathbf{B}^k(x) \neq x$ and $\mathbf{B}^k(x) \in \mathbf{A}$ for $i \neq 1$. Assume that $y \neq \mathbf{U}_\mathbf{B}(\mathbf{B}^k)(\mathbf{A})$. Then the points $y_0 = \mathbf{B}^k(\mathbf{A}) \notin \mathbf{A}$ for all $i = 0, \ldots, m - 1$, but $m > \frac{d - 2}{2}$ and therefore $\text{card}(\mathbf{U}_\mathbf{B}(\mathbf{A})) > m = \frac{d - 2}{2} > d - 1$. This is contradiction. Proposition is proved.


Now, return back to the theorem. Let N be min(D(L(D)), L(D)). Under assumption and by using the lemma 1 we find a rational map $R \in A(R) \subset W_j$ such that

1. the respective periodic domains of $F(R)$, again denoted by $D_i$ and $D_2$ are attractive;
2. there is no two critical points in forward orbits of $D_i$ and $D_2$ having common full orbit;
3. If $\ell_j$ is number of punctures of $S_{D_j}$, then

$$2\ell_j > d_1^j \ldots d_{1(D_j)-1} - 1.$$

Let $S_{D_j} = S_{D_j}'$, then it is easy to see that all accesses to $x$ and $y$ from $D_1$ and $D_2$ are $R_i ^{n_i}$ invariant. If $h$ is the Riemann map from $D_i$ onto unit circle $\Delta$ and $B = h \circ R_i ^{n_i} \circ h^{-1}$, then $B$ is a Blaschke model for $R_i$.

**PROOF.** The direction $\Leftarrow$ is obvious.

Theorem A we complete proof of the case, that $B$ satisfies to assumptions of the proposition 3.1 and by using this proposition and theorem A we complete proof of the case.

Let $S_{D_1}$ and $S_{D_2}$ be different. If we show that all accesses from $D_1$ and $D_2$ to the points $x$ and $y$ are invariant, then by the argument above we complete proof of the corollary 1.

It is clear (by using linearization near the point $x_j$) that if one access from either $D_1$ or $D_2$ is invariant, then all accesses from $D_1$ and $D_2$ are invariant.

Let $B$ be component of $C(D_1)$ containing $D_2$, then $R_i ^{n_1}(D_2) \subset B$ for all $k$. Therefore accesses from $D_1$ to $x$ and $y$ are invariant. The theorem 1 and corollary 1 are proved.

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**§4 PROOF OF THE THEOREM 2 AND THE COROLLARY 2**

Let $R : \Delta \to \Delta$ be a continuous endomorphism. Fix natural order on $\Delta$, that is $x = \exp(2\pi i t) \leq y = \exp(2\pi i \theta) \iff 0 \leq \theta < t < 1$. Let $X = (x_1, x_2, ..., x_n)$ be a periodic point of $R$ of a period $n$ and no $k \in \mathbb{N}$ such that $R^n(X) = x_i$, for any $i$. Denote by $< x$, $y$ > the arc between $x$ and $y$ going from $x$ to $y$ in counter clockwise direction.

Now, fix a periodic point $X = (x_1, x_2, ..., x_n)$ of the given endomorphism $R$. A map $f_X$ is called isometric if $f(x_j) = R(x_j)$ and on arcs $< x_i, x_{i+1}, x_j >$ the map $f_X$ is linear in coordinates $0 \leq \theta < 2\pi$, where $x = \exp(2\pi i \theta) \in \Delta$. It is clear if $R(z) = z^k$, for some $k > 1$ and $R < x_i, x_{i+k}, x_j >$ is one-to-one on $\Delta$ then $f_X < x_i, x_{i+k}, x_j > = R < x_i, x_{i+k}, x_j >$.

Further we say that a periodic point $X = (x_1, ..., x_n)$ of a continuous endomorphism $R$ of unit circle has a combinatorial number (or a right combinatorial rotation number, but we already occupy this term) $e = \frac{k}{n}$, where $e$ is non-reducible iff for any $i = 1, ..., n$ the interior of the arc $< x_i, x_j >$ contains $p - 1$ points of $X$. It is clear that linearization of a periodic point is homeomorphism if $R$ has a combinatorial rotation number. In this case $f_X$ preserves the natural orientation of unit circle $\Delta$.

**THEOREM 4.1.** Let $R(z) = z^d, d \geq 2$ and $X = (x_1, ..., x_n, n > 1$ a periodic point. Then $X$ has combinatorial number $d$ and only if there exists a Blaschke map $B$ and a homeomorphism $h : \Delta \to \Delta$ such that $B = h \circ R(z) \circ h^{-1}$ and the periodic point $X' = h(X)$ is geodesic accessible from unit disk $\Delta$.

**PROOF.** The direction $\Leftarrow$ is obvious.

Suppose that $X$ has combinatorial rotation number $\frac{1}{n}$, then $R(z) = z^{1+\frac{1}{n}}$ for all $i$. Let $f$ be a linearization of $X$. Denote by $g$ the map $z \to |z|^d \arg(z)$. If $r_i$ is radius going from the point $x_i$ to zero, then $g(r_i) = R(r_i)$.

Fix $0 < t < 1$ and rings $C^{\theta}_t = \{z, |z| \leq t \}$ and $C^{\theta}_t = \{z, |z| < t \}$. Then $R(C^{\theta}_t) = g(C^{\theta}_t) = C^\theta_t$. Denote by $\alpha^i$ the segments $r_i \subset C^\theta_t$, then $R(\alpha^i) = g(\alpha^i)$. Further denote by $D^i$ the disk components in $\Delta \setminus C^\theta_t$.

Now, our goal is constructing of a branched covering $F$ from $\Delta$ to $\Delta$ such that

1. $F = R$ on ring component of $\Delta \setminus C^\theta_t$ and
2. $F|_{\alpha^i} = g|\alpha^i}$.

**THE CONSTRUCTION.**

Denote by $\Delta^i$ the piece of $C^\theta_t$ between $\alpha^i$ and $\alpha^{i+1}$ in counterclockwise direction.

1. Set $F := R$ on $\Delta \setminus C^\theta_t$.
2. If $R(\Delta^i) = \Delta^i_{1+n}$, then by definition of $g$ we have $R(\Delta^i) = g(\Delta^i)$. In this case we set

$$F := g$$

3. Let $R(\Delta^i) \not= \Delta^i_{1+n}$, but $\Delta^i_{1+n}$, then $R(\Delta^i) = C^\theta_t$. Let $n_i$ be number of times of covering of $\Delta^i_{1+n}$ by $R(\Delta^i)$. Take any point $z^i \subset \Delta^i$ and fix point $x^i \in \Delta^i_{1+n}$ such that $R(x^i) = x^i$. Let $\ell_i \subset \Delta^i$ be an arc going from $z^i$ to point $x^i$ of intersection of $\alpha^i$ and the circle $|z| = t$ (see picture 2). Let $\alpha^i_{1+n} < \alpha^i < \alpha^i_{1}$ be points of $\Delta$ such that $\Delta^i \subset [\alpha^i_{1+n}, \alpha^i_{1}]$ are going from point $\alpha^i_{1}$ to $\alpha^i_{1+n}$ respectively (see picture 1). Let $\alpha^i_{1+n} \subset |z| = t$ be arcs $< \alpha^i_{1+n}, \alpha^i_{1+n+1} >$ and $\gamma_n \subset |z| = t$ be arcs $< \alpha^i \cap |z| = t >$.

Consider partition of $\Delta^i$ into sets as shown in picture 3. Set $F$ on this set as follows.

a. Such as $g(V) = V$, we set $F := g$ on $V$.

b. Consider $W$. Let $\Psi$ be quasiregular branched covering from $W$ onto $W_1$ mapping $\ell_i$ onto $\ell_i$ such that on any component of $\partial W_1 \setminus \partial D^i$ the map $\Psi$ is one-to-one and $\ell_i$ is unique point of branching for $\Psi$ of the local degree $n_i$ (see picture 3). Then we set $F := \Psi$ on $\partial W_1$.

c. Consider $U_k$, then $\partial U_k = \beta_k \cup \alpha^i$, where $\beta_k \in \partial W$. The map $R|_{U_k} \setminus \Psi|_{U_k}$ maps $\partial U_k$ onto $\partial W_1 \cup \partial D^i$. Let $\phi_k$ be map $\Psi$ from $U_k$ onto $D^i \setminus W_1$ quasi-conformal on the interior of $U_k$ and coinciding on $\alpha^i$ and $\beta_k$ with $R$ and $\Psi$, respectively. Then we set $F := \phi_k$ on $\partial U_k$, respectively.

d. At the end look on $\Lambda$. The boundary of $\Lambda$ consists of $\gamma_n \cup \alpha \cup \beta$, where $\alpha \subset \partial W$ and $\beta \subset \partial W$. The map $R|_{\Lambda} \setminus \Psi|_{\Lambda}$ maps $\partial \Lambda = \partial \Lambda_1 \setminus \partial \Lambda_1 = \partial (\Delta_{1+n} \setminus W_1 \setminus V_1)$. Let $\Phi$ be a map $\Psi$ from $\Lambda$ to $\Lambda_1$ coinciding with $R$ and $\Psi$ on $\gamma_n$, $\alpha$, and $\beta$, respectively and quasi-conformal homeomorphism on the interior of $\Lambda$. Then we set $F|_{\Lambda} = \Phi$. 

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[Page numbers listed for context and navigation]
So, by induction we have constructed a map from $\Delta D^2$ onto $\Delta$. Now consider the ring $C^t_i$, then $F$ maps the circle $\{ |z| = t^i \}$ onto $\{ |z| = t^{i+1} \}$ by one-to-one. Let $\delta(z) = t^iz$ be map from $D^2$ onto $D^1$ and $H$ is a quasiconformal automorphism of $C^t_i$ such that

$$H \circ F \circ H^{-1} \circ \delta = \delta \circ \delta.$$

Then union $\cup_i \cup_n q^i_1(H(\alpha_i))$ forms a arcs $\beta_1, ..., \beta_n$ such that $H(\alpha_i) \subset \beta_1, \beta_2, \ldots, \beta_n$. Now, by using gluing construction of quasiconformal surgery that is glue $D^2$ with $\delta$ to $\Delta D^2$ with $F$ by $H$ to produce a topological disk $D$ and an branched covering $\Phi : D \to D$. Construct an invariant conformal structure $\mu$ on $\{D^2 \cup \{\Delta D^2\}\}/H$ by $\Phi$ beginning with standard one on $\delta(D^2)$. Because $F$ is conformal on $\Delta D^2$ the structure $\mu$ has bounded distortion. The application of the Riemann measurable theorem gives a conformal map $\phi : (D, \mu) \to (\Delta, \sigma_0)$, where $\sigma_0$ is the standard conformal structure of $\Delta$. Then $B = \phi \Phi \phi^{-1}$ is a Blaschke map and arcs $\gamma_i = \phi(\beta_i \cup \{ z \in r, r^i < |z| \})$ gives arcs going from an attractive point to periodic point $\phi(X)$. By construction the arcs drop onto a closed Jordan curve $\gamma' : CS_{D^2} \subset S_{D^2}$ then the geodesic $\gamma$ in homotopical class of $\gamma'$ is desired geodesic. The theorem is proved. 

REMARK 4.1. In theorem above we have constructed the Blaschke map possible having critical points of the local power bigger than 2. Then the quasiconformal surgery shows that for any structurally stable Blaschke map $B$ any periodic point of $B$ having combinatorial number is geodesic accessible.

Now, finish proof of the theorem 2. Let $B$ be the Blaschke model for $R^k : D \to D$, where $k = L(D)$ and $h$ is conjugating conformal map. Let $X = (x_1, ..., x_n) \subset \partial D$ be a periodic point satisfying $\lim_{n \to \infty} h(x_n) = x$ for any $i$. Then $X$ has a combinatorial number $m = \tau x$ or in other words $m = \tau x$, where $\tau x$ is combinatorial rotation number of $x$. Further, the assumption (2) of the theorem and the lemma 1 allow think that $S_{D^2} \subset S_{D^2}$ is $d - 1$ punctured torus, where $d = d_{\alpha}$. Therefore by using the theorem 3.1, the remark 3.1 and theorem A we complete the proof of the theorem 2.

The corollary 2.

By theorem A one can think that the Julia set of $R$ is connected.

Assume that all fixed points are repulsive, then by result of Eremenko and Levin see [EL] we are know that all fixed points are accessible from the domain $D$. Thus at least one from them, say $x$, has non-trivial combinatorial rotation number. Then by using the lemma 1 and theorem 2 we complete the case.

Now, let $x$ be attractive fixed point and $D_1$ respective periodic invariant domain, then $\partial D$ is quasicircle. Further, let $Y = (y_1, y_2) \subset \partial D$ be a periodic point of period two, then $Y$ has trivial combinatorial rotation number respect to $D \cup D_1$.

Let $B$ and $B_1$ be Blaschke models for $R : D \to D$ and $R : D_1 \to D_1$ and $h_1$ and $h_2$ are respective conformal maps. Then points $X = (x_1, x_2)$ and $X' = (x_1', x_2') \in \partial D$ satisfying $\lim_{n \to \infty} h_1(x_n) = \lim_{n \to \infty} h_2(x_n) = y_i$ are periodic for $B$ and $B_1$, respectively and have combinatorial numbers $e_i$. Then again by theorem 2 and remark 3.1 we complete the case together with corollary 2.

REMARK 4.2. It is clear that the arguments above work for parabolic case also, except when $deg(R|_{D_1}) = 2$. 

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REFERENCES


The region $\Delta_1^1$, here $\overrightarrow{V_1} = \gamma(\overrightarrow{V_1})$.

Picture 1.

The region $\Delta_1^1$, here $\overrightarrow{V_1} = \gamma(\overrightarrow{V_1})$.

Picture 2.
The homeomorphisms $w_1$ and $w_2$ are quasiconformal.

Picture 3