IC/92/294

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

GENERALIZED SPIN SYSTEMS AND σ-MODELS

S. Randjbar-Daemi

Abdus Salam

and

J. Strathdee

INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

MIRAMARE-TRIESTE

IC/92/294

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

GENERALIZED SPIN SYSTEMS AND σ-MODELS

S. Randjbar-Daemi
International Centre for Theoretical Physics, Trieste, Italy,

Abdus Salam
International Centre for Theoretical Physics, Trieste, Italy
and
Department of Theoretical Physics, Imperial College, London, United Kingdom

and

J. Strathdee
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

A generalization of the $SU(2)$-spin systems on a lattice and their continuum limit to an arbitrary compact group G is discussed. The continuum limits are, in general, non-relativistic σ-model type field theories targeted on a homogeneous space G/H, where H contains the maximal torus of G. In the ferromagnetic case the equations of motion derived from our continuum Lagrangian generalize the Landau-Lifshitz equations with quadratic dispersion relation for small wave vectors. In the antiferromagnetic case the dispersion law is always linear in the long wavelength limit. The models become relativistic only when G/H is a symmetric space. Also discussed are a generalization of the Holstein-Primakoff representation of the $SU(N)$ algebra, the topological term and the existence of the instanton type solutions in the continuum limit of the antiferromagnetic systems.

MIRAMARE – TRIESTE
September 1992

1. INTRODUCTION

Spin systems have been studied for many years and continue to provide new insights into the behaviour of quantum mechanical systems with many degrees of freedom [1]. In this paper the notion of spin is generalized to include the representations of groups larger than the three-dimensional rotation group [2].

The usual spin system is a collection of $SU(2)$ spin operators associated with the sites of a lattice and coupled to their neighbours. The Hamiltonian is

$$H = \frac{1}{2} \sum_{m,n} J_{mn}\, S^{(m)} \cdot S^{(n)} + \ldots \qquad (1.1)$$

where lattice sites are labelled by sets of integers, $m, n \ldots$, and the spin operators satisfy the $SU(2)$ commutation rules,

$$[\, S^{(m)}_\alpha , S^{(n)}_\beta \,] = i\, \delta_{mn}\, \varepsilon_{\alpha\beta\gamma}\, S^{(n)}_\gamma$$

In (1.1) only quadratic terms are indicated but higher order terms could be added. Non-isotropic versions of (1.1), in which the $SU(2)$ symmetry is broken, may also be considered.

If the lattice is one-dimensional and the spin operators are represented by Pauli matrices then the problem can be solved exactly [3]. Otherwise, the rigorous results are only partial and it becomes necessary to use approximate or numerical methods [1]. Two regimes are particularly suitable for approximation: large spins and long wavelengths [4]. In the large spin or *correspondence theory limit*, the spin operators are represented by unit 3-vectors,

$$S^{(m)}_\alpha \sim s\, \phi^{(n)}_\alpha , \qquad \phi^{(n)} \cdot \phi^{(n)} = 1 , \quad s \gg 1$$

and the system becomes classical. In the long wavelength limit the system approaches a continuum field theory appropriate for the study of low energy excitations. When both approximations are used in conjunction the system is described by a non-linear σ-model [5],

$$S^{(m)}_\alpha \sim s\, \phi_\alpha(x)$$
$$\mathcal{L} \sim \frac{1}{2f} (\partial \phi_\alpha)^2 + \ldots$$

Corrections to the *correspondence theory limit* can be computed in the form of a loop expansion in which the system is represented by a so-called "quantum" non-linear σ-model. It is widely believed that the quantum non-linear σ-model provides an accurate description of the long wavelength, low energy properties of the spin system, even in the small spin regime [6].

A primary goal in the investigation of spin systems is to find indications of phase transitions and critical behaviour. Thus, at sufficiently low temperatures, it may be asked, is there long range order? Does one find spontaneous magnetization in the ground state? In the classical regime this is certainly true but, when quantum effects are taken into account the ground state may well be disordered. For example, it is known that for the one–dimensional ($D = 1$) system there is no long range order in the ground state. On the other hand, for $D = 3$ there can be long range (antiferromagnetic) order. This order is destroyed at some finite critical temperature. For $D = 2$ the ground state has been shown to exhibit long range antiferromagnetic order for $s \geq 1$ but the situation for $s = 1/2$ is not clear, at least in the isotropic models. Although there cannot be any long range order for $D = 2$ at finite temperature, it is interesting to consider the behaviour of the correlation length as the temperature goes to zero. Using renormalization group methods it is possible to distinguish two regimes according to the coupling strength [6]. If the coupling exceeds a certain critical value then the correlation length remains finite for $T \to 0$, indicating a disordered ground state. If the coupling is less than the critical value then the correlation length diverges exponentially, indicating long range order in the ground state [7].

Most of the work in this field is concerned with the $SU(2)$ spin models since it is physically motivated. Our aim here is to extend some of these ideas to "spin" systems based on an arbitrary Lie group. Instead of associating $SU(2)$ generators with the sites of a lattice we shall use the generators from any classical Lie algebra. We shall not attempt to generalize any of the rigorous theorems from the $SU(2)$ literature. Rather, our purpose at this stage is to develop general formalism for treating the long wavelength and correspondence theory limits. Our view is that while such models may not have any immediate physical application, they may eventually serve to caste some light on general features of the $SU(2)$ models by placing them in a broader context.

From a mathematical point of view the generalized models are interesting in themselves. In going to groups of rank > 1 the structure becomes much richer. For instance, one may contemplate new varieties of long range order, beyond the familiar ferromagnetic and antiferromagnetic. The $SU(3)$ analogue of the $s = 1/2$, $SU(2)$ system would employ the triplets 3 and 3^*. In the ground state one might expect to find an orderly arrangement of these states on the lattice. On the other hand, the large quantum number or correspondence limit will lead to σ–models on one or other of the manifolds, $SU(3)/SU(2) \times U(1)$, $SU(3)/U(1) \times U(1)$, etc. depending on the ground state.

To generalize the well–known $SU(2)$ spin wave formalism it is necessary first of all to generalize the Holstein–Primakoff representation of spin matrices. The approach followed here starts from the coherent state method used by Haldane [8] which is easy to generalize. In this method, the finite dimensional vector spaces on which the spin matrices act are provided with an over–complete basis labelled by coordinates on a coset space. Transition amplitudes in this basis are represented by path integrals whose meaning becomes unambiguous in the limit of large quantum numbers, and which can be used to extract a correspondence limit Lagrangian. In principle it would also be possible to use this Lagrangian in the sub–correspondence regime, applying canonical quantization procedures to find corrections. However, the underlying path integral is subject to ordering ambiguities which make the passage to quantum theory less straightforward. This problem is solved, at least in the examples we have examined, by extrapolating directly from the coherent basis expectation values of the generators to operator expressions whose correctness

is then verified by using the canonical commutation rules. The procedure will be illustrated in Sec.6 for the case of $\mathbb{C}P^N$ where a realization of the Holstein–Primakoff type is obtained for the generators of $SU(N+1)$.

At the classical level, where ordering problems are suppressed, it is possible to give an explicit expression for the Lagrangian. The dynamical variables in terms of which it is expressed can be interpreted as coordinates on a coset space, G/H, where H includes the Cartan subgroup or maximal torus of G. If H is precisely the Cartan subgroup then G/H is a flag manifold. The detailed structure of the Lagrangian depends on the coherent state basis from which one starts. This basis is generated by applying finite transformations, L_ϕ, belonging to the group G, to some chosen reference state, $|\Lambda>$,

$$|\phi> = L_\phi|\Lambda>, \qquad L_\phi \in G .$$

The stability group, H, is defined as the set of transformations which leave the reference state invariant, up to a phase

$$h|\Lambda> = |\Lambda> e^{i\psi(h)}, \qquad h \in H \subset G .$$

The coordinates ϕ^μ, $\mu = 1, 2, .., \dim G/H$, may be chosen in any convenient way to parametrize representatives, L_ϕ, from the cosets G/H. The Cartan components of the spin connection on G/H are defined by the 1–form

$$L_\phi^{-1}\, dL_\phi = -A^j(\phi)\, H_j + \ldots$$

where the operators H_j are generators of the Cartan algebra. These are among the generators of the stability group, H, and their eigenvalues serve to label the reference state,

$$H_j|\Lambda> = |\Lambda> \Lambda_j .$$

The coherent basis system is established at each site on the lattice and their direct products define an over–complete basis for the Hilbert space of the model. It will be shown in Sec.2 that the correspondence limit Lagrangian takes the form

$$L = \frac{\hbar}{i} \sum_n A_\mu(\phi_n)\, \partial_t \phi_n^\mu - H(\phi) \qquad (1.2)$$

where

$$A_\mu(\phi_n) = A_\mu^j(\phi_n)\, \Lambda_{nj}$$

and

$$\begin{aligned} H(\phi) &= <\phi|H|\phi> \\ &= \frac{1}{2} \sum_{m,n} J_{mn}\, Q_\alpha(\phi_m)\, Q_\alpha(\phi_n) + \ldots \end{aligned}$$

where the functions $Q_\alpha(\phi)$ are defined as coherent state expectation values of the generators of G,

$$Q_\alpha(\phi_m) = <\phi|Q_\alpha^{(m)}|\phi> .$$

Generalized spin waves are obtained by examining weak excitations of the system described by (1.2). These are defined, at the classical level, when the coherent state at each site on the lattice is close to its reference value. If the Lagrangian is translation invariant then the usual Fourier techniques can be used to extract the spectrum. Some examples of this will be given in Sec.3 where the question of classical stability is considered.

Still at the classical level, it is straightforward to make the restriction to long wavelength configurations, turning the lattice model into a continuum field theory. What emerges is a kind of non–linear σ-model with fields targeted on the coset space, G/H. The Lagrangian will be first order in the time derivative but second (and higher) order in the space derivatives. However, if the (classical) ground state is "antiferromagnetic" in the sense that the reference state weights, Λ_{nj}, alternate in sign across the lattice, then some of the dynamical variables are algebraic and can be eliminated to give a Lagrangian which is second order in the time derivative. This Lagrangian is generally non–relativistic, even when higher order space derivatives are neglected, unless G/H happens to be a symmetric space. This will be discussed in Sec.4.

In passing from the lattice to the continuum description, a term appears in the Lagrangian density (for the antiferromagnetic case) which is a total derivative. Such a term has no relevance at the classical level since it makes no contribution to the equations of motion. It is only an artefact of the method and may be safely discarded. In the quantized continuum theory, on the other hand, such a term may not be negligible. If it makes a finite contribution to the action functional then it will affect the sum over configurations. Since our discussion, in Sec.4, of the passage to the continuum theory is couched in classical terms – the factor ordering question is ignored – we cannot determine whether such terms are actually needed for the long wavelength description of spin systems at the quantum level. Haldane observed this term in his treatment of $SU(2)$ spin systems in one space dimension ($D = 1$) and he noted that it has a topological interpretation, giving an alternating sign factor in the sum over configurations when the spin is a half–integer [9),2)]. Within the limits of our discussion we have confirmed that this effect persists in the generalized antiferromagnetic $D = 1$ spin systems. (For $D > 1$ our total derivative term does not seem to have a topogical significance and should therefore probably be suppressed in the quantum theory [10)].) Topological aspects of the $D = 1$ systems are discussed in Sec.5 where first order equations for generalized instantons are derived.

The classical non–linear σ–models obtained by this approach can be quantized in the usual way. It is not at all obvious that such quantum non–linear σ–models have much relevance to the original spin problem but, as mentioned above, there is a common belief in the $SU(2)$ case that they are good for describing the long wavelength, low energy features of the spin problem. Some support for this view may perhaps derive from the standard treatment of quantized σ–models by dimensional regularization. In such treatments it is prescribed that quantum corrections associated with factor ordering should be discarded. Recall that it is precisely these ordering contributions – defined by the Holstein–Primakoff operator valued expressions for the generators $Q_\alpha(\phi_n)$ – which distinguish the sub–correspondence regime on the lattice. If they are indeed not relevant in the

continuum limit then the quantum non–linear σ–model should be appropriate for the long wave behaviour of the lattice model.

Finally, it may be remarked that the loop expansions of the quantized theory comprise quantum corrections to the classical spin wave amplitudes. Since both $A_\mu(\phi)$ and $Q_\alpha(\phi)$ in (1.2) are linear in the weights, Λ_j, it is clear that the loop expansions can be read as expansions in powers of $1/\Lambda$ if the couplings J_{mn} are redefined to absorb one power of Λ, $J_{mn} = J'_{mn}/\Lambda$. With this interpretation, the classical theory emerges when $\Lambda \to \infty$. This is what is meant by the expression, large quantum number or correspondence limit.

2. THE SEMICLASSICAL REGIME

Our purpose is to discuss the large quantum number, or correspondence limit of generalized spin systems. Typically, the Hamiltonian is given as a sum of terms in which operators associated with sites on a lattice are coupled,

$$H = \frac{1}{2}\sum_{m,n} J^{\alpha\beta}_{mn}\, Q^{(m)}_\alpha\, Q^{(n)}_\beta + \ldots \tag{2.1}$$

where $m, n, \ldots$ label the sites and the coefficients $J^{\alpha\beta}_{mn}$ are coupling parameters. The matrices $Q^{(m)}_\alpha$ are generators of some representation of a Lie group, G. They satisfy the commutation rules

$$[\, Q^{(m)}_\alpha , Q^{(n)}_\beta \,] = \delta_{mn}\, c_{\alpha\beta}{}^{\gamma}\, Q^{(n)}_\gamma \,. \tag{2.2}$$

The couplings are assumed to be G–invariant, i.e. proportional to the Killing metric,

$$J^{\alpha\beta}_{mn} = J_{mn}\, g^{\alpha\beta} \tag{2.3}$$

and also translation invariant,

$$J_{mn} = J_{m-n} \,. \tag{2.4}$$

Higher order and/or non–invariant coupling terms could be included in the Hamiltonian (2.1) but we shall not consider such possibilities.

We are not looking for exact solutions. Following Affleck [2)], our aim is to find a Lagrangian description which can be used for semi–classical approximations, generalized spin waves, and which leads to various σ–model type theories in the continuum limit. Our approach is to generalize the method used by Haldane [3)]. As outlined in Sec.1 this involves the introduction of an over–complete basis of coherent states in the finite dimensional vector space on which the Q_α act, followed by the construction of a path integral representation for transition amplitudes in the coherent basis. In the $SU(2)$ case the coherent states in question are associated with the points of the manifold $SU(2)/U(1)$ and they describe the orientation of the spin vector in the correspondence theory limit. This picture generalizes easily to the cosets G/H.

Let G be a Lie group and H one of its subgroups. We are interested in those subgroups which include at least the Cartan subgroup (maximal Abelian subgroup). If H is precisely the Cartan subgroup then G/H is a flag manifold, but we may consider larger, non–Abelian subgroups so that G/H becomes a subspace of the flag manifold.

Suppose that G is decomposed into left cosets with respect to the subgroup, H. From each coset one can choose a representive element $L_\phi \in G$ where $\phi = \{\phi^\mu\}$ labels a point on the manifold G/H. This means that, for arbitrary $g \in G$, there is a map

$$\phi \rightarrow \phi' = \phi'(\phi, g)$$

defined by

$$g L_\phi = L_{\phi'}\, h \tag{2.5}$$

where $h = h(\phi, g) \in H$. The detailed form of the functions $\phi'(\phi, g)$ and $h(\phi, g)$ depends on the choice of representative elements, L_ϕ.

In the vector space that carries one of the irreducible representations of G, choose one vector, $|\Lambda>$, that is invariant, up to a phase, under the action of H,

$$h|\Lambda> = |\Lambda> e^{i\psi(h)} . \tag{2.6}$$

This vector will be referred to as the "reference state". Now, corresponding to the points $\phi \in G/H$, define the coherent states,

$$|\phi> = L_\phi\, |\Lambda> . \tag{2.7}$$

Under the action of G these states are transformed according to

$$g|\phi> = |\phi'> e^{i\psi(h)} \tag{2.8}$$

where ϕ' and h are determined by (2.5).

The coherent states can be expanded in an orthonormal basis,

$$|\phi> = \sum_\lambda |\lambda> < \lambda|L_\phi|\Lambda> \tag{2.9}$$

and it is possible to project the orthonormal basis vectors from the coherent states by integrating over the coset manifold,

$$|\lambda> = \int d\mu(\phi)\; |\phi> < \Lambda|L_\phi^{-1}|\lambda> \tag{2.10}$$

where $d\mu$ is a suitably normalized G–invariant measure on G/H. The coherent states therefore constitute an over–complete basis.

Of particular interest is the overlap between neighbouring states,

$$\begin{aligned} <\phi + d\phi|\phi> &= <\Lambda|L_{\phi+d\phi}^{-1}\, L_\phi\, |\Lambda> \\ &= \Lambda|(1 - L_\phi^{-1}\, dL_\phi)|\Lambda> \\ &= 1 + A^j\, \Lambda_j \end{aligned} \tag{2.11}$$

where the 1–forms, $A^j = d\phi^\mu\, A^j_\mu(\phi)$, are the Cartan components of the spin connection on G/H. To obtain this we have expanded the Maurer–Cartan form in a basis of the Lie algebra of G,

$$\begin{aligned} L_\phi^{-1}\, dL_\phi &= e^\alpha\, Q_\alpha \\ &= -A^j\, H_j + e^{\tilde\alpha}\, Q_{\tilde\alpha} + e^{\dot\alpha}\, Q_{\dot\alpha} \end{aligned} \tag{2.12}$$

where the operators Q_α satisfy the commutation rules

$$[Q_\alpha, Q_\beta] = c_{\alpha\beta}{}^\gamma\, Q_\gamma . \tag{2.13}$$

The generators of the Cartan algebra are denoted, H_j. These operators are all in the algebra of H together (in general) with non–Abelian elements, $Q_{\tilde\alpha}$. The remaining generators are denoted by Q_α. Their coefficient 1–forms, e^α, in the Maurer–Cartan form define the frames on G/H. The requirement (2.6) that $|\Lambda>$ be invariant, up to a phase, under the action of H means

$$\begin{aligned} H_j\, |\Lambda> &= |\Lambda>\, \Lambda_j \\ Q_{\tilde\alpha}\, |\Lambda> &= 0 . \end{aligned} \tag{2.14}$$

The operators $Q_{\dot\alpha}$ belong to some representation of H. This representation is often reducible but it cannot contain any components that are neutral with respect to the Cartan algebra. Hence $<\Lambda|Q_\alpha|\Lambda> = 0$ and the result (2.11) follows.

The infinitesimal formula (2.11) can be integrated to obtain a path integral representation for the finite overlap. To do this one makes repeated use of the completeness condition (2.10) or

$$1 = \int d\mu(\phi)\; |\phi><\phi| . \tag{2.15}$$

Thus, one writes firstly

$$\begin{aligned} <\phi'|\phi> = \int &<\phi_N|\phi_{N-1}>\; d\mu(\phi_{N-1})\; <\phi_{N-1}|\phi_{N-2}>\; d\mu(\phi_{N-2}) \ldots \\ &\ldots d\mu(\phi_1)\; <\phi_1|\phi_0> \end{aligned}$$

where $\phi_N = \phi'$ and $\phi_0 = \phi$. The intermediate points, $\phi_{N-1}, \phi_{N-2}, \ldots, \phi_1$ are integrated over G/H. In the limit $N \rightarrow \infty$ this gives rise in the usual formal way to a path integral over configurations $\phi(t)$. The factors in the integrand are given by (2.11),

$$\begin{aligned} <\phi + d\phi|\phi> &= \exp(A^j\, \Lambda_j) \\ &= \exp(dt\, \dot\phi^\mu\, A^j_\mu(\phi)\, \Lambda_j) \end{aligned}$$

to first order in dt. Hence, the finite overlap is

$$<\phi'|\phi> = \int (d\mu) \exp \int dt\, \dot\phi^\mu\, A^j_\mu(\phi)\, \Lambda_j . \tag{2.16}$$

Finally, the coherent state transition amplitudes are represented by the path integrals

$$< \phi' | \exp \left(- \frac{i}{\hbar} \, tH \right) | \phi > = \int (d\mu) \exp \left(\frac{i}{\hbar} \int dt \, L \right) \tag{2.17}$$

where the Lagrangian is given by

$$L = \frac{\hbar}{i} \, \dot{\phi}^\mu \, A^j_\mu(\phi) \, \Lambda_j - H(\phi) \; . \tag{2.18}$$

The classical Hamiltonian, $H(\phi)$, is defined by the diagonal elements of H in the coherent basis,

$$H(\phi) = < \phi | H | \phi > \; . \tag{2.19}$$

The Euler Lagrange equations deriving from (2.18) take the form

$$\frac{\hbar}{i} \, \Lambda_j \, F^j_{\mu\nu} \, \dot{\phi}^\nu = \frac{\partial H}{\partial \phi^\mu} \tag{2.20}$$

where $F^j_{\mu\nu} = \partial_\mu A^j_\nu - \partial_\nu A^j_\mu$ are the Cartan components of the curvature tensor on G/H.

The path integral representation (2.17) is of course only formal. To give it a precise meaning one must apply canonical quantization methods to the classical Lagrangian (2.18) and invent a suitable prescription for interpreting the ambiguous factor ordering in $H(\phi)$. The canonical momenta are define by

$$\begin{aligned} \pi_\mu &= \frac{\partial L}{\partial \dot{\phi}^\mu} \\ &= \frac{\hbar}{i} \, A^j_\mu(\phi) \, \Lambda_j \; . \end{aligned} \tag{2.21}$$

To proceed further with this it will be necessary to choose some parametrization of G/H and obtain explicit formulae for the components A^j_μ. Once the commutation rules have been determined for the ϕ^μ, it becomes possible to consider the structure of $H(\phi)$. For the systems we are concerned with, the Hamiltonian is given as a polynomial in the generators Q_α. This means that the classical Hamiltonian will be expressed in terms of the functions

$$Q_\alpha(\phi) = < \phi | Q_\alpha | \phi > \tag{2.22}$$

These functions are unambiguous at the classical level but when the dynamical variables ϕ^μ are quantized it is necessary to determine the factor ordering such that the algebra is realized,

$$[\, Q_\alpha(\phi), Q_\beta(\phi) \,] = c_{\alpha\beta}{}^\gamma \, Q_\gamma(\phi) \; .$$

This is precisely the problem solved by Holstein and Primakoff [11] for the case of $SU(2)$. In Sec.6 we shall give a generalization for the case of $SU(N+1)$ with coordinates ϕ^μ on $\mathbf{C}P^N$. There it will be verified that the ϕ-commutators vanish like $1/\Lambda$ in the limit $\Lambda \to \infty$ (see Eq.(6.10)). But first we shall continue with the classical approximation (i.e. the large quantum number limit).

3. WEAK EXCITATIONS AND CLASSICAL STABILITY

Suppose that the Hamiltonian for the generalized spin system on a lattice is given as a sum over pairs of sites, etc., as described in Sec.1. In the classical approximation,

$$H(\phi) = \frac{1}{2} \, \Sigma \, J_{mn} \, g^{\alpha\beta} \, Q_\alpha(\phi_m) \, Q_\beta(\phi_n) + \ldots \tag{3.1}$$

where $g^{\alpha\beta}$ is the Killing metric and the functions $Q_\alpha(\phi)$ are defined by (2.22). It is convenient to express these functions in terms of matrix elements of L_ϕ in the adjoint representation. These matrices, $D_\alpha{}^\beta$, are defined by

$$g^{-1} \, Q_\alpha \, g = D_\alpha{}^\beta(g) \, Q_\beta , \quad g \in G \, .$$

If follows from the definition (2.7) of the coherent states that

$$\begin{aligned} Q_\alpha(\phi) &= < \Lambda | L^{-1}_\phi \, Q_\alpha \, L_\phi | \Lambda > \\ &= D_\alpha{}^\beta(L_\phi) \; < \Lambda | Q_\beta | \Lambda > \\ &= D_\alpha{}^j(L_\phi) \, \Lambda_j \; . \end{aligned} \tag{3.2}$$

Hence (3.1) takes the form

$$\begin{aligned} H(\phi) &= \frac{1}{2} \, \Sigma \, J_{mn} \, g^{\alpha\beta} \, D_\alpha{}^j(L_{\phi_m}) \, D_\beta{}^k(L_{\phi_n}) \, \Lambda_{mj} \, \Lambda_{nk} + \ldots \\ &= \frac{1}{2} \, \Sigma \, J_{mn} \, \Lambda^j_m \, D_j{}^k(L^{-1}_{\phi_m} \, L_{\phi_n}) \Lambda_{nk} + \ldots \end{aligned} \tag{3.3}$$

where the invariance of the metric has been used,

$$\begin{aligned} g^{\beta\alpha} \, D_\alpha{}^j(L) &= g^{j\alpha} \, D_\alpha{}^\beta(L^{-1}) \\ &= g^{jk} \, D_k{}^\beta(L^{-1}) \; . \end{aligned}$$

The indices j, k refer to components in the Cartan algebra, and we take a basis such that the metric tensor is block diagonal.

Since the coset manifold is homogeneous, no generality is lost by assuming that ϕ_n becomes independent if n in the ground state, i.e.

$$< L^{-1}_{\phi_m} \, L_{\phi_n} > = 1 \; . \tag{3.4}$$

With this convention the ground state value of the Hamiltonian (3.3) reduces to

$$< H(\phi) > = \frac{1}{2} \sum_{mn} J_{mn} \, \Lambda_m \cdot \Lambda_n + \ldots \tag{3.5}$$

Minimization of this expression must be our guide in choosing the weights Λ_n that define the ground state configuration. To be more specific, we shall suppose that the pattern of weights takes the form of a finite number of translation invariant sublattices. Write the variables in the form

$$\{\phi_n\} = \{\phi_{\underline{n}\nu}\} \tag{3.6}$$

where $\underline{n} = (n_1, n_2, \ldots, n_D)$ is a D-vector with integer components and $\nu = 1, \ldots, f$ indicates the sublattice. The weights are independent of $\underline{n}$,

$$\Lambda_{\underline{n}\nu} = \Lambda_\nu .$$

The translation invariant coupling parameters $J_{\underline{m}\mu,\underline{n}\nu}$ depend on $\underline{m} - \underline{n}$. The ground state energy per cell is then given by

$$\frac{1}{2} \sum_{\underline{n}',\nu,\nu'} J_{\underline{n}\nu,\underline{o}\nu'} \Lambda_\nu \cdot \Lambda_{\nu'} . \tag{3.7}$$

To minimize the ground state energy density in, for example, a one–dimensional system with nearest neighbour couplings one should choose $\Lambda_n \cdot \Lambda_{n+1}$ to be positive (negative) if $J_{n,n+1}$ is negative (positive). In this way one is led to the most obvious generalizations of ferromagnetic (antiferromagnetic) order. In more than one dimension there are new possibilities. For example, in two dimensions with a triangular lattice each site has six nearest neighbours. If all nearest neighbours are coupled with the same positive strength then one can make an ordered ground state with three distinct sublattices such that the energy density is

$$J(\Lambda_1\Lambda_2 + \Lambda_2\Lambda_3 + \Lambda_3\Lambda_1) = \frac{J}{2}\left[(\Lambda_1 + \Lambda_2 + \Lambda_3)^2 - \Lambda_1^2 - \Lambda_2^2 - \Lambda_3^2\right] .$$

Hence the configuration with $\Lambda_1 + \Lambda_2 + \Lambda_3 = 0$ would be favoured. Such a generalized "antiferromagnet" becomes a possibility in the $SU(3)$ models.

To test the stability of conjectured ground state configurations one can compute the energy associated with weak perturbations and find the excitation spectrum. Write

$$\phi_n = \phi + \Delta\phi_n \tag{3.8}$$

where ϕ is a constant and $\Delta\phi_n$ is small. Substitute into (3.3) and collect the bilinear terms. To carry out this computation we need the formula

$$\begin{aligned} L_\phi^{-1} L_{\phi+\Delta\phi} &= 1 + \Delta\phi^\mu L^{-1}\partial_\mu L + \frac{1}{2}\Delta\phi^\mu\Delta\phi^\nu L^{-1}\partial_\mu\partial_\nu L + \ldots \\ &= 1 + \Delta\phi^\mu e_\mu{}^\alpha Q_\alpha + \frac{1}{2}\Delta\phi^\mu\Delta\phi^\nu(\partial_\mu e_\nu{}^\alpha Q_\alpha + e_\mu{}^\alpha e_\nu{}^\beta Q_\alpha Q_\beta) + \ldots \end{aligned}$$

which derives from (2.12). Hence,

$$\begin{aligned} L_{\phi_m}^{-1} L_{\phi_n} = 1 + \Big[&(\Delta\phi_n{}^\mu - \Delta\phi_m{}^\mu) e_\mu{}^\alpha + \frac{1}{2}(\Delta\phi_n{}^\mu\Delta\phi_n{}^\nu - \Delta\phi_m{}^\mu\Delta\phi_m{}^\nu)\partial_\mu e_\nu{}^\alpha - \\ &- \frac{1}{2}\Delta\phi_m{}^\mu\Delta\phi_n{}^\nu e_\mu{}^\beta e_\nu{}^\gamma c_{\beta\gamma}{}^\alpha\Big] Q_\alpha \\ &+ \frac{1}{2}(\Delta\phi_n{}^\mu - \Delta\phi_m{}^\mu)(\Delta\phi_n{}^\nu - \Delta\phi_m{}^\nu) e_\mu{}^\alpha e_\nu{}^\beta Q_\alpha Q_\beta + \ldots \end{aligned}$$

and, in the adjoint representation,

$$D_j{}^k(L_{\phi_m}^{-1} L_{\phi_n}) = \delta_j^k + \frac{1}{2}(\Delta\phi_n{}^\mu - \Delta\phi_m{}^\mu)(\Delta\phi_n{}^\nu - \Delta\phi_m{}^\mu) e_\mu{}^\alpha e_\nu{}^\beta c_{j\alpha}{}^\gamma c_{\gamma\beta}{}^k + \ldots$$

With this approximation the Hamiltonian (3.3) takes the form

$$H(\phi) = \frac{1}{2}\Sigma J_{mn}\left[\Lambda_m \cdot \Lambda_n + \frac{1}{2}(\Delta\phi_n{}^\mu - \Delta\phi_m{}^\mu)(\Delta\phi_n{}^\nu - \Delta\phi_m{}^\nu) k_{\mu\nu}^{mn}(\phi) + \ldots\right] \tag{3.9}$$

where $k_{\mu\nu}$ is a symmetric G–invariant tensor on G/H defined by

$$k_{\mu\nu}^{mn} = e_\mu{}^\alpha e_\nu{}^\beta \Lambda_m^j c_{j\alpha}{}^\gamma c_{\gamma\beta}{}^k \Lambda_{nk} . \tag{3.10}$$

The ground state will be stable against weak classical perturbations if the matrix, $J_{mn} k_{\mu\nu}^{mn}$, is positive. To examine this question it is helpful to express the algebra of G in the Cartan–Weyl basis with generators H_j and E_α, where α denotes a root. The commutation rules include

$$\begin{aligned} [H_j, E_{\pm\alpha}] &= \pm\,\alpha_j\, E_{\pm\alpha}, \\ [E_\alpha, E_{-\alpha}] &= \alpha^j H_j . \end{aligned}$$

Since we are dealing with unitary representations we can choose the basis such that

$$H_j{}^+ = H_j \quad \text{and} \quad E_\alpha{}^+ = E_{-\alpha} .$$

The root vectors are then real. However, it is important to remark that since $L^{-1}dL$ is antihermitian, the frame components in the Cartan–Weyl basis must satisfy

$$(\Delta\phi^\mu e_\mu{}^\alpha)^* = -\Delta\phi^\mu e_\mu{}^{-\alpha} .$$

The tensor (3.10) reduces in this basis to a sum over roots,

$$k_{\mu\nu}^{mn} = \sum_{roots} e_\mu{}^\alpha e_\nu{}^{-\alpha} \Lambda_m \cdot \alpha\,\alpha \cdot \Lambda_n . \tag{3.11}$$

Therefore,

$$\Delta\phi^\mu\Delta\phi^\nu k_{\mu\nu}^{mn} = -\sum_{roots} |\Delta\phi^\mu e_\mu{}^\alpha|^2 \Lambda_m \cdot \alpha\,\alpha \cdot \Lambda_n .$$

It follows that the conjectured ground state will be stable if the matrix, $J_{mn}\Lambda_{mj}\Lambda_{nk}$, is negative definite. This is a sufficient condition. It would be straightforward in principle to sharpen this criterion for a specific model but it would not be very useful to pursue the question in generality.

We conclude this discussion with brief remarks about the excitation spectrum. To simplify the notation define the frame components of the weak fluctuations,

$$\Delta\phi_n{}^\mu e_\mu{}^\alpha = \psi_n^\alpha = -(\psi_n^{-\alpha})^* . \tag{3.12}$$

In terms of these variables the bilinear part of the Hamiltonian (2.31) becomes

$$H_2 = -\frac{1}{2} \sum_{\alpha>0} \sum_{m,n} J_{mn}\, \Lambda_m \cdot \alpha\, \alpha \cdot \Lambda_n\, |\psi_n^\alpha - \psi_m^\alpha|^2 \tag{3.13}$$

where the sum is restricted to positive roots. With the translation invariant background, described above, comprising a finite number of sublattices, $\{n\} = \{\underline{n}, \nu\}$, it is natural to use Fourier series. Define the Fourier components,

$$\tilde{\psi}_\nu^\alpha(\underline{k}) = \sum_{\underline{n}} \psi_{\underline{n}\nu}^\alpha\, e^{-i\underline{k}\cdot\underline{n}}$$

$$\psi_{\underline{n}\nu}^\alpha = \int \left(\frac{dk}{2\pi}\right)^D \tilde{\psi}_\nu^\alpha(\underline{k})\, e^{i\underline{k}\cdot\underline{n}} \tag{3.14}$$

where the integration ranges over a cell of volume $(2\pi)^D$. The Hamiltonian (3.13) then takes the form

$$H_2 = -\frac{1}{2} \sum_{\alpha>0} \int \left(\frac{dk}{2\pi}\right)^D \sum_{\underline{n}} \sum_{\nu,\nu'} \cdot J_{\underline{n}\nu,\underline{o}\nu'}\, \Lambda_\nu \cdot \alpha\, \alpha \cdot \Lambda_{\nu'} \left|\tilde{\psi}_\nu^\alpha(\underline{k})\, e^{i\underline{k}\cdot\underline{n}/2} - \tilde{\psi}_{\nu'}^\alpha(\underline{k})\, e^{-i\underline{k}\cdot\underline{n}/2}\right|^2$$

$$= \sum_{\alpha>0} \int \left(\frac{dk}{2\pi}\right)^D \sum_{\nu,\nu'} \tilde{\psi}_\nu^\alpha(\underline{k})^*\, H_{\nu\nu'}^\alpha(\underline{k})\, \tilde{\psi}_{\nu'}^\alpha(\underline{k})\ . \tag{3.15}$$

Stability requires that the matrix, $H_{\nu\nu'}^\alpha(\underline{k})$, should be positive definite for each root.

To obtain the frequency spectrum it is necessary to extract the bilinear part of the Lagrangian (2.18),

$$L = \frac{\hbar}{i} \sum_n \dot{\phi}_n^\mu\, A_\mu^j(\phi_n)\Lambda_{nj} - H(\phi)\ .$$

Substituting (3.8) and discarding a total derivative gives

$$L_2 = \frac{\hbar}{2i} \sum_n \Delta\dot{\phi}_n^\mu\, \Delta\phi_n^\nu\, F_{\nu\mu}^j(\phi)\Lambda_{nj} - H_2$$

$$= \sum_{\alpha>0} \left[\frac{\hbar}{2i} \sum_n \alpha \cdot \Lambda_n\, \psi_n^{\alpha *}(\overleftarrow{\partial}_t - \overrightarrow{\partial}_t)\psi_n^\alpha + \frac{1}{2} \sum_{mn} J_{mn}\, \Lambda_m \cdot \alpha\, \alpha \cdot \Lambda_n |\psi_n^\alpha - \psi_m^\alpha|^2\right]. \tag{3.16}$$

To obtain this expression we have used the Maurer–Cartan expression for the curvature tensor,

$$F_{\mu\nu}{}^j = e_\mu{}^\alpha\, e_\nu{}^\beta\, c_{\alpha\beta}{}^j\ .$$

In the Cartan–Weyl basis this gives

$$F_{\mu\nu}{}^j\, \Lambda_{nj} = \sum_{roots} e_\mu{}^\alpha\, e_\nu{}^{-\alpha}(\alpha \cdot \Lambda_n)\ . \tag{3.17}$$

The linearized equations of motion are easily obtained,

$$-\alpha \cdot \Lambda_\nu\, \frac{\hbar}{i}\, \partial_t\, \psi_\nu^\alpha(\underline{k}) = \sum_{\nu'} H_{\nu\nu'}^\alpha(\underline{k})\, \psi_{\nu'}^\alpha(\underline{k})\ .$$

The frequencies are therefore given by the zeroes of the determinant

$$\det\left(\alpha \cdot \Lambda_\nu\, \delta_{\nu\nu'}\, \hbar\omega - H_{\nu\nu'}^\alpha(\underline{k})\right)\ . \tag{3.18}$$

To illustrate, we consider the one–dimensional chain with nearest neighbour coupling where there are two familiar cases, ferromagnetic and antiferromagnetic, for which the spectrum is easily obtained.

(a) *Ferromagnetic ground state* ($J < 0$)

In this case we have a simple lattice with Λ independent of n. The fluctuation Hamiltonian (3.13) takes the form

$$H_2 = -J \sum_{\alpha>0} (\alpha \cdot \Lambda)^2 \sum_n |\psi_n^\alpha - \psi_{n+1}^\alpha|^2$$

$$= -J \sum_{\alpha>0} (\alpha \cdot \Lambda)^2 \int \frac{dk}{2\pi}\, \tilde{\psi}^\alpha(k)^*\, |e^{ik/2} - e^{-k/2}|^2\, \psi^\alpha(k)$$

i.e.

$$H^\alpha(k) = -4J(\alpha \cdot \Lambda)^2 \sin^2 k/2 \tag{3.19}$$

so that

$$\hbar\omega = -4J\, \alpha \cdot \Lambda\, \sin^2 k/2 \tag{3.20}$$

The sign of ω is not significant. If $\alpha \cdot \Lambda < 0$ we can interpret ψ^α as the creation operator and $\psi^{-\alpha}$ as the annihilation operator for an excitation of energy $\hbar|\omega|$.

(b) *Antiferromagnetic ground state* ($J > 0$)

In this case we have two sublattices with $\Lambda_1 = \Lambda = -\Lambda_2$. The Hamiltonian (3.13) now takes the form

$$H_2 = J \sum_{\alpha>0} (\alpha \cdot \Lambda)^2 \sum_n \left[|\psi_{n1}^\alpha - \psi_{n2}^\alpha|^2 + |\psi_{n1}^\alpha - \psi_{n-1,2}^\alpha|^2\right]$$

$$= J \sum_{\alpha>0} (\alpha \cdot \Lambda)^2 \int \frac{dk}{2\pi} \left[|\psi_1^\alpha(k) - \psi_2^\alpha(k)|^2 + |e^{ik/2}\, \psi_1^\alpha(k) - e^{-ik/2}\, \psi_2^\alpha(k)|^2\right]$$

i.e.

$$H^{\alpha}(k) = 2J(\alpha\cdot\Lambda)^2 \begin{bmatrix} 1 & -e^{-ik/2}\cos k/2 \\ -e^{ik/2}\cos k/2 & 1 \end{bmatrix} . \tag{3.21}$$

The secular determinant (3.18) is

$$\det\begin{vmatrix} \alpha\cdot\Lambda\ \hbar\omega - 2J(\alpha\cdot\Lambda)^2 & 2J(\alpha\cdot\Lambda)^2\, e^{-ik/2}\cos k/2 \\ 2J(\alpha\cdot\Lambda)^2\, e^{ik/2}\cos k/2 & -\alpha\cdot\Lambda\ \hbar\omega - 2J(\alpha\Lambda)^2 \end{vmatrix}$$

$$= (\alpha\cdot\Lambda)^2\left(-\hbar^2\omega^2 + 4J^2(\alpha\cdot\Lambda)^2\sin^2 k/2\right)$$

so that

$$\hbar\omega = \pm\, 2J(\alpha\cdot\Lambda)\sin\, k/2\ . \tag{3.22}$$

In contrast to the ferromagnetic case (3.20), the spectrum is linear in k near $k = 0$. For symmetric spaces like $SU(N+M)/SU(N)\times SU(M)\times U(1)$, it can be shown that $\alpha\cdot\Lambda$ is independent of α and hence, so are the frequencies. The spectrum is relativistic for small k. In this case the corresponding low energy field theory will turn out to be the well known relativistic non–linear σ–model targeted on the appropriate symmetric space.

4. THE CONTINUUM LIMIT

To extract a continuum description of the long wavelength behaviour of the system one supposes that the dynamical variables are slowly varying across the lattice,

$$\phi_m - \phi_n \ll \phi_n\ .$$

In the correspondence theory limit this means

$$\begin{aligned} Q_{\alpha}(\phi_m)\, Q^{\alpha}(\phi_n) &\simeq \Lambda_m^j\, D_j{}^k(L_{\phi_m}^{-1} L_{\phi_n})\Lambda_{nk} \\ &= \Lambda_m\cdot\Lambda_n + \\ &\quad + \frac{1}{2}(\phi_m-\phi_n)^{\mu}(\phi_m-\phi_n)^{\nu}\, k_{\mu\nu}^{mn}(\phi_n) + \dots \end{aligned} \tag{4.1}$$

where $k_{\mu\nu}$ is the tensor introduced above,

$$k_{\mu\nu}^{mn}(\phi) = \sum_{roots} e_{\mu}{}^{\alpha}(\phi)\, e_{\nu}{}^{-\alpha}(\phi)\,(\Lambda_m\cdot\alpha)(\alpha\cdot\Lambda_n)\ .$$

The idea is to express quantities like $H(\phi)$ as functionals of smooth interpolating fields, $\phi(x)$,

$$H(\phi) = \int d^D x\, \mathcal{H}(\phi(x), \partial\phi(x), \dots) \tag{4.2}$$

defining thereby a Hamiltonian density. Since the interpolating fields are supposed to be slowly varying one aims to find the leading terms in an expansion in powers of $\partial\phi$, $\partial^2\phi$, etc. The lattice sums one meets are usually simple enough that this expansion can be found by inspection. However, it is probably worth pointing out that there is a systematic procedure that employs Fourier expansions. The lattice variables, ϕ_n may be represented by Fourier integrals,

$$\phi_n = \int \left(\frac{dk}{2\pi}\right)^D \tilde{\phi}(\underline{k})\, e^{i\underline{k}\cdot\underline{n}} \tag{4.3}$$

where the wave vectors are integrated over a cell of volume $(2\pi)^D$. The Fourier components $\tilde{\phi}$ are periodic in $\underline{k}$ with period 2π. The Hamiltonian may be regarded as a functional of $\tilde{\phi}$, e.g.

$$H(\phi) = \sum_N \frac{1}{N!}\int\left(\frac{dk_1}{2\pi}\right)^D \dots \left(\frac{dk_N}{2\pi}\right)^D H_N(k_1\dots k_N)\,\tilde{\phi}(k_1)\dots\tilde{\phi}(k_N) \tag{4.4}$$

where the coefficient functions, H_N, are of course periodic in the wave vectors $\underline{k}_1, \underline{k}_2, \dots \underline{k}_N$. If ϕ_n is slowly varying then $\tilde{\phi}(\underline{k})$ is non–vanishing only in the neighbourhood of $\underline{k} = 0 \pmod{2\pi}$. In this case it is therefore legitimate to expand the coefficient functions in powers of $\underline{k}_1, \dots, \underline{k}_N$, e.g.

$$H_N(k_1\dots k_N) = (2\pi)^D\,\delta(\Sigma k)\left(h_N^{(0)} - \Sigma\, \underline{k}_j\cdot\underline{k}_{\ell}\, h_N^{(2)} + \dots\right) .$$

The periodicity in $\underline{k}$ is now irrelevant and one may substitute

$$(2\pi)^D\,\delta(\Sigma k) = \int d^D x\, e^{i\Sigma \underline{k}\cdot\underline{x}}\ .$$

Define the interpolating fields $\phi(x)$ by the continuum Fourier integral,

$$\phi(x) = \int\left(\frac{dk}{2\pi}\right)^D \tilde{\phi}(\underline{k})\, e^{i\underline{k}\cdot\underline{x}}\ . \tag{4.5}$$

One then obtains $H(\phi)$ in the form (4.2) with

$$\begin{aligned} \mathcal{H} &= \sum_N \frac{1}{N!}\left[h_N^{(0)}\,\phi(x)^N + \binom{N}{2} h_N^{(2)}\,\phi(x)^{N-2}\,(\underline{\partial}\phi)^2 + \dots\right] \\ &= V(\phi) + Z(\phi)(\underline{\partial}\phi)^2 + \dots\ . \end{aligned} \tag{4.6}$$

In the following we shall discard terms containing more than two derivatives of the interpolating fields. Since the expression (4.1) is bilinear in $\phi_m - \phi_n$ it will be sufficient to keep only the first derivative in this quantity. However, a minor complication arises from the need to keep track of the various translation invariant sublattices. We write

$$\phi_{\underline{n}\nu} = \phi_{\underline{n}} + \xi_{\underline{n}\nu} \tag{4.7}$$

where $\underline{n}$ is a D-vector with integer-valued components and $\nu = 1, \ldots, f$ labels the sublattice. The variables $\xi_{\underline{n}\nu}$ are constrained to satisfy

$$\sum_\nu \xi_{\underline{n}\nu} = 0 \tag{4.8}$$

and they are small quantities of order, $\phi_{\underline{n}} - \phi_{\underline{m}}$. Hence, to leading order,

$$\phi_{\underline{n}\nu} - \phi_{\underline{n}'\nu'} = \left[(\underline{n} - \underline{n}') \cdot \underline{\partial}\phi(x) + \xi_\nu(x) - \xi_{\nu'}(x)\right]_{\underline{x}=\underline{n}} \tag{4.9}$$

which is to be substituted into the formula for the Hamiltonian,

$$H(\phi) = \frac{1}{2} \sum_{n\nu n'\nu'} J_{\underline{n}\nu,\underline{n}'\nu'} \Big[\Lambda_\nu \cdot \Lambda_{\nu'} + \frac{1}{2}(\phi_{\underline{n}\nu} - \phi_{\underline{n}'\nu'})^\sigma (\phi_{\underline{n}\nu} - \phi_{\underline{n}'\nu'})^\rho k^{\nu\nu'}_{\sigma\rho}(\phi_{\underline{n}}) + \ldots \Big]$$

Using the translation invariance of $J_{\underline{n}\nu,\underline{n}'\nu'}$, one obtains the density,

$$\mathcal{H} = \frac{1}{2} \sum_{\underline{n}\nu\nu'} J_{\underline{n}\nu,\underline{o}\nu'} \Big[\Lambda_\nu \cdot \Lambda_{\nu'} + \frac{1}{2}(\underline{n} \cdot \underline{\partial}\phi + \xi_\nu - \xi_{\nu'})^\sigma (\underline{n} \cdot \underline{\partial}\phi + \xi_\nu - \xi_{\nu'})^\rho k^{\nu\nu'}_{\sigma\rho}(\phi) + \ldots \Big] \tag{4.10}$$

It is generally assumed that the coupling strengths fall off rapidly with distance between sites so that the sum over $\underline{n}$ in this expression contains only a few terms. Defining the moments,

$$\begin{aligned} \sum_{\underline{n}} J_{\underline{n}\nu,\underline{o}\nu'} &= J_{\nu\nu'} = J_{\nu'\nu} \\ \sum_{\underline{n}} n^i J_{\underline{n}\nu,\underline{o}\nu'} &= J^i_{\nu\nu'} = -J^i_{\nu'\nu} \\ \sum_{\underline{n}} n^i n^j J_{\underline{n}\nu,\underline{o}\nu'} &= J^{ij}_{\nu\nu'} = J^{ij}_{\nu'\nu} \end{aligned} \tag{4.11}$$

one obtains

$$\mathcal{H} = \frac{1}{2} \sum_{\nu\nu'} \Big[J_{\nu\nu'} \Lambda_\nu \cdot \Lambda_{\nu'} + \frac{1}{2}\Big(J^{ij}_{\nu\nu'} \partial_i\phi^\rho \partial_j\phi^\sigma + 2 J^i_{\nu\nu'} \partial_i\phi^\rho(\xi_\nu - \xi_{\nu'})^\sigma + J_{\nu\nu'}(\xi_\nu - \xi_{\nu'})^\rho(\xi_\nu - \xi_{\nu'})^\sigma \Big) k^{\nu\nu'}_{\rho\sigma}(\phi) + \ldots \Big] \tag{4.12}$$

Although this general result may look rather forbidding, it can simplify in specific applications. For example, if the couplings are isotropic and $J^i = 0$, $J^{ij} \sim \delta^{ij}$. In the ferromagnetic case there are no sublattices so the variables ξ_ν disappear along with the label, ν. In the antiferromagnetic case there are usually only two sublattices and $\xi_2 = -\xi_1, \Lambda_2 = -\Lambda_1$ and, moreover, one usually assumes $J_{11} = J_{22} = 0$, etc.

To complete the discussion of the continuum limit, it is necessary to consider the kinetic term,

$$\frac{\hbar}{i} \sum_{\underline{n}\nu} \dot\phi^\mu_{\underline{n}\nu} A^j_\mu(\phi_{\underline{n}\nu}) \Lambda_{\nu j} \tag{4.13}$$

where A_μ is the spin connection on G/H. Substituting the expression (4.7) for $\phi_{\underline{n},\nu}$ and expanding in powers of ξ, one obtains after discarding a total derivative,

$$\begin{aligned} &(\dot\phi + \dot\xi_\nu)^\rho A_\rho(\phi + \xi_\nu) = \\ &= \dot\phi^\rho A_\rho(\phi) + \xi^\rho_\nu \left(\dot\phi^\lambda + \frac{1}{2}\dot\xi^\lambda_\nu\right) F_{\rho\lambda} + \frac{1}{2}\xi^\rho_\nu \xi^\sigma_\nu \dot\phi^\lambda \partial_\sigma F_{\rho\lambda} + \ldots \end{aligned} \tag{4.14}$$

where $A_\rho = A^j_\rho \Lambda_{\nu j}$ and $F_{\rho\lambda}$ is the corresponding curvature tensor.

In the ferromagnetic case there is nothing further to do. Only the first term in (4.14) is present and the equations of motion will imply $\dot\phi/\phi \sim O(k^2)$ to the approximation used in (4.12). In other cases, if we impose the condition,

$$\sum_\nu \Lambda_{\nu j} = 0\ , \tag{4.15}$$

then the first erm in (4.14) is absent and the most important term becomes

$$\frac{\hbar}{i} \sum_\nu \xi^\rho_\nu \dot\phi^\lambda F^j_{\rho\lambda} \Lambda_{\nu j}\ . \tag{4.16}$$

Here the equations of motion imply that both $\dot\phi/\phi$ and ξ_ν are $O(k)$. It is therefore consistent in this approximation to discard all other terms in (4.14). Notice, in particular, that $\dot\xi_\nu$ disappears from the Lagrangian. This means that ξ_ν becomes an algebraic variable and can be eliminated from the dynamics *).

To conclude, the Lagrangian for the ferromagnetic spin waves is given by

$$L = \int d^D x \left[\frac{\hbar}{i} \partial_t \phi^\mu A_\mu(\phi) - \frac{1}{4} J^{ij} \partial_i \phi^\mu \partial_j \phi^\nu k_{\mu\nu}(\phi) \right] \tag{4.17}$$

where $A_\mu = A^j_\mu \Lambda_j$ and

$$k_{\mu\nu}(\phi) = \sum_{\text{roots}} e_\mu{}^\alpha e_\nu{}^{-\alpha} (\alpha\Lambda)^2\ .$$

*) If the condition (4.15) is not imposed then ξ_ν becomes a true dynamical variable associated with optical branches in the spectrum, i.e. $\dot\xi/\xi$ remains finite in the limit $k \to 0$.

The tensor J^{ij} is assumed to be negative definite.

The equations of motion derived from (4.17) are generalizations of the well known Landau–Lifshitz [13] equations for ferromagnetic spin waves. They are

$$\frac{\hbar}{i} F_{\mu\nu}\, \partial_t \phi^\nu = \frac{J}{2} k_{\mu\nu} \Delta \phi^\nu$$

where we have assumed an isotropic coupling, i.e.

$$J^{ij} = J\,\delta^{ij} \quad J < 0$$

and where the "Laplacian" is given by

$$\Delta \phi^\nu = \partial_i^2 \phi^\nu + \frac{1}{2} k^{\nu\rho} (\partial_\lambda k_{\rho\mu} + \partial_\mu k_{\rho\lambda} - \partial_\rho k_{\lambda\mu})\ \partial_i \phi^\lambda\, \partial_i \phi^\mu .$$

It is not difficult to verify that for $G/H = SU(2)/U(1)$ these equations assume their familiar form [13]

$$\hbar\, \partial_t \vec{n} = -\frac{J}{2} s\, \vec{n} \times \nabla_i{}^2\, \vec{n}$$

where s is the spin and $\vec{n}$ is a unit vector describing the target space S^2.

For the antiferromagentic case (with two sublattices),

$$\Lambda_1 = \Lambda = -\Lambda_2$$
$$\xi_1 = \xi = -\xi_2$$

the Lagrangian is given by

$$L = \int d^D x \Big[\frac{\hbar}{i} 2\xi^\mu F_{\mu\nu}\, \partial_t \phi^\nu - \\ - \left(J^{ij}\, \partial_i\phi^\mu\, \partial_j\phi^\nu + 4J^i\, \partial_i\phi^\mu\, \xi^\nu + 4J\, \xi^\mu\xi^\nu \right) k_{\mu\nu}(\phi) \Big] \qquad (4.18)$$

where, for simplicity, we have taken $J_{11}^{ij} = J_{22}^{ij} = 0$ and we have suppressed the sublattice indices, writing

$$J_{12}^{ij} = J_{21}^{ij} = J^{ij}, \qquad J_{12}^i = -J_{21}^i = J^i$$
$$J_{12} = J_{21} = J$$

and

$$k_{\mu\nu}^{12} = k_{\mu\nu}^{21} = k_{\mu\nu} = -\sum_{roots} e_\mu{}^\alpha\, e_\nu{}^{-\alpha} (\alpha\Lambda)^2 . \qquad (4.19)$$

The tensor J^{ij} is assumed to be positive definite. The auxiliary variable ξ^μ can be eliminated from (4.18) by solving the Euler Lagrange equation $\delta L/\delta \xi^\mu = 0$,

$$\frac{\hbar}{i} F_{\mu\nu}\, \dot{\phi}^\nu - 2k_{\mu\nu}(J^i\, \partial_i\phi^\nu + 2J\,\xi^\nu) = 0 .$$

This gives

$$L = \int d^D x \Big[\frac{\hbar^2}{4J} g_{\mu\nu}\, \partial_t\phi^\mu\, \partial_t\phi^\nu - \frac{\hbar}{i}\frac{J^i}{J}\, \partial_i\phi^\mu\, F_{\mu\nu}\, \partial_t\phi^\nu \\ - \left(J^{ij} - \frac{J^i J^j}{J} \right) \partial_i\phi^\mu\, \partial_j\phi^\nu\, k_{\mu\nu} \Big] \qquad (4.20)$$

where the tensor $g_{\mu\nu}(\phi)$ is defined by

$$g_{\mu\nu} = -F_{\mu\lambda}\, F_{\nu\rho}\, (k^{-1})^{\lambda\rho} \\ = -\sum_{roots} e_\mu{}^\alpha\, e_\nu{}^{-\alpha} \qquad (4.21)$$

(The formulae (4.19) and (3.17) have been used.) The tensor $g_{\mu\nu}$ is G–invariant and positive definite.

For symmetric spaces there is a unique G–invariant second rank symmetric tensor on G/H implying that

$$k_{\mu\nu}(\phi) = v^2\, g_{\mu\nu}(\phi)$$

when v is independent of ϕ. If in addition we also have invariance under space rotations, then

$$J^{ij} - \frac{J^i J^j}{J} \sim \delta^{ij}$$

and the theory becomes fully relativistic. This obtains automatically for $D = 1$ for which we recover the well known two–dimensional σ–models targeted on symmetric spaces.

The Lagrangian (4.20) gives a sensible description of antiferromagnetic spin waves if

$$J > 0, \quad J^{ij} - \frac{J^i J^j}{J} > 0 \qquad (4.22)$$

The middle term in (4.20),

$$-\frac{\hbar}{i}\frac{J^i}{J}\, \partial_i\phi^\mu\, F_{\mu\nu}\, \partial_t\phi^\nu \qquad (4.23)$$

is of a topological nature. It makes no contribution to the classical equations of motion. Indeed, it can be expressed as a total derivative. The integral of this quantity over a spacetime region, M, reduces to a boundary integral,

$$-\frac{\hbar}{i}\frac{J^i}{J}\int_M dt\, d^Dx\, \partial_i\phi^\mu\, \partial_t\phi^\nu\, F_{\mu\nu} = \\ = -\frac{\hbar}{i}\frac{J^i}{J}\int_M dt\, d^Dx \left[\partial_i(\partial_t\phi^\mu A_\mu) - \partial_t(\partial_i\phi^\mu A_\mu) \right] . \qquad (4.24)$$

If M is a compact two–dimensional manifold this integral is a topological invariant – an integer multiple of $2\pi\hbar$ for the 2–sphere as will be shown in the following section. In general the integral

(4.24) has no topological significance. For compact manifolds of higher dimension it has been argued that this term should vanish [10].

It was shown by Haldane, who discovered the term in the case of a one-dimensional antiferromagnetic spin system with nearest neighbour couplings, that the topological contribution gives an alternating sign in the sum over configurations for half-integer spin systems [9]. In the following we consider some classical aspects of the one-dimensional case.

5. INSTANTONS

The continuum theory that emerges from the one-dimensional antiferromagnetic chain is described by a Lagrangian of the type (4.20). Consider now the Euclidean version of this theory. The Lagrangian density is

$$\mathcal{L} = \frac{1}{4J}\, g_{\mu\nu}\, \partial_t\phi^\mu\, \partial_t\phi^\nu + J'\, k_{\mu\nu}\, \partial_x\phi^\mu\, \partial_x\phi^\nu \tag{5.1}$$

where we take $\hbar = 1$ and discard the topological term. The tensors $g_{\mu\nu}$ and $k_{\mu\nu}$, given by (4.21) and (4.19), respectively, are positive definite and we shall assume that J and J' are positive.

There is a simple argument that suggests the possible existence of finite action solutions of the equations of motion, i.e. the instantons. Consider the expressions

$$\begin{aligned}
&\frac{1}{4J}\, g_{\mu\nu}\left(\partial_t\phi^\mu \pm 2i\sqrt{JJ'}\, F^\mu{}_\lambda\, \partial_x\phi^\lambda\right)\left(\partial_t\phi^\nu \pm 2i\sqrt{JJ'}\, F^\nu{}_\rho\, \partial_x\phi^\rho\right)\\
&= \frac{1}{4J}\, g_{\mu\nu}\, \partial_t\phi^\mu\, \partial_t\phi^\nu \pm i\sqrt{\frac{J'}{J}}\, F_{\nu\lambda}\, \partial_x\phi^\lambda\, \partial_t\phi^\nu\\
&\quad - J'\, g_{\mu\nu}\, F^\mu{}_\lambda\, F^\nu{}_\rho\, \partial_x\phi^\lambda\, \partial_x\phi^\rho\\
&= \frac{1}{4J}\, g_{\mu\nu}\, \partial_t\phi^\mu\, \partial_t\phi^\nu \pm i\sqrt{\frac{J'}{J}}\, F_{\nu\lambda}\, \partial_x\phi^\lambda\, \partial_t\phi^\rho + J' k_{\mu\nu}\, \partial_x\phi^\mu\, \partial_x\phi^\nu
\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
g_{\mu\nu}\, F^\nu{}_\rho &= F_{\mu\rho}\\
&= \sum_{\text{roots}} e_\mu{}^\alpha\, e_\rho{}^{-\alpha}\, \alpha\cdot\Lambda\,.
\end{aligned} \tag{5.3}$$

In a basis of real coordinates the components $F_{\mu\nu}$ are pure imaginary. Hence the expresssions (5.2) are real and positive, i.e.

$$\begin{aligned}
\mathcal{L} &= \frac{1}{4J}\, g_{\mu\nu}\, \partial_t\phi^\mu\, \partial_t\phi^\nu + J'\, k_{\mu\nu}\, \partial_x\phi^\mu\, \partial_x\phi^\nu\\
&\geq \left|\sqrt{\frac{J'}{J}}\, F_{\mu\nu}\, \partial_x\phi^\mu\, \partial_t\phi^\nu\right|.
\end{aligned}$$

This means that the Euclidean action has a lower bound,

$$\begin{aligned}
&\int d^2x \left[\frac{1}{4J}\, g_{\mu\nu}\, \partial_t\phi^\mu\, \partial_t\phi^\nu + J'\, k_{\mu\nu}\, \partial_x\phi^\mu\, \partial_x\phi^\nu\right]\\
&\qquad \geq \left|\sqrt{\frac{J'}{J}} \int d^2x\, F_{\mu\nu}\, \partial_x\phi^\mu\, \partial_t\phi^\nu\right|.
\end{aligned} \tag{5.4}$$

It remains to determine whether there are configurations for which the integral on the right-hand side is finite.

Suppose that the two-dimensional Euclidean spacetime has the topology of a sphere. The coordinate functions, $\phi^\mu(x)$, define the image of this sphere in the manifold, G/H. It is not difficult to see that such maps have a topological classification, The manifold is generally covered by more than one coordinate patch. If the coordinates in two patches that intersect are associated with the group elements $L_\phi^{(1)}$ and $L_\phi^{(2)}$, as described above in Sec.2, then at points in the overlap there is a relation,

$$L_\phi^{(2)} = L_\phi^{(1)}\, h^{(1,2)}(\phi) \tag{5.5}$$

where $h^{(1,2)} \in H$. Only the Abelian part of $h^{(1,2)}$ is of interest because this is what determines the relation between $A^j(\phi)$ in the two patches. With

$$h^{(1,2)}(\phi) = e^{i\psi^j(\phi)H_j} \tag{5.6}$$

it follows from the definition (2.12) that

$$A^j_{(2)} = A^j_{(1)} - i\, d\psi^j\,. \tag{5.7}$$

If the overlap between the two patches is not simply connected then the angles $\psi^j(\phi)$ can be multiple valued, i.e.

$$\Delta\psi^j = \oint d\psi^j \neq 0$$

where the integral is taken around a closed path in the overlap. However, it is essential that the group element $h^{(1,2)}$ be single valued and this implies

$$\Delta\psi^j\, \Lambda_j \in 2\pi\mathbf{Z}\,. \tag{5.8}$$

On the other hand, the flux of $F^j\,\Lambda_j$ associated with a compact 2-space must reduce to a sum of such terms. With the 2-sphere suppose that the northern hemisphere maps into patch 1 while the southern maps into patch 2. Then the flux is given by

$$\begin{aligned}
\int F^j\, \Lambda_j &= \int dA^j\, \Lambda_j\\
&= \oint (A^j_{(1)} - A^j_{(2)})\Lambda_j\\
&= i\,\Delta\psi^j\, \Lambda_j
\end{aligned} \tag{5.9}$$

where the contour runs around the image of the equator. The flux is an integer multiple of $2\pi i$. This integer, N, serves to classify the configurations and, in each class the Euclidean action is bounded below by $|2\pi N\sqrt{J'/J}\,|$.

To saturate the bound one needs to solve the first order differential equations,

$$g_{\mu\nu}\,\partial_t\phi^\nu + 2i\sqrt{JJ'}\,F_{\mu\nu}\,\partial_x\phi^\nu = 0$$

or, more explicitly,

$$(\partial_t\phi^\mu - 2i\sqrt{JJ'}\,\alpha\cdot\Lambda\;\partial_x\phi^\mu)e_\mu{}^\alpha = 0\,. \tag{5.10}$$

The solutions of these equations comprise the instantons. If G/H is a complex symmetric space it can be shown that Eq.(5.10) reduces to the well studied case [12)]

$$\partial_{\bar z}\,\zeta^i = 0$$

where z is a complex coordinate in the 2–dimensional Euclidean spacetime and $(\zeta^i, \bar\zeta^i)$ is a set of complex coordinates on the target space G/H.

Returning to Haldane's observation, it will be seen that our result (4.20) appears to differ in some respects. The Euclidean version of (4.20) with topological term included is

$$\begin{aligned}\mathcal{L} = \frac{1}{4J}\,g_{\mu\nu}\,\partial_t\phi^\mu\,\partial_t\phi^\nu + \frac{J^x}{J}\,F_{\mu\nu}\,\partial_x\phi^\mu\,\partial_t\phi^\nu + \\ + \left(J^{xx} - \frac{J^xJ^x}{J}\right)k_{\mu\nu}\,\partial_x\phi^\mu\,\partial_x\phi^\nu\end{aligned} \tag{5.11}$$

where we have again set $\hbar = 1$. The coefficients J, J^x and J^{xx} are given in terms of the spin couplings, $J_{n\sigma;n'\sigma'}$, by the lattice sums (4.11). In particular,

$$J = \sum_n J_{n,1;o,2} \quad \text{and} \quad J^x = \sum_n n\,J_{n,1;o,2}\,. \tag{5.12}$$

Due to the presence of the topological term the Euclidean action contains the imaginary contribution

$$2\pi i\;N\;J^x/J$$

where N is the instanton number. Haldane's result, obtained for $SU(2)$ spin models with nearest neighbour couplings, is simply $i\pi N$. In fact, this result does agree with ours since it can be shown quite generally that $J^x = J/2$. The relevant coupling parameters are

$$\begin{aligned}J_{n,1;n,2} &= J_{n+1,1;n2} &= K_1\,,\\ J_{n-1,1;n2} &= J_{n+2,1;n2} &= K_3\,,\\ J_{n-2,1;n2} &= J_{n+3,1;n2} &= K_5\,,\end{aligned} \tag{5.13}$$

etc., corresponding to nearest, third nearest, fifth nearest, . . . neighbours on the lattice. Substituting into (5.12) gives

$$\begin{aligned}J &= 2K_1 + 2K_3 + 2K_5 + \ldots\\ J^x &= K_1 + K_3 + K_5 + \ldots\end{aligned} \tag{5.14}$$

which confirms Haldane's result in the more general context.

To summarize, the Euclidean path integral for the generalized antiferromagnetic spin chains includes the sign factor, $e^{i\pi N}$, where N is an integer defined by the flux integral,

$$\begin{aligned}N &= \frac{1}{2\pi}\int F^j\,\Lambda_j\\ &= \frac{1}{2\pi}\int F_j\,\Lambda^j\\ &= N_j\,\Lambda^j\,.\end{aligned} \tag{5.15}$$

The weights $\Lambda^j = g^{jk}\,\Lambda_k$ are integers for any finite dimensional unitary representation of G. The coefficients, N_j are integers that characterize the configuration, $\phi^\mu(x)$. Models in which the Λ^j are all even will not be sensitive to the classes of configurations – like the integer spin $SU(2)$ models. Other models, in which one or more of the Λ^j are odd, will be sensitive – they must generalize, in various ways, the half–integer spin $SU(2)$ models.

6. OPERATOR REALIZATION FOR $SU(N+1)$

To resolve the factor ordering ambiguities in the quantized theory it is necessary to construct a realization of the generators $Q_\alpha(\phi)$ in terms of the dynamical variables ϕ_n. The details of such a realization will depend on the nature of the target manifold G/H and the quantum numbers of the reference state, $|\Lambda\;>$. We are not able to give a general solution but we can illustrate a method which works in at least some cases. Here we discuss the realization of $SU(N+1)$ in terms of operators associated with the coordinates of $\mathbf{C}P^N$.

The algebra of $SU(N+1)$ is spanned by the $N(N+2)$ generators, $Q_A{}^B, A, B = 1,2,\ldots,N+1$, which satisfy the commutation rules

$$[Q_A{}^B, Q_C{}^D] = \delta_A^D\,Q_C{}^B - \delta_C^B\,Q_A{}^D \tag{6.1}$$

and the constraint, $Q_A{}^A = 0$. The fundamental representation of this algebra is generated by the $(N+1)$–dimensional traceless hermitian matrices,

$$(Q_A{}^B)_C{}^D = \delta_A^D\,\delta_C^B - \frac{1}{N+1}\,\delta_A^B\,\delta_C^D\,. \tag{6.2}$$

We shall consider cosets of $SU(N+1)$ with respect to the subgroup $SU(N) \times U(1)$ generated by the N^2 operators, $Q_i{}^j$, $i, j = 1, \ldots, N$. The finite transformations, L_ϕ, that represent the cosets can be chosen such that, in the fundamental representation,

$$(L_\phi)_C{}^D = \begin{bmatrix} \sqrt{1-\phi\phi^+} & \phi \\ -\phi^+ & \sqrt{1-\phi^+\phi} \end{bmatrix} \qquad (6.3)$$

where ϕ is an N–component column vector. The coordinates, ϕ_i, on the manifold, $\mathbb{C}P^N$, belong to the fundamental representation of $SU(N)$. It is not difficult to verify that the matrix (6.3) is both unitary and unimodular.

According to the general discussion of Sec.2, the reference state, $|\Lambda>$, from which the coherent states are generated, must be invariant up to a phase under the action of the stability group, $SU(N) \times U(1)$. In particular, it must be a singlet of $SU(N)$. We are therefore restricted to those representations of $SU(N+1)$ which contain such a singlet. In terms of Young tableaux these representations (n_1, n_2) are characterized by n_1 columns with N boxes and n_2 columns with 1 box. Each of these representations contains just one singlet of $SU(N)$. We shall find that the realizations to be obtained depend on $n_1 - n_2$, only.

The Cartan subalgebra of $SU(N+1)$ is spanned by the N elements, $Q_1{}^1, Q_2{}^2, \ldots, Q_N{}^N$, whose eigenvalues define the components of the weight vectors

$$Q_j{}^j\,|\Lambda> = |\Lambda>\,\Lambda_j, \qquad j = 1, \ldots, N.$$

Since the reference state is a singlet of $SU(N)$ these components are all equal

$$\Lambda_1 = \Lambda_2 = \ldots = \Lambda. \qquad (6.4)$$

It is convenient to define the hypercharge operator

$$Y = Q_{N+1}{}^{N+1} = -\sum_1^N Q_j{}^j \qquad (6.5)$$

to stand for the $SU(N)$ singlet generator. In the fundamental representation it is a diagonal matrix,

$$Y_C{}^D = \begin{bmatrix} -1/(N+1) & & & \\ & \ddots & & \\ & & -1/(N+1) & \\ & & & N/(N+1) \end{bmatrix} \qquad (6.6)$$

The value of the hypercharge in the reference state is given by

$$Y|\Lambda> = -\sum_j Q_j{}^j\,|\Lambda> = -N\Lambda\,|\Lambda>.$$

On the other hand, since this state is the singlet component of a representation contained in the direct product of n_2 fundamental representations and n_1 conjugates, it follows that

$$Y = (n_2 - n_1)\,\frac{N}{N+1}$$

or

$$\Lambda = \frac{n_1 - n_2}{N+1}. \qquad (6.7)$$

The $U(1)$–component of the spin connection on $\mathbb{C}P^N$ is defined by

$$L_\phi^{-1}\,dL_\phi = i\,A\,Y + \ldots$$

With the parametrization chosen in (6.3) this gives

$$A = \frac{1}{i}\,\frac{N+1}{2N}\,(\phi^+\,d\phi - d\phi^+\phi). \qquad (6.8)$$

The overlap between neighbouring coherent states is therefore,

$$\begin{aligned} <\phi + d\phi|\phi> &= <\Lambda|L_{\phi+d\phi}^{-1}\,L_\phi\,|\Lambda> \\ &= 1 - i\,A\,<\Lambda|Y|\Lambda> \\ &= 1 + \frac{1}{2}\,(N+1)\Lambda\,(\phi^+\,d\phi - d\phi^+\phi) \end{aligned}$$

where Λ is given by (6.7). Hence the Lagrangian (2.18) in this case takes the form

$$L = \frac{\hbar}{2i}\,(N+1)\Lambda\,(\phi^+\,\partial_t\phi - \partial_t\phi^+\phi) - H(\phi, \phi^+). \qquad (6.9)$$

Canonical quantization gives the commutation rules

$$\begin{aligned} [\phi_i, \phi_j] &= 0 \\ [\phi_i, \phi_j^+] &= -\frac{1}{(N+1)\Lambda}\,\delta_{ij} \\ [\phi_i^+, \phi_j^+] &= 0 \end{aligned} \qquad (6.10)$$

where $(N+1)\Lambda$ is an integer.

To construct the generators out of these operators it is helpful to consider firstly their coherent state expectation values,

$$\begin{aligned} <\phi|Q_A{}^B\,|\phi> &= <\Lambda|\,L_\phi^{-1}\,Q_A{}^B\,L_\phi\,|\Lambda> \\ &= (L_\phi)_A{}^C\,<\Lambda|\,Q_C{}^D\,|\Lambda>\,(L_\phi^{-1})_D{}^B \\ &= -(N+1)\Lambda\,(L_\phi\,Y\,L_\phi^{-1})_A{}^B \end{aligned}$$

where Y is the hypercharge matrix in the fundamental representation (6.6). With the matrices $(L_\phi)_A{}^B$ given by (6.3) one obtains,

$$< \phi |\, Q_A{}^B \,| \phi > \; = \begin{bmatrix} \Lambda - (N+1)\Lambda\, \phi\phi^+ & -(N+1)\Lambda\sqrt{1-\phi^+\phi}\,\phi \\ -(N+1)\Lambda\sqrt{1-\phi^+\phi}\,\phi^+ & -N\Lambda + (N+1)\Lambda\,\phi^+\phi \end{bmatrix} \qquad (6.11)$$

using the elementary identities

$$\sqrt{1-\phi\phi^+}\;\phi = \phi\sqrt{1-\phi^+\phi}$$

and

$$\phi^+\sqrt{1-\phi\phi^+} = \sqrt{1-\phi^+\phi}\;\phi^+ \,.$$

The classical expressions on the right-hand side of (6.11) suggest the following operator realization for $\Lambda < 0$,

$$\begin{aligned} Q_i{}^j(\phi) &= \Lambda\,\delta_i^j - (N+1)\Lambda\;\phi_j^+\,\phi_i \\ &= (\Lambda - 1)\delta_i^j - (N+1)\Lambda\;\phi_i\,\phi_j^+ \\ Q_i{}^{N+1}(\phi) &= -(N+1)\Lambda\;\sqrt{1-\phi^+\phi}\;\phi_i \\ &= -(N+1)\Lambda\;\phi_i\sqrt{1-\frac{1}{(N+1)\Lambda}-\phi^+\phi} \\ Q_{N+1}{}^j(\phi) &= -(N+1)\Lambda\;\phi_j^+\;\sqrt{1-\phi^+\phi} \\ &= -(N+1)\Lambda\sqrt{1-\frac{1}{(N+1)\Lambda}-\phi^+\phi}\;\phi_j^+ \\ Q_{N+1}{}^{N+1}(\phi) &= -N\Lambda + (N+1)\Lambda\;\phi^+\phi\,. \end{aligned} \qquad (6.12)$$

It is easy to verify, using the commutation rules (6.10) that these operators satisfy the algebra of $SU(N+1)$. The reference state is annihilated by ϕ_i. The basis for a finite dimensional unitary representation of $SU(N+1)$ is given by

$$|\Lambda>,\quad \phi_i^+\,|\Lambda>,\quad \phi_{i_1}^+\,\phi_{i_2}^+\,|\Lambda>,\ldots,\phi_{i_1}^+\cdots\phi_{i_p}^+\,|\Lambda> \qquad (6.13)$$

where $p = -(N+1)\Lambda = n_2 - n_1$ is a positive integer. The rest of the Fock space, spanned by vectors with $p+1, p+2, \ldots$ creation operators, carries a non–unitary infinite dimensional representation of $SU(N+1)$.

Similarly, for $\Lambda > 0$ the operator realization is given by

$$\begin{aligned} Q_i{}^j(\phi) &= \Lambda\;\delta_i^j - (N+1)\Lambda\;\phi_i\,\phi_j^+ \\ &= (\Lambda + 1)\delta_i^j - (N+1)\Lambda\;\phi_j^+\,\phi_i \\ Q_i{}^{N+1}(\phi) &= -(N+1)\Lambda\;\phi_i\sqrt{1+\frac{N}{(N+1)\Lambda}-\phi^+\phi} \end{aligned}$$

$$\begin{aligned} &= -(N+1)\Lambda\;\sqrt{1+\frac{1}{\Lambda}-\phi^+\phi}\;\phi_i \\ Q_{N+1}{}^j(\phi) &= -(N+1)\Lambda\;\sqrt{1+\frac{N}{(N+1)\Lambda}-\phi^+\phi}\;\phi_i^+ \\ &= -(N+1)\Lambda\;\phi_i^+\;\sqrt{1+\frac{1}{\Lambda}-\phi^+\phi} \\ Q_{N+1}{}^{N+1}(\phi) &= -(\Lambda+1)N + (N+1)\Lambda\;\phi^+\phi\,. \end{aligned} \qquad (6.14)$$

Here the reference state is annihilated by ϕ_i^+ and the basis for a finite dimensional unitary representation is given by

$$|\Lambda>,\quad \phi_i|\Lambda>,\quad \phi_{i_1}\phi_{i_2}\,|\Lambda>,\ldots,\phi_{i_1}\cdots\phi_{i_p}\,|\Lambda> \qquad (6.15)$$

where $p = (N+1)\Lambda = n_1 - n_2$.

It appears from the structure of the basis sets, (6.13) and (6.15) that the operators (6.12) and (6.14) realize, respectively, the unitary representations $(0, n_2 - n_1)$ and $(n_1 - n - 2, 0)$ but not the general case (n_1, n_2). Our approach clearly leaves something to be desired. This should not be a surprise since we began with the assumption that commutators should be ignored in the first approximation – an assumption that is appropriate to the correspondence theory limit. The resulting formula should therefore be read as corrections to this limiting case valid, we expect, for large n_1 and n_2 such that

$$|n_1 - n_2| \ll n_1 + n_2\,.$$

We believe that it should be possible to construct exact expressions from ϕ and ϕ^+ that realize the general case (n_1, n_2) and which reduce to the forms (6.12), (6.14) when higher powers of $|n_1 - n_2|/(n_1 + n_2)$ are neglected. Probably these expressions will have branch point singularities at $\phi = 0$. In support of the conjecture we have examined a Holstein–Primakoff type realization for the $SU(2)$ case which depends explicitly on the spin, s, and helicity, λ, and which reduces to the standard Holstein–Primakoff form when $\lambda = \pm s$. This realization is discussed in the appendix.

Finally, in applying these formulae to an $SU(N+1)$ lattice model one must construct the Hamiltonian operator, $H(\phi, \phi^+)$ and set up the Feynman rules. Typically, the free propagator, $< T\,\phi_n\,\phi_m^+ >$, will have the Fourier representation

$$G(\omega, k) = \frac{1}{(n_1 - n_2)\hbar\omega - H(k)}$$

and the higher order contributions will be suppressed by powers of $(n_1 - n_2)^{-1}$. Corrections to the classical theory are obtained in the usual way as a loop expansion. Of course, at any finite order the approximate charge operators will give rise to transitions into the unphysical (non–unitary) part of the representation. This defect is endemic to the semiclassical treatment of spin waves.

7. OUTLOOK

In this paper we have begun the investigation of a class of spin systems based on an arbitrary Lie group. The purpose of this study is to develop a better understanding of the role of symmetry in the behaviour of spin systems in general. We believe that it may eventually be possible to generalize some of the rigorous work that has traditionally been concentrated on the $SU(2)$ systems, particularly that which concerns the nature of the ground state: the question of long range order and its dependence on parameters such as spin, coupling strength and the dimensionality of space. At this point we have not addressed such questions. The work described in this paper is of a preliminary nature, being concerned with kinematics and only the simplest approximations: the correspondence theory limit and the naive long wavelength limit. Even so, we believe that the results have some interest.

Firstly, we have shown that the classical theory which emerges in the correspondence theory limit is associated with a flag manifold or one of its submanifolds. These manifolds are not, in general, symmetric spaces although they are homogeneous and can be analyzed by group theoretic methods. At least the preliminary features of the semi-classical theory can be discerned in that it is possible to identify dynamical variables and formulate a canonical quantization programme. The problem of factor ordering, however, has been only partially resolved. To express interesting Hamiltonians as well defined self-adjoint and group invariant operators it will probably be necessary to obtain more general versions of the Holstein–Primakoff realization than we have found. Our version, discussed in Sec.6, is able to realize only a limited class of representations, those for which the weights are not degenerate. These representations are characterized by a single quantum number, $n_1 - n_2$ in the case of $SU(N+1)$. One of our motivations for going beyond $SU(2)$ was to have a richer variety of quantum numbers on which interesting physical quantities might depend. This remains to be achieved.

Secondly, we have shown that the long wavelength properties of the classical theory are described by generalized non-relativistic σ–models in which the fields take their values on a coset space G/H in which H includes the maximal torus. The Lagrangian is of first order in the time derivative but, depending on the nature of the ground state ordering, it may be equivalent to a second order theory. Such second order systems resemble the relativistic σ–models except that the space and time derivatives are coupled to independent tensors, $k_{\mu\nu}$ and $g_{\mu\nu}$, respectively. The flag manifold generally admits more than one invariant tensor of rank 2. This is to be contrasted with the homogeneous symmetric spaces which admit only one: σ–models on symmetric spaces such as $\mathbb{C}P^N$ are inevitably relativistic.

A question that calls for investigation is the long wavelength behaviour of the quantized theory. At present it is not clear whether the factor ordering ambiguities are significant in the continuum limit. If they are not, then it should be possible to set up renormalization group equations and search for fixed points. In particular, it would be interesting, in the case of second order σ–models to see whether the tensors $k_{\mu\nu}$ and $g_{\mu\nu}$ evolve towards the relativistic form, $k_{\mu\nu} = c^2 g_{\mu\nu}$, at

the fixed point. This idea is pursued in a separate paper.

Finally, for generalized spin systems in one space dimension we have shown that the Haldane topological term emerges in the continuum limit of the antiferromagnetic chain, As in the $SU(2)$ case it appears that representations for which the contravariant weight components, Λ^j, are odd integers are distinguished. They contribute an alternating sign to the path integrals.

APPENDIX THE REALIZATION OF $SU(2)$

A generalization of the Holstein–Primakoff realization can be obtained by applying the method discussed in Sec.2. The structures here are sufficiently simple that all formulae can be exhibited in detail and so provide a useful illustration of the method. In particular, we shall be able to see how the Holstein–Primakoff formulae emerge as a special case.

Consider the $(2s+1)$–dimensional representation of $SU(2)$ defined by

$$\begin{aligned} J_3\,|m> &= |m>\, m \\ J_+\,|m> &= |m+1>\,\sqrt{(s-m)(s+m+1)} \\ J_-\,|m> &= |m-1>\,\sqrt{(s+m)(s-m+1)} \end{aligned} \qquad (A.1)$$

where $m = -s, -s+1, \ldots, s$ and $<m|m'> = \delta_{mm'}$. The coherent states are defined by

$$\begin{aligned} |\theta\varphi> &= L_{\theta\varphi}\,|\lambda> \\ &= e^{-i\varphi J_3}\, e^{-\theta J_2}\, e^{i\varphi J_3}\,|\lambda> \end{aligned} \qquad (A.2)$$

where $0 \le \theta < \pi, 0 \le \varphi \le 2\pi$ and $|\lambda>$ is one of the eigenstates of J_3. The expectation value of $\underline{J}$ is a 3–vector of length $|\lambda|$,

$$\begin{aligned} <\theta\varphi|J_\pm|\theta\varphi> &= \lambda \sin\theta\, e^{\pm i\varphi} \\ <\theta\varphi|J_3|\theta\varphi> &= \lambda\cos\theta\,. \end{aligned} \qquad (A.3)$$

Applying the methods of Sec.2, computing the infinitesimal everlap of coherent states and expressing the transition amplitudes in path integral notation one arrives at the classical Lagrangian

$$L = \hbar\lambda(\cos\theta - 1)\,\partial_t\varphi - H(\theta,\varphi) \qquad (A.4)$$

where $(\cos\theta - 1)\,d\varphi$ is the spin connection on the 2–sphere in polar coordinates. The canonical variables are

$$\varphi \quad \text{and} \quad \pi = \hbar\lambda(\cos\theta - 1)\ .$$

In the quantized theory they are represented by operators subject to the commutation rule

$$[\pi,\varphi] = \frac{\hbar}{i}$$

or, more precisely, since states should be represented by periodic functions of φ,

$$e^{i\varphi}\,\cos\theta\, e^{-i\varphi} = \cos\theta - \frac{1}{\lambda}\,. \qquad (A.5)$$

In the basis which diagonalizes φ the eigenstates of J_3 are represented by

$$<\varphi|m> = e^{im\varphi}\,. \qquad (A.6)$$

The canonical momemtum can be represented by the differential operator

$$\pi = -\hbar(i\partial_\varphi + \lambda) \qquad (A.7)$$

which gives $\lambda\cos\theta = -i\partial_\varphi$. With this choice one finds, using (A.1) and (A.6),

$$\begin{aligned} J_3 &= \lambda\cos\theta \\ J_+ &= e^{i\varphi}\,\sqrt{(s-\lambda\cos\theta)(s+\lambda\cos\theta+1)} \\ &= \sqrt{(s-\lambda\cos\theta+1)(s+\lambda\cos\theta)}\; e^{-i\varphi}\,. \end{aligned} \qquad (A.8)$$

It is straightforward to verify that the operators (A.8) satisfy the commutation rules of $SU(2)$ as well as the constraint

$$\frac{1}{2}(J_+J_- + J_-J_+) + J_3^2 = s(s+1)\ . \qquad (A.9)$$

It is clear form (A.5) that the operators φ and θ become commutative in the limit $|\lambda| \to \infty$. Also the large quantum number behaviour of the operators (A.8) approximates the classical formulae (A.3). For example, under the assumptions

$$s^2 - \lambda^2 \lesssim O(|\lambda|) \quad \text{and} \quad |\lambda|\sin^2\theta \gg 1 \qquad (A.10)$$

as s and $|\lambda|$ become large, one finds the expansion

$$J_+ \simeq \lambda\, e^{i\varphi}\sin\theta\left[1 + \frac{s(s+1) - \lambda^2 - \lambda\cos\theta}{2\lambda^2\sin^2\theta} - \frac{1}{8}\left(\frac{s(s+1) - \lambda^2 - \lambda\cos\theta}{\lambda^2\sin^2\theta}\right)^2 + \ldots\right]. \qquad (A.11)$$

To recover the usual form of the Holstein–Primakoff realization define the operator

$$\begin{aligned} \phi &= e^{i\varphi}\sqrt{\frac{1-\cos\theta}{2}} \\ &= \sqrt{\frac{1-\frac{1}{\lambda}-\cos\theta}{2}}\; e^{i\varphi} \end{aligned} \qquad (A.12)$$

and its hermitian conjugate, ϕ^+. The commutation rule becomes

$$[\phi,\phi^+] = -\frac{1}{2\lambda} \qquad (A.13)$$

which is the $SU(2)$ version of (6.10). The definition (A.12) can be inverted to give

$$\cos\theta = 1 - 2\phi^+\phi \quad \text{and} \quad e^{-i\varphi} = \phi\,\frac{1}{\sqrt{\phi^+\phi}} = \frac{1}{\sqrt{\phi\phi^+}}\,\phi\,. \qquad (A.14)$$

Hence, substituting into (A.8),

$$\begin{aligned} J_3 &= \lambda - 2\lambda\phi^+\phi \\ J_+ &= \phi^+ \sqrt{\frac{(s+\lambda-2\lambda\phi\phi^+)(s-\lambda+2\lambda\phi^+\phi)}{\phi\phi^+}} \\ &= \sqrt{\frac{(s+1-\lambda+2\lambda\phi^+\phi)(s+\lambda-2\lambda\phi^+\phi)}{\phi^+\phi}}\,\phi^+ \\ J_- &= J_+^+ \,. \end{aligned} \tag{A.15}$$

The singularity at $\phi = 0$ in these formulae is eliminated by choosing $s = \pm\lambda$. For example, with $\lambda = -s$, (A.15) reduces to the familiar form

$$\begin{aligned} J_3 &= -s + 2s\phi^+\phi \\ J_+ &= 2s\,\phi^+\sqrt{1-\phi^+\phi} \\ &= 2s\sqrt{1+\frac{1}{2s}-\phi^+\phi}\,\phi^+ \end{aligned} \tag{A.16}$$

with $[\phi, \phi^+] = 1/2s$. This is the $SU(2)$ version of the realization (6.12). On the basis of this simple example we are encouraged to believe that it may be possible to generalize the operators (6.12) and (6.14) to yield the representation (n_1, n_2) of $SU(N+1)$.

REFERENCES

1) For reviews see, for example,
I. Affleck, in *Strings, Fields and Critical Phenomena*, Les Houches Summer School, 1988, Session XLIX, edited by E. Brezin and J. Zinn–Justin (North Holland, 1990);
E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison–Wesley, Redwood City, 1991);
E. Manousakis, Rev. Mod. Phys. **63**, 1 (1991).

2) I. Affleck, Nucl. Phys. **B257**, 397 (1985);
I. Affleck, Nucl. Phys. **B265**, 409 (1986).

3) H.A. Bethe, Z. Phys. **71**, 205 (1931);
R.L. Orbach, Phys. Rev. **112**, 309 (1958).

4) C. Kittel, *Quantum Theory of Solids* (Wiley, New York, 1963).

5) F.D.M. Haldane, Phys. Lett. **93A**, 464 (1983); Phys. Rev. Lett. **50**, 1153 (1983);
See also Ref.2.

6) See, for example, the discussion in Sec.1 of
S. Chakrabarty, B.I. Halperin and D.R. Nelson, Phys. Rev. **B39**, 2344 (1989).

7) See, for example, the review by Manousaki, Ref.1.

8) F.D.M. Haldane, Phys. Rev. Lett. **57**, 1488 (1986);
See also A. Perelomov, *Generalized Coherent States and their Applications* (Springer, 1986).

9) F.D.M. Haldane, J. Appl. Phys. **57**, 3359 (1985).

10) E. Fradkin and M. Stone, Phys. Rev. **B38**, 7215 (1988);
F.D.M. Haldane, Phys. Rev. Lett. **61**, 1029 (1988);
X.G. Wen and A. Zee, Phys. Rev. Lett. **61**, 1025 (1988);
D. Dombre and N. Read, Phys. Rev. **B38**, 7181 (1988).

11) T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

12) A. Belavin and A. Polyakov, JETP Lett. **22** (1975) 245;
A. D'Adda, P. di Vecchia and M. Lüscher, Nucl. PHys. **146B**, 63 (1978);
M. Lüscher, Nucl. Phys. **135B**, 1 (1978);
H. Eichenherr, Nucl. PHys. **146B**, 250 (1978): **155B**, 544 (1979).

13) E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics, Part 2* (Pergamon Press, Oxford, 1980).