THERMODYNAMICS OF A DEFORMED BAG
AND QUARK-GLUON DECONFINEMENT PHASE TRANSITION

A. Ansari

and

M.G. Mustafa

MIRAMARE-TRIESTE
I. Introduction

In heavy ion collisions with now available projectile energies up to 200 GeV/particle [1] a hot central region is formed which is generally believed to consist of a quark-gluon plasma. As the hot zone expands and cools down, a phase transition to a hadron gas (deconfined to confined quark-hadron [Q-H] phase) takes place around a temperature, $T_c$. Initially, the energy density is stored mostly in hadronic resonances [2]. A dynamic theory of this Q-H phase transition is a very interesting problem, not properly understood in view of the very small space-time dimensions involved. The physics of Q-H phase transition is presently an object of intense studies by many authors [3-5].

However, even in view of the present day ultrarelativistic heavy ion collisions, it is not yet unambiguously established that hadron to quark-gluon plasma (QGP) phase transition does take place. Recent extensive lattice gauge calculations [6] have cast doubt about it. In a very recent model calculation Brown, Reche and Pizzochero [7] have shown that the hadron and quark/gluon do not coexist at a common temperature, which means no transition to a QGP phase. Thus, it still remains an open problem if there is at all a QGP phase transition. To find out the value of the QGP transition temperature $T_c$, usually a two phase model is considered [3,4,8,9,10] where, according to the Gibbs criterion, pressure of the relativistic quark-gluon gas is assumed to be equal to that of the hadron gas at $T_c$. Ultimately $T_c$ is related to the bag pressure constant, $B$. But there have also been a kind of single...
phase models directly in terms of the MIT bag model [11-14]. In such studies the effects of finite size of the system have been considered important by many authors [13-17]. Restriction to colour singletness, the most dramatic effect of the QCD interaction, has also been emphasised by many [8,14,18,19].

In our recent work [14], we have been able to take into account simultaneously the effects of colour neutrality, finite size and non-zero chemical potential, in the statistical approach constructing a grand canonical partition function in terms of quark (massless) and gluon (Abelian) eigenmodes inside a spherical bag. Instead of going to the continuum limit all the quantities are computed numerically for the finite size bag. In a variational sense, we look for the extrema of the free energy, \( F(T,R) \) as a function of the bag radius, \( R \), at various values of \( T \). A stable bag radius \( R_0(T) \) is defined corresponding to the minimum of \( F(T,R) \) for a given \( T \). The lowest value of \( T \) for which \( F(T) \) has no more a minimum but for \( R \rightarrow \infty \), is defined as \( T_x \) because now hadron ceases to exist as a finite bound system.

Recently Ellis and Olive [3] have extended their effective Lagrangian approach [20] to \( \mu \neq 0 \) case and have calculated the Q - H phase boundary in the \( \mu - T \) plane. Besides getting a temperature \( T_x \), corresponding to the appearance/disappearance of the gluon condensate, they find an other value \( T_C \) where the local minimum in the effective potential for the gluon condensate parameter disappears. We will discuss more about it later on to see if there is any common feature between this \( T - T_x \) band and the \( T - T_x \) region in the \( \mu - T \) plane which we get.

In the relativistic heavy ion collisions, where hot and compressed matter zone is formed, it is likely that the intrinsic shape of the hadrons will get deformed. Also temperature induced deformation can develop when quarks and gluons populate large angular momentum states during their intrinsic excitations. In fact, even at zero temperature it is expected [21,22] that the intrinsic shape of the bag would be deformed due to non-isotropic pressure of gluons on the bag surface. It will also introduce more degrees of freedom in the hadronic phase [7].

Now coming to the present work, here we have extended the approach of Ref.[14] to the case of a deformed bag with spheroidal shape. For this, simply the quark and gluon single-particle (sp) energies are needed for a deformed bag. Then a deformation parameter dependent \( T_s \) and \( T_c \) can be calculated.

In the next section we sketch very briefly the computational method of quark and gluon sp energies in a spheroidal cavity following the work of Viollier, Chin and Kerman [22]. Sec. III contains a brief outline of our calculations within the statistical approach including the constraint of colour singletness. In Sec.IV we discuss our results, and present a brief conclusion in Sec.V.

II. Quark and gluon eigen modes in a spheroidal cavity.

Single particle energies and wave functions for massless quarks confined in a spheroidal cavity have been calculated
by Hahn, Goldflam and Wilets as well as by Vasak et al. [23] using the linear boundary condition and writing a deformed sp wavefunction as a superposition of spherical solutions in the spirit of Nilsson model calculations in nuclear physics [24]. Around the same time Viollier, Chin and Kerman [22] developed an elegant formalism using the so-called affine transformation [25] for the sp energies of quarks and gluons both in the spheroidal cavity. In this approach the problem is reduced to diagonalization of a real non-symmetric matrix.

We need not give details of the formalism here as these are rather lengthy, but very clearly presented in Ref.22. However, in order to introduce the deformation parameter, \( D \), we will write down some of the equations. In the body fixed frame, the boundary of a spheroidal cavity is given by

\[ \frac{x^2}{D^2} + \frac{y^2}{D^2} + \frac{z^2}{R^2} = 1 \]  

where, \( x^2 + y^2 + z^2 = R^2 \)  

With the above choice, the semi-major and semi-minor axes of the spheroid are \( D R \) and \( R / D \), respectively, with a volume clearly independent of \( D \). A radial coordinate at the surface of the spheroid is given by

\[ R(\phi) = R_0 \left[ 1 + \alpha^2 P \left( \cos \phi \right) \right] \]

where, \( P \) is the standard Legendre polynomial of order 2.

Now we come to the problem of sp energies and would like to make some remarks without giving any equations. As already indicated by the authors [22], some complex eigenvalues are not ruled out. In the range of angular momentum states computed in Ref.22 and, in fact, up to much beyond, the eigenvalues are real. However, for the purpose of present thermodynamic calculations where we want to go up to temperature \( T \sim 200 \text{ MeV} \), very high energy and large angular momentum states of the bag need to be populated; e.g. \( j = 41/2 \). For \( D > 2 \) or \( D < 0.6 \) occasionally some eigenvalues are coming out to be complex with very small imaginary components; e.g. \( 20.038 \pm 0.008 \) which is in the units of \( \hbar c / R \) with \( \hbar c = 197.31 \text{ MeV} \). The worst number is \( 7.916 \pm 0.030 \) for \( D = 2.4 \). This is true for quarks as well as gluons. In such cases we have simply ignored the imaginary component hoping that in hundreds of states involved, this should not introduce any serious error. One could check how much these states contribute to the partition function and so on.

We are looking at the possible causes of complex eigenvalues at large angular momenta and large deformations so that, in future, the effects of higher deformations, particularly on the oblate side, could be explored.
III. Statistical mechanics for the deformed bag.

First of all we should clarify a few things regarding the deformed shape of the hadronic bag. According to the above Eq. 5, and keeping the volume independent of the deformation parameter $\alpha^2$ (defined in Eq. 6) for a given spherical radius $R_0$, we rewrite it as

$$R(\theta) = R_0 f_V(\alpha^2) [1 + \alpha^2 \beta_2(\cos\theta)] \quad (7)$$

where,

$$f_V(\alpha^2) = \left[1 + \frac{3}{5} \alpha^2 + \frac{2}{15} \alpha^2 \right]^{1/3} \quad (8)$$

It is known that for a fixed volume, if the shape is deformed the rms radius is increased for which we define

$$\langle R^2 \rangle = \alpha^2 \int_0^\pi d\phi \sin\phi f_V(\alpha^2) \frac{1}{2} \int_0^{2\pi} R(\phi) \sin\phi d\phi \quad (9)$$

$$= R_0^2 \int_0^{2\pi} \frac{1}{2} R(\phi) d\phi \quad (10)$$

where,

$$f_V(\alpha^2) = \frac{1 + \frac{1}{5} \alpha^2 + \frac{3}{25} \alpha^2 + \frac{15}{125} \alpha^2}{1 + \frac{1}{5} \alpha^2} \quad (11)$$

As will be discussed later, the rms value $\langle R^2 \rangle^{1/2}$ will be used for an approximate computation of the Casimir energy of the deformed bag.

In the following subsection we wish to write down some of the useful formulae without any restriction to colour singlet states. Then the colour projection will be included in the next subsection.

A. COLOUR UNPROJECTED SCHEME

In the grand canonical formalism, a partition function for the bag containing quarks and gluons can be written as

$$Z_{\text{vac}} = \frac{Z_{\text{vac}}}{T} \quad (12)$$

where $Z_{\text{vac}}$ takes care of the temperature $T \to 0$ limit of the bag, and we take

$$-T \ln Z_{\text{vac}} = B V + d/R \quad (13)$$

$B V$ is the volume energy and $d/R$ is the Casimir energy without any explicit $T$ dependence. As will be seen later, $R$ can depend on $T$ implicitly. $B$ is, of course, the usual bag pressure constant. $Z_{\text{vac}}$ refers to the gluon part and $f$ denotes the number of quark flavours, presently $u$ and $d$ without any distinction. The thermodynamic potential $\Omega$ at $T (= 1/\beta)$ with chemical potential $\mu$ for quarks (baryon chemical potential $\mu_b = 3\mu$) and $-\mu$ for antiquarks is given by

$$\Omega_{\text{vac}} = \ln Z_{\text{vac}} = \sum_i \ln (1 + e^{(\xi_i^q - \mu)}) + \sum_i \ln (1 + e^{(-(\xi_i^q + \mu)})$$

$$- \sum_i \ln (1 + e^{-(\xi_i^q)}) \quad (14)$$

where $\xi_i^q$ and $\xi_i^q$ are, respectively, the quark and gluon sp energies in the bag which is axially deformed if $D \neq 1$.

A baryon number $b = 1$ is imposed by adjusting $\mu_b$ such that the excess of the number of quarks over the number of antiquarks is $3b$. Namely

$$N = N_q - N_{\bar{q}} = 3b \quad (15)$$

Then the free energy of the whole system can be written as

$$F(T, R) = -T \Omega_{\text{vac}} + \mu N + B V + d/R \quad (16)$$
The pressure generated by the quark - gluon gas \( P = \left( \frac{3 \pi^2}{aV} \right) T^4 / N \)
is balanced by the bag pressure \( B \).

As in Ref.14, to study the stability of the system, the variation of the free energy, \( F \), for a fixed \( T \) and \( D \), as a function of the radius parameter \( R \), can be made. Since with \( D \neq 1 \), \( R \) is not actually the radius of the bag we will call it a radius parameter, as entered in, for instance, Eq.7. It may be reminded that sp energies are in the units of \( R \) and the volume of the bag can be changed by varying \( R \) only. However, there is some problem in computing the Casimir energy term \( d / R \) in Eq.16. For a fixed value of \( R \), the surface area as well as the rms radius of the bag increases with the increase of \( D \). It is not clear how to account for this. We consider \( d \) to be independent of \( T \). One choice is to go ahead with \( 1 / R \) in place of \( 1/R \) in Eq.16, or take the inverse of an rms value computed as a function of \( D \). Clement and Maamache [26] have studied the dependence of zero-point energy on the deformation parameter \( D \). They do find it decreasing with the increase of prolate or oblate deformation (e.g., see Fig.1 in Ref.26). Thus, we have computed the Casimir energy as \( d / R \) in Eq.16. Using expression (2) in (9) is a bit complicated in the sense that the required integral has to be computed numerically. So, for the time being we have just used Eq.10 to get the inverse of the rms radius, \( < R^2 >^{1/2} \). Numerical results will be discussed in a separate section.

Here we just want to mention that at a \( T \) when quarks and gluons are still confined \( F \) vs \( R \) plot gives a minimum at a value, say, \( R_{\text{min}} \) (\( T \)). But finally at a certain larger value of \( T \), when \( R \) (\( T \)) \(-\to\infty \), then obviously the bag becomes unstable, and we can call this as the deconfining transition temperature, \( T_C \).

**R. COLOUR SINGLET PARTICION FUNCTION**

The most important effect of the QCD interaction is that only those states exist which are colour singlet. Thus, imposing this restriction on the total partition function of the quark - gluon system should retain a large part of the non-perturbative aspects of the interaction. The partition function is essentially given as in Eq.12, but it is no longer a simple product of individual colour singlet terms. The colour projection has to be carried out simultaneously for quarks and gluons both leading to an over-all colour correlation.

The method we follow here is that of Gorenstein et al [27] and Auberson et al [28] which was followed in Ref.14. Without giving any formal details, we first write down the thermodynamic potential as a function of the invariant group parameters \( \theta_i \) obeying the constraint \( \sum_i \theta_i = 0 \).

\[
\Omega(\theta) = \sum_{i} \sum_{K} \frac{1}{K} \{ \left[ (N_{c}^{-1} + 2 \cos K(\theta_{i} - \theta_{j}) \right] \\
+ 2 \cos K(2\theta_{1} + \theta_{2}) \cos K(\theta_{1} + 2\theta_{2}) \}^{K/2} \\
- \frac{(-1)^{K}}{e^{K}} \left[ e^{-Kx_{1}^{2}/N} + e^{-Kx_{2}^{2}/N} \right] \\
+ \sum_{K} \left[ e^{Kx_{1}^{2}/N} + e^{Kx_{2}^{2}/N} \right]
\]

(17)

In the above \( N \) is the number of colours, equal to 3 here.

Then the colour singlet partition function \( Z_{C} \) is evaluated after performing the integration with the Haar measure \( d\theta(g) \) of the invariant group

\[
Z_{C} = \int d\theta(g)^{N} \Omega(\theta)
\]

(18)

with
\[
\int d\theta (\theta) = \frac{1}{2\pi} \int d\theta_1 d\theta_2 \int d\theta_1' d\theta_2' \left( 2 \sin \frac{1}{2} (\theta_1 - \theta_1') \right)^2 \quad (19)
\]

Now most of the physical quantities can be calculated using this \( Z \). However, as should be obvious, for the computation of every quantity a separate numerical integration has to be carried out. The colour singlet thermodynamic potential is given by

\[
\Omega_0 = \ln \left( \frac{Z}{\theta_0} \right) \quad (20)
\]

The total energy of the system in general is given by

\[
\mathcal{E}(T) = T^2 \sqrt{2 \beta c} + \beta N + B V + d < R^2 > \quad (21)
\]

At the extremum of \( \mathcal{F} \) where the pressure vanishes, the energy is given by

\[
\mathcal{E}(T) = 4 B V = \frac{16}{3} B R(T) \quad (22)
\]

Putting this value of \( R(T) \) in Eq.10 the deformation dependent rms radius of the bag can be determined.

**IV. Results and Discussion**

The physical behaviour of the system at a temperature \( T \) is governed by the properties of its free energy \( F(T) \). Treating the quark - gluon bag at a high \( T \) like a many - body system, its stability features are studied by plotting \( F(T) \) as a function of \( R \); very much like the variation of free energy of a hot fissionable nucleus as a function of quadrupole deformation parameter. Corresponding to a rather large value of the deformation parameter \( D = 2 \), a plot of \( F(T) \) vs \( R \) is displayed in Fig.1 for \( B = 200 \) MeV. Out of the three curves shown, marked by the values of \( T = 143, 144, 145 \) (all in MeV), the two with lower values of \( T \) have got two extremum points; a minimum at a smaller value of \( R \) and a maximum at a larger one. If the value of \( T \) is quite small ( \( T < 50 \) MeV) then the maximum point is essentially at \( R \to \infty \) and we have a stable hadron. Physically a meaningful maximum appears at \( R = 2 \) fm with \( \beta \to 0 \) for \( T = 125 \) MeV (for \( D = 2 \)). We define this value of \( T \) for which \( \beta \to 0 \) as \( T_c \). The corresponding temperature for a spherical shape \( (D = 1) \) is about 140 MeV \( (14) \). As is clear from Fig.1, the maximum disappears at \( T = 145 \) MeV and one may say that suddenly the minimum moves to \( R \to \infty \) leading to deconfinement of the quarks and gluons of the bag. Thus for \( D = 2 \) we get \( T_c \simeq 145 \) MeV compared to 165 MeV for \( D = 1 \) (spherical shape).

We may note that for \( D = 2 \) the difference between \( T \) and \( T_c \) is about 20 MeV, rather comparable to the spherical case of about 25 MeV (165 MeV - 140 MeV).

In Table.1 the variation of transition temperature as a function of \( D \) is enumerated for the colour unprojected as well as colour projected (singlet) case, the latter being indicated by a superscript '0'. There is a sizable decrease of \( T \) for the values of \( D \) away from unity. For the sake of visual appreciation a plot of \( T \) vs \( \alpha_2 \) ( \( \alpha_2 = D - 1 \) ) is displayed in Fig.2. Around \( \alpha_2 \simeq 2 \) the rate of decrease of \( T \) becomes negligible. At this stage we would like to point out that already for \( \alpha_2 \geq 0.50 \), the ratio of the major to minor axes of a spheroid becomes about 2:1 \( (24) \). Then for this order of deformation the decrease in \( T \) is of the order of 5 - 10 MeV (see Table.1). Now, the question is as to how significant is this decrease. The most relevant quantity for the physical consideration is
the energy density which goes like the fourth power of temperature. Therefore, a small uncertainty in the value of $T$ or $\mu$ at the boundary of the stability region ($P = 0$), means a sizable uncertainty in the value of the critical energy. In view of the same argument, one may be again reminded that the colour singlet restriction causes the increase in the predicted value of $T$ by as much as about 20 MeV (see Table I).

Next we come to another important result showing the standard plot of $\mu$ vs $T$ (values of $\mu$ and $T$ at the boundary $P = 0$ line where $F$ has extrema) in Fig. 3. Like in Fig. 1 these points also correspond to $D = 2.0$ and the singlet and colour unprojected cases are labelled as I and II, respectively. On curve I the arrow indicates the value of temperature ($T$) for which $R \rightarrow \infty$ and the bag boundary completely melts away. In the min temperature range $T < T < T^c$ there are two possible values of $\mu$ for each $T$, the lower one corresponding to the maximum point of $F$ implying a large metastable bag (baryon). We should emphasize that the increase of $T - T^c$ to about 20 MeV, producing this metastable region, is entirely due to the restriction to colour singletness.

In Fig. 3 we have indicated by a dashed line A-B a constant $-T^c$ curve in the range $T \rightarrow T^c$. In an effective lagrangian approach recently Ellis and Olive [3] have studied the nature of the boundary between the hadronic and QGP phase in the $\mu - T$ plane (see Fig. 1 in Ref. 3). They find a temperature $T^c$ (denoted by $T^c$ in Ref. 3) at which the gluon condensate disappears and a higher value $T^c$ beyond which no hadron can exist. Thus, they get a $T^c - T$ band (model dependent) in the $\mu - T$ plane which may belong to a superheated mixed hadron phase. To us there seems to be a parallel between this $T^c - T$ band and our $T^c - T$ region with $T$ equivalent to our $T^c$. In Fig. 3 the two arrows around the line A-B indicate the path of superheating/supercooling.

As mentioned in the introduction, Brown, Bethe and Pizzochero [7] have found, in a model study, that even if the pressure between the hadronic and QGP phases are equal, the temperatures are not. On the hadronic side they get $T = 128$ MeV and on the QGP side $T = 172$ MeV. Thus, without a common $T$ the two QG phases cannot coexist. For the feasibility of a phase transition they suggest the scenario of a phase jump through non-equilibrium processes. In our calculation we predict a transition temperature. However, if the hadron temperature can rise to $T^c$ (around point B in Fig. 3) then it may absorb enough latent heat to reach the point A on the left. Depending upon the dynamical conditions, like in a nuclear fission, the quarks and gluons can become deconfined, and with a further deposit of energy their temperature can rise to $T^c$.

Once the QGP is formed, the hadronization is possible following just the reverse path. But this is only showing the path. A real theory has to contain many complex dynamic processes for nucleation and hadronization.
V. Conclusions:

In a single phase model studying the thermodynamics of a finite size hadronic bag we have seen the effects of restriction to colour singlet states and that of shape deformation on the QGP phase transition temperature.

The effect of deformation is to decrease the value of $T\text{ }_{\text{C}}$, though not drastically. On the other hand, restriction to colour singlet states leads to an enhanced $T\text{ }_{\text{C}} - T\text{ }_{\text{S}}$ temperature region of the metastable states (the temperature difference not much affected by deformation) which may play a significant role for the QGP phase transition, as well for the reverse process of hadronization. Thus according to these findings, the possibility of quark-gluon deconfinement exists even without hadron's temperature rising to $T\text{ }_{\text{C}}$.

ACKNOWLEDGEMENTS.

We are most grateful to Drs. Jishnu Dey and Mira Dey for suggesting us to think about the deformed bag case and then for many discussions in the process. AA is grateful to Prof. Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

References:

4. H. reaves, Phys. Rep. 201 (1991) 335, and references there in
   65 (1990) 2491
   B 263 (1991) 337
8. J. Rafelski and M. Danos, Lecture Notes in Physics, vol.231
   (Springer-Verlag, 1985) 361
    J. I. Kapusta, NUCLEAR PHASE TRANSITION AND HEAVY ION REACTIONS
    (World-Scientific, 1987) eds. T.T.S. Kuo, D. Strottman and
    S. S. Wu, p.113
    C 22 (1984) 189
14. A. Ansari, J. Dey, M. Dey, P. Ghose and M. A. Matin, Hadronic
    J. Suppl. 5 (1990) 233
Transition temperature $T$ as a function of the $C$ deformation parameter $D$. The shape is oblate, spherical, or prolate, respectively as $1 > D > 1$.

<table>
<thead>
<tr>
<th>$D$</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>139.0</td>
<td>145.0</td>
<td>146.0</td>
<td>145.0</td>
<td>142.0</td>
<td>137.0</td>
<td>132.5</td>
<td>129.0</td>
</tr>
</tbody>
</table>

$C$

$T$  | 157.0 | 163.5 | 165.0 | 163.5 | 160.0 | 155.0 | 150.0 | 145.0 |

$C$
Figure Captions

Fig. 1. Variation of the colour projected free energy, $F(T)$, as a function of the radial parameter $R$ for the value $D = 2$. For each curve the corresponding temperature (in MeV) is indicated. The bag pressure constant $B^{1/4} = 200$ MeV.

Fig. 2. Variation of the deconfinement temperature $T$ with the deformation parameter $\alpha_c = D - 1$. The colour projected (singlet) and unprojected curves are labelled as I and II, respectively. At a given value of $\alpha_c (D)$ the singlet value is higher than the other one by about 15 - 20 MeV.

Fig. 3. Chemical potential, $\mu$ vs temperature, $T$, plot defining the boundary $P = 0$ line for the colour projected as well as unprojected case with $D = 2$. On curve I the top most point marked by an arrow and $T$ denotes the value of the temperature for which minimum $C$ in $F(T)$ vs. $R$ moves to $R --> \infty$ (see Fig. 1, $T = 145$). The line A - B designates a fixed temperature path in the metastable region of the temperature $(T, T_S)$. For more details see the text.
Fig. 2

Fig. 3