DISTRIBUTIVELY GENERATED MATRIX NEAR RINGS

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ABSTRACT

It is known that if $R$ is a near ring with identity then $(I, +)$ is abelian iff $(I^+, +)$
is abelian and $(I, +)$ is abelian iff $(I^*, +)$ is abelian [S.J. Abbasi, J.D.P. Meldrum, 1991].
This paper extends these results. We show that if $R$ is a distributively generated near ring
with identity then $(I, +) \subseteq Z(R)$, the center of $R$, iff $(I^+, +) \subseteq Z(M_n(R))$, the center of
matrix near ring $M_n(R)$. Furthermore $(I, +) \subseteq Z(R)$ iff $(I^*, +) \subseteq Z(M_n(R))$.

MIRAMARE – TRIESTE
April 1993

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1 Introduction

(R, +) is a right near ring if (R, +) is a group (not necessarily abelian), (R, *) is a semigroup and (x + y)z = xz + yz \forall x, y, z \in R. R is zero-symmetric if x0 = 0 \forall x \in R. A normal subgroup (I, +) of (R, +) is an ideal of R if IR \subseteq R and x(a + y) - xy \in I \forall x, y \in R and a \in I. A group G is an R-module if R is homomorphic to M(G), the set of all functions from G to itself. A normal subgroup (H, +) of G is called an R-ideal of G if r(h + g) - rg \in H \forall r \in R, h \in H and g \in G. If R is zero-symmetric and G is an R-module then any R-ideal of G is an R-submodule.

R is distributively generated (d.g.) if ((R, +) is generated as a group by (S, *), a semi-group of distributive elements of R. Every d.g. near ring is zero-symmetric [A. Fröhlich, 1958]. If X \subseteq (R, S), then the ideal I of (R, S), generated by X, is the normal subgroup of (R, +) generated by SXR = \{sxr, sxr, x : s \in S, x \in X, r \in R\}.

(I1, I2), the commutator of ideals I1 and I2 of a d.g. near ring R, is an ideal of R.

All these results are available in [J. D. P. Meldrum 1985].

Mn(R), the near ring of nxn matrices, is a subnear ring of M(Rn), generated by the set \{fij : r \in R, 1 \leq i, j \leq n\} where Rn = \oplus^n(R, +), the direct sum of n copies of (R, +), \(f_{ij}^r = \begin{pmatrix} ix \end{pmatrix} s \in S, x \in X, r \in R\}. (I1, I2), the commutator of ideals I1 and I2 of a d.g. near ring R, is an ideal of R.

2 On d.g. matrix near rings

2.1 Lemma

Let R be a d.g. near ring. Then \(f^{(x, y)}_{ij} = (f_{ij}^x, f_{ij}^y)\) where x, y \in R and 1 \leq i, j \leq n.

Proof: Since \(f_{ij}^{x+y} = f_{ij}^x + f_{ij}^y\) and \(f_{ij}^{-x} = -f_{ij}^x\), therefore the result follows immediately by definition and simple calculation.

2.2 Lemma

Let I1 and I2 be ideals of a d.g. near ring R. Then \((I_1, I_2) \subseteq (I_1^+, I_2^+)\).

Proof: Let A \in (I_1, I_2). We use induction on the weight, w(A), of A. If w(A) = 1, and A = f_{ij}^x where x \in (I_1, I_2), 1 \leq i, j \leq n then A \in (I_1^+, I_2^+), by simple calculation and lemma 1.

Now suppose that the result is true for all elements of (I_1, I_2) of weight less than m, m \in N, m \geq 2. If w(A) = m then A = A_1 + A_2 or A = A_1A_2, where w(A_1), w(A_2) < w(A).

Case 1 follows simply by induction. For case 2, since (I_1^+, I_2^+),M_n(R) \subseteq (I_1^+, I_2^+), therefore A \in (I_1^+, I_2^+). The result now follows by induction.

Remark: For a d.g. near ring, I^+ = GP < S\bar{T}M_n(R) > M_n(R), where S = \{f_{ij}^s : s \in S, 1 \leq i, j \leq n\}.
2.3 Lemma

\((I_1^+, I_2^+) \supseteq (I_1, I_2)^+\).

Proof: As \((I_1^+, I_2^+)\) is an ideal of \(M_n(R)\) so \((I_1^+, I_2^+) \supseteq S(I_1^+, I_2^+)M_n(R) \supseteq S(I_1, I_2)M_n(R),\) by lemma 2. Now since \(((I_1^+, I_2^+), +)\) is a normal subgroup of \((M_n(R), +)\), therefore \((I_1^+, I_2^+) \supseteq S(\overline{I_1, I_2})M_n(R)\) iff \((I_1^+, I_2^+) \supseteq G_p < S(\overline{I_1, I_2})M_n(R) > M_n(R) = (I_1, I_2)^+\).

2.4 Lemma

\((I_1^*, I_2^*) \subseteq (I_1, I_2)^*\).

Proof: Let \((A,B) \in (I_1^*, I_2^*)\) where \(A \in I_1^*, B \in I_2^*.\) Take \(\alpha \in R^n\). Then \((A, B)\alpha \in (I_1, I_2)^n\), by definition of commutator and the fact that \((I_1^*, I_2^*) = (I_1, I_2)^n\). So \((A, B) \in (I_1, I_2)^*\).

Let \(U \in (I_1^*, I_2^*)\). Then \(U = \pm C_1 \pm \ldots \pm C_k \in (I_1, I_2)^*\) as \(C_t \in (I_1, I_2)^* \forall t, 1 \leq t \leq k.\) This completes the proof.

2.5 Theorem

Let \(R\) be a d.g. near ring with identity. If \(I \subseteq Z(R)\), the centre of \(R\), then \(I^* \subseteq Z(M_n(R)).\)

Proof: Since \((R^*, I^*) \subseteq (R, I)^* = \{0\}^* = 0\) and \(R^* = M_n(R)\), therefore \((M_n(R), I^*) = 0\).

2.6 Theorem

Let \(R\) be a d.g. near ring with identity. If \(I \subseteq Z(R)\), then \(I^+ \subseteq Z(M_n(R)).\)

Proof: \(I^+ \subseteq I^* \subseteq Z(M_n(R)).\)

2.7 Theorem

Let \(R\) be a d.g. near ring with identity. \(I \subseteq Z(R)\) iff \(I^+ \subseteq Z(M_n(R)).\)

Proof: Since \((I, R)^+ \subseteq (I^+, R^+) = (I^+, M_n(R)) = 0 = \{0\}^+,\) therefore \((I, R) = \{0\}.\) Hence \(I \subseteq Z(R).\)

2.8 Theorem

Let \(R\) be a d.g. near ring with identity. \(I \subseteq Z(R)\) iff \(I^* \subseteq Z(M_n(R)).\)

Proof: Since \((I, R)^+ \subseteq (I^+, R^+) \subseteq (I^*, M_n(R)) = 0 = \{0\}^+,\) therefore \((I, R) = \{0\}.\) This completes the proof.

3 Open Problem

The author does not know whether these results are true or false for non d.g. near rings, and for d.g. near rings without identity.
4 Acknowledgement

Many thanks are due to Professor Mohammad Abdus Salam, IAEA and UNESCO for the hospitality and financial support at the International Centre for Theoretical Physics, Trieste.

The author also wishes to express her deep gratitude to Professor J. Meldum for his useful comments and to her Parents for all their encouragement and prayers.

References


