ABSTRACT

The modules of the orthosymplectic Lie superalgebra $osp(3/2)$, induced from finite-dimensional irreducible submodules of the stability subalgebra $so(3)\mathfrak{gl}(1)$ are investigated. The corresponding to them infinite-dimensional irreducible or indecomposable modules, the Kac modules and the related typical and atypical modules are studied in details. Every such module is decomposed into a direct sum of either indecomposable or irreducible modules of the even subalgebra $so(3)\mathfrak{sp}(2)$. For each of these (infinite-dimensional or finite-dimensional, irreducible or indecomposable) modules relations are written down, giving the transformations of the basis under the action of the algebra generators.
I. INTRODUCTION

The orthosymplectic Lie superalgebras (LS's) have various applications in quantum physics. Already such fundamental objects as the position and the momentum operators $q_i, p_i$ of an n-dimensional quantum system are the odd generators of the orthosymplectic Lie superalgebra (LS) $osp(1/2n) = B(0/n)$, the even generators being all their anticommutators.\(^1\) The creation and the annihilation operators (CAO's) of any Bose field are odd generators of an infinite-dimensional LS $B(0/\infty)$; a pair of mutually anticommuting Bose and Fermi fields generate $B(\infty/\infty)$ if the Fermi field is postulated to be from the even sector.\(^2,3\) Several other applications of orthosymplectic LS's in various superfield theories,\(^4\) supersymmetric quantum mechanics\(^5\), nuclear physics, etc. are also available.

In the applications one usually needs the representations of the algebra under consideration. Unfortunately the purely mathematical problem to determine the representations (or, say, only the finite-dimensional irreducible representations) of the orthosymplectic LS's is at present far from being solved. It is much less developed even in comparison to the other big class of basic\(^7\) LS's, namely $sl(m/n)$, where by now several questions concerning the finite-dimensional irreducible representations (matrix elements, structure of the atypical modules, branching rules, character formulas, etc.) have been answered (for some of the results along this line see Refs.8-33).

Historically the first representations of a LS, namely of the algebra $osp(1/2)$ were found by Wigner\(^34\) in 1950. These are infinite-dimensional irreducible representations and the corresponding to them modules are state spaces of an one-dimensional noncanonical harmonic oscillator. About the same time the parastatistics\(^35\) have been invented. Later it became clear that any $n$ pairs of para-Bose CAO's generate (in a sense of a free algebra with relations) the universal enveloping algebra of $osp(1/2n)$.\(^1\) Consequently, the representation theory of the para-Bose statistics is simply another language for the representations of the orthosymplectic LS's. In particular the known representations of $n$ pairs ($n$ could be $\infty$) of para-Bose CAO's $p_i^\dagger$ with order of the statistics $p_i^\dagger$ describe a small (but important for physical applications) class of infinite-dimensional irreducible highest weight representations of $osp(1/2n)$. Other infinite-dimensional oscillator representations of $osp(m/n)$ are also available (see Ref.37 and the references therein).

Certain finite-dimensional modules of certain $osp(m/2n)$ algebras were investigated by several authors. Explicit expressions however for the matrix elements of the generators for all finite-dimensional irreducible modules (fidirmods) have been obtained so far only for the low rank algebras $osp(1/2)^3, osp(2/2)^{36,39}$ and $osp(3/2)^{38,40}$. A comprehensive analysis of the transformation properties of the $osp(m/2n)$ fidirmods was recently carried out by Le Blanc and Rowe\(^38\). Using the vector coherent state theory these authors succeeded to go far ahead in expressing the matrix elements of the generators. Still their results are given in terms of certain unknown jet entities. The reduced matrix elements of the even subalgebra $so(m)\otimes sp(2n)$ depend on the unknown branching rules in the decomposition $osp(m/2n)\otimes so(m)\otimes sp(2n)$ and the related to it multiplicity problem. Even in the multiplicity free case of the para-Bose algebra $osp(1/2n)$ one needs in addition the $sp(2n)\otimes gl(n)$ reduced Wigner coefficients, the Clebsch-Gordon coefficients of $sp(2n)$ or at least of $gl(n)$, a basis and its transformation within the $sp(2n)$ fidirmods - problems that are still waiting to be solved. Let us mention in this respect that the last problem was solved more than 40 years ago for all other classical Lie algebras.\(^41,42\) It is natural therefore that Le Blanc and Rowe\(^40\) have written explicit analytical expressions only for the matrix elements of $osp(1/2)$, $osp(2/2)$ and $osp(3/2)$ fidirmods, the LS's for which all above mentioned problems are settled.

The present paper deals also with the representations of the LS $osp(3/2)$. Our results can be considered to a large extent as complimentary to the the analysis of Van der Jeugt\(^40\) of the finite- and infinite-dimensional irreducible representations of $osp(3/2)$. The approaches are however different. Van der Jeugt\(^40\) has used the shift operator technique, developed by Hughes and Yadegar\(^43\). In our case we begin with the induced representation method of Kac\(^47\), which, depending on the representation of the even subalgebra,
leads to an infinite-dimensional irreducible or indecomposable
osp(3/2) module \( \tilde{W}(p,q) \), labelled with the numbers \( p, q \). Any such
module has a natural basis (we call it induced basis); one easily
writes its transformations under the action of the generators. The
transformations of the reduced (with respect to the even subalgebra
so(3) \( \oplus \) sp(2)) basis, which we introduce later, are consequence of
(i) the transformations of the induced basis and (ii) the relations
between the induced and the reduced bases. In terms of the reduced
basis one easily describes the invariant subspace \( \tilde{W}_{\text{inv}}(p,q) \) of
\( \tilde{W}(p,q) \) carrying an infinite-dimensional irreducible or
indecomposable representation of \( \text{osp}(3/2) \), the finite-dimensional
Kac module\(^{28} \) \( W(\rho,\gamma) = \tilde{W}(p,q) / \tilde{W}(p,q)_{\text{inv}} \) (carrying also an
irreducible or indecomposable representation of \( \text{osp}(3/2) \)) and,
finally, the irreducible \( \text{osp}(3/2) \) submodule of \( W(\rho,\gamma) \) (which
differs from the Kac module only in the atypical case\(^{9} \)).

In Sec.II we recall the definition of the orthosymplectic
LS’s and introduce the notation. Section III is devoted to the
representations of \( \text{osp}(3/2) \). First we define the modules, induced
from finite-dimensional irreducible representations of the
subalgebra so(3) \( \oplus \) sl(1), the stability subalgebra in the terminology
of Blanc and Rowe,\(^{38} \) introduce a basis, the induced basis, and
write down its transformations under the action of the Chevalley
generators (Sec.III.A). In Sec.III.B we define a new, reduced with
respect to the even subalgebra basis. Next (Sec.III.C) we analyze
the irreducible induced modules and those indecomposable modules,
which, after factorizing with respect to the maximal invariant
subspace, lead to infinite-dimensional irreducible modules. In
Sec.III.D we write down an orthonormed basis for the infinite-
dimensional star representation. Section III.E is devoted to the
finite-dimensional indecomposable and irreducible representations.
For all these cases we write explicit analytical expressions for
the transformations of the basis under the action of the Chevalley
generators.

Throughout we use the following abbreviations and notation:
LS, LS’s - Lie superalgebra, Lie superalgebras,
LA, LA’s - Lie algebra, Lie algebras,
CAO’s = creation and annihilation operators,
lin.env.(X) - the linear envelope of X,
fidirmod(s) - finite-dimensional irreducible module(s)
\( \mathbb{Z} \) - all integers,
\( \mathbb{Z}_+ \) - all nonnegative integers,
\( \mathbb{Z}_0 = \{0,1\} \) - the ring of all integers modulo 2,
\( \mathbb{N} \) - all positive integers,
\( \mathbb{C} \) - the complex numbers,
\( \mathbb{R} \) - the real number,
\( \text{[x,y]} = xy - yx; \{x,y\} = xy + yx \),
\( \theta(x) = 0 \) for \( x < 0 \) .

II. PRELIMINARIES
A. The orthosymplectic Lie superalgebras and some subalgebras

One can define the orthosymplectic Lie superalgebras in
different equivalent ways: in a matrix form\(^7 \), through the super-
commutation relations of its generators (for a detailed description
of \( \text{osp}(m/n) \) in this way we refer to Le Blanc and Rowe\(^{39} \)), etc. Here
we give a realization (see Ref.2, eq.(18)), which is easy to deal
with and is more familiar to the physicists.

Let \( c^\mu_\nu \), be Fermi CAO’s for \( i=1,\ldots,m \) and Bose CAO’s for
\( i=m+1,\ldots,m+n \). Assume the somewhat unusual property that the Bose
and Fermi operators mutually anticommute,

\[ \{c^\mu_\nu, c^\xi_\sigma\} = 0, \quad \text{if } \mu = \nu \text{ and } \xi = \sigma, \quad \text{or} \quad \text{if } \mu \neq \nu \text{ and } \xi \neq \sigma. \quad (2.1) \]

Set

\[ (j) = \begin{cases} 0 & \text{for } i < sm \\ 1 & \text{for } i > sm \end{cases} \quad (2.2) \]

and let

\[ \deg(c^\mu_\nu) = (j) c^\mu_\nu, \quad (2.3) \]

i.e. the Fermi operators are even and the Bose - odd elements
(which is also unusual grading). Define

\[ \{c^\mu_\nu, c^\xi_\sigma\} = (-1)^{(i)(j)} c^\xi_\sigma c^\mu_\nu, \quad \text{for } i=1,\ldots,m+n, \xi, \sigma = 1, \ldots, m+n. \quad (2.4) \]

Then one has the following realizations\(^2,3,15\) of some LS’s (in the
Kac notation\(^9 \)) and LA’s:

\[ \text{osp}(2m+1/2n) \# B(m/n) = \text{lin.env.}(\{c^\mu_\nu, c^\xi_\sigma\}, c^\mu_\nu | i, j = 1, \ldots, m+n, \xi, \sigma = 1, \ldots, m+n \}. \quad (2.5) \]
osp(2m/2n) = D(m/n) = lin.env. \{[c^i, c^j] | i, j = 1, \ldots, m+n, i \neq j\},
\text{gl}(m/n) = lin.env. \{[c^i, c^j] | i, j = 1, \ldots, m+n, i \neq j\},
so(2m+1) = B = lin.env. \{[c^i, c^j, c^k] | i, j, k = 1, \ldots, m, i \neq j \neq k\},
sp(2n) = C = lin.env. \{[c^i, c^j] | i, j = 1, \ldots, m+n, i \neq j\}.

(2.6)
(2.7)
(2.8)

The grading in each of the above algebras is induced from (2.3); the supercommutator between any two homogeneous elements is defined in the usual way,

\[ [x, y] = xy - (-1)^{\text{deg}(x)\text{deg}(y)}yx. \]

The direct sum \( \text{so}(2m+1) \oplus \text{sp}(2n) \) is the even subalgebra of \( \text{osp}(2m+1/n) \).

B. The algebra \( \text{osp}(3/2) \)

In a matrix form the orthosymplectic LS \( \text{osp}(3/2) \) can be defined as a subset of all 5-by-5 matrices, namely

\[ \begin{pmatrix}
  a & b & x & u \\
  -b & a & y & v \\
  -c & -b & 0 & z \\
  u & v & d & e \\
  -y & -x & -z & f & -d
\end{pmatrix}, \]

where the nonzero entries are arbitrary complex numbers. The even subalgebra \( \text{osp}(3/2)_e \) consists of the matrices for which \( x = y = z = u = v = 0 \); it is isomorphic to \( \text{so}(3) \oplus \text{sp}(2) \).

The odd subspace \( \text{osp}(3/2)_o \) of \( \text{osp}(3/2) \) corresponds to the case in (2.11) when \( a+b+c+d+e+f = 0 \). The product \([\cdot, \cdot]\) on \( \text{osp}(3/2) \) is a matrix anticommutator between any odd matrices and a matrix commutator in all other cases.

It is convenient to describe \( \text{osp}(3/2) \) as a subalgebra of the LS \( \text{gl}(3/2) \). As generators of \( \text{gl}(3/2) \) take the Weyl generators \( e_{ij} \), \( i, j = 1, \ldots, 5 \), which satisfy the supercommutation relations

\[ [e_{ij}, e_{kl}] = \delta_{ik} e_{jl} - \delta_{il} e_{kj} - (-1)^{\text{deg}(e_{ij})\text{deg}(e_{kl})} \delta_{jk} e_{il} + (-1)^{\text{deg}(e_{ij})} \delta_{ij} e_{kl}, \]

(2.13)

where \( \langle 1 \rangle = \langle 2 \rangle = \langle 3 \rangle = 0 \), \( \langle 4 \rangle = \langle 5 \rangle = 1 \); \( \text{deg}(e_{ij}) = \langle i \rangle + \langle j \rangle \) mod 2. In the 5-by-5 matrix realization \( e_{ij} \) is a matrix with 1 on the \( i \)th row and the \( j \)th column and 0 elsewhere.

In the table below we express the generators of \( \text{osp}(3/2) \) in two different ways: in terms of Weyl generators and in terms of one pair Fermi operators \( c^a b^c \) and one pair Bose operators \( c^a b^c \) see (2.14). For convenience of the reader we give also the relation of our notation to those of Van der Jeugt, Le Blanc and Rowe and the standard notation of Kac.

In the \( Z_2 \) column we indicate the \( Z_2 \)-grade of the corresponding generator.

<table>
<thead>
<tr>
<th>Our notation</th>
<th>Jeugt</th>
<th>Blanc</th>
<th>CAO's</th>
<th>Kac</th>
<th>( Z_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H = e_{54} )</td>
<td>( -t )</td>
<td>( J )</td>
<td>( \frac{1}{2}b^*b )</td>
<td>( G_{2} )</td>
<td>( \hat{0} )</td>
</tr>
<tr>
<td>( e_{24} + e_{42} )</td>
<td>( -R_{1/2} )</td>
<td>( F_{-1/2} )</td>
<td>( f^* - f )</td>
<td>( G_{1} )</td>
<td>( \hat{1} )</td>
</tr>
<tr>
<td>( e_{31} + e_{43} )</td>
<td>( R_{0,1/2} )</td>
<td>( F_{0} )</td>
<td>( \frac{1}{\sqrt{2}}b^* )</td>
<td>( G_{2} )</td>
<td>( \hat{1} )</td>
</tr>
<tr>
<td>( e_{12} + e_{23} )</td>
<td>( R_{1,1/2} )</td>
<td>( F_{-1/2} )</td>
<td>( b^* )</td>
<td>( G_{1} )</td>
<td>( \hat{1} )</td>
</tr>
<tr>
<td>( F = a^* = \sqrt{2}(e_{31} - e_{23}) )</td>
<td>( s )</td>
<td>( \sqrt{2}L_{-1} )</td>
<td>( -s^* )</td>
<td>( G_{0} )</td>
<td>( \hat{1} )</td>
</tr>
<tr>
<td>( H = \frac{1}{2}(e_{55} - e_{44}) )</td>
<td>( t )</td>
<td>( \sqrt{2}L_{-1} )</td>
<td>( -t^* )</td>
<td>( G_{2} )</td>
<td>( \hat{1} )</td>
</tr>
</tbody>
</table>

The subspaces \( G_{2}, G_{-1}, G_{1}, G_{2} \) and the subalgebras \( G_{0}, N^2, H, P, F_3 \).
are linear envelopes of the generators indicated on the table. \( H \) is
the Cartan subalgebra, \( N' \) and \( N' \) are the nilpotent subalgebras of
the positive and the negative root vectors, respectively; \( \text{osp}(3/2) \)
is a Z-graded algebra,7

\[
\text{osp}(3/2) = F_{i=1}^n G^i, \quad [G^i, G^j] = 2 \delta_{ij}, \tag{2.15}
\]

with Z-graded homogeneous subspaces \( G^i, G^0, G^1, G^2, \) and \( G^3 = 0 \)
for \( |ij| ≥ 2 \). The supercommutation relation on \( \text{osp}(3/2) \) are
completely computed from the Bose-Fermi realization or from (2.13),
using the expressions of the generators in terms of \( e^i_\dagger \). According
to (2.15) the zero grade component \( G^0 \) is a subalgebra, the stability
subalgebra in the terminology of Blanc and Rowe; it is isomorphic
to \( \text{so}(3) \circ \text{gl}(1) = \text{gl}(2) \). The even subalgebra is a direct space sum of
the even Z-graded spaces,

\[
\text{so}(3/2) = \text{so}(3) \circ \text{sp}(2) = G^2 + G^0 + G^2. \tag{2.16}
\]

The generators \( F^2 = F^2_1 = F^2_2 \) of \( \text{so}(3) \) and \( H_0, H^* \) of \( \text{sp}(2) \) satisfy
the usual for \( \text{sl}(2) = \text{so}(3) \circ \text{sp}(2) \) commutation relations:

\[
[F_i^2, F_j^2] = i \epsilon^{ij} e^2, \quad [F_i^2, \epsilon^2] = 2 \epsilon^0, \quad [H^0, H^*] = 2 \epsilon^2. \tag{2.17}
\]

The algebra \( \text{osp}(3/2) \) is completely defined with its Chevalley
generators8

\[
h_i = F^2_i = 2 \epsilon^0 = \epsilon^1_1 - \epsilon^2_2 + \epsilon^4_4, \quad h_i = 2(\epsilon^1_1 - \epsilon^2_2),
\]

\[
e_i = \epsilon^i_1 - \epsilon^2_2, \quad e_i = \epsilon^i_1 + \epsilon^2_2, \quad f_i = \epsilon^i_1 - i \epsilon^2_2, \quad f_i = \epsilon^i_1 + i \epsilon^2_2, \tag{2.18}
\]

which satisfy

1. The Cartan-Kac relations:

a) \( [h_i, h_j] = 0 \); \tag{2.19a}

b) \( [h_i, e_i] = 0, \quad [h_i, f_i] = 0 \),

\[
[h_i, e_j] = \delta^i_j e_i, \quad [h_i, f_j] = -\epsilon^{ij} e_j. \tag{2.19b}
\]

c) \( e_i f_i = h_i \), \( e_i f_i = h_i \), \( e_i f_i = h_i \), \( [e_i, f_j] = 0 \); \tag{2.19c}

2) The Serre relations

\[
[e_i, e_j] = 0, \quad [f_i, f_j] = 0, \quad [e_i, [e_i, e_j]] = 0, \quad [f_i, [f_i, f_j]] = 0. \tag{2.19d}
\]

Taking into account (2.19c) one can express all equations
(2.19) in terms of only four generators \( a_i^c = f_i, \ a_i^a = e_i, \ a_i^d = f_i, \ a_i^d = e_i \)
(throughout below \( e = \pm 1 \)):

1) \( [a_i^c, a_i^c] = [a_i^d, a_i^d] = 0 \); \tag{2.20a}

2) \( [[a_i^a, a_i^a], a_i^c] = 0, \quad [[a_i^a, a_i^d], a_i^d] = 0, \quad
\quad [[a_i^a, a_i^c], a_i^c] = 0 \); \tag{2.20b}

3) \( [a_i^c, [a_i^c, [a_i^c, a_i^c]]] = 0, \quad [a_i^d, [a_i^d, [a_i^d, a_i^d]]] = 0 \). \tag{2.20c}

The relations (2.20) define uniquely \( \text{osp}(3/2) \). More precisely, the
free \( \mathbb{Z}_2 \)-graded associative algebra with unity, generators \( a_i^c, i = 1, 2 \)
and relations (2.20) is the universal enveloping algebra \( \mathfrak{u} \)
\( \mathfrak{u}[\text{osp}(3/2)] \) of \( \text{osp}(3/2) \). \( \mathfrak{u} \) is a Lie superalgebra under the super-commutator (2.10). The subspace spanned on the elements (\( c = \pm \))

\[
a_i^c, \quad a_i^a, \quad a_i^c, \quad [a_i^a, a_i^a], \quad [a_i^a, a_i^c], \quad [a_i^a, a_i^c], \tag{2.21}
\]

is a subalgebra of the Lie superalgebra \( \mathfrak{u} \), isomorphic to \( \text{osp}(3/2) \).

The latter can be taken as another definition of the LS \( \text{osp}(3/2) \); this
approach plays a relevant role in the quantum algebras and in particular in the quantum orthosymplectic algebras.7 For us the relevance of the relations (2.20) stems from the observation that
they provide the most economical way of checking all super-
commutation relations in the algebra; in particular, given set of
operators $a^*_1$, $a^*_2$ in a linear space $V$ generate a representation of
osp(3/2) in $V$ if and only if the equations (2.20) hold. The
expressions of the generators in terms of $a^*_1$, $a^*_2$ then read (see also (2.18))

$$
e_{35}e_{44}=[a^*_1,a^*_1]-\frac{1}{2}[a^*_2,a^*_2], \quad e_{31}e_{22}=\frac{1}{2}[a^*_2,a^*_1]
$$

$$
e_{34}e_{23}=-\frac{1}{\sqrt{2}}[a^*_2,a^*_2], \quad e_{36}e_{43}-\frac{1}{\sqrt{2}}[a^*_2,a^*_1],
$$

$$
e_{44}e_{32}=\frac{1}{2}[a^*_2,a^*_1], \quad e_{45}e_{41}=\frac{1}{2}[a^*_2,[a^*_1,a^*_1]],
$$

$$
e_{54}e_{45}=\frac{1}{4}[a^*_1,[a^*_2,a^*_1]], \quad e_{45}e_{41}=\frac{1}{4}[a^*_1,[a^*_1,a^*_1]].
$$

III. REPRESENTATIONS OF osp(3/2)

A. Induced modules

We now proceed to introduce, following Kac,$^7$ the osp(3/2)
module $\widetilde{W}$, induced from a module $V_Q$ of the stability subalgebra
$G_Q=so(3)\otimes gl(1)$. To this end turn first $V_Q$ into a $P$ module, setting
$P^Q=0$, i.e.,

$$(e_{34}e_{45})V_0=(e_{34}e_{45})V_0=(e_{34}e_{45})V_0=0.\quad (3.1)$$

The osp(3/2) module, induced from the $G_Q$ module $V_Q$, is defined to
be the factor space

$$\tilde{W}=(U[osp(3/2)]V_Q)/I\tag{3.2}
$$

of the tensor product of the osp(3/2) universal enveloping algebra
$U[osp(3/2)]\otimes V_Q$ with $V_Q$ and subsequently factorized by the subspace

$$I=\text{lin.env.}(ue_{12}uv,ue_{12}uv,ue_{12}uv).\tag{3.3}
$$

The linear space $\tilde{W}$ is equipped with a structure of an osp(3/2)
module in a natural way:

$$g(uv)=gumv, geosp(3/2), uuv\tilde{W}.
$$

The Poincaré-Birkhoff-Witt theorem$^7$ yields that $U$ is a linear span
of all elements of the form

$$g=(e_{15}+e_{42})^0(e_{35}+e_{43})^0(e_{25}+e_{44})^0(e_{45})^p, \quad \eta_1, \eta_2, \eta_3=0,1, \eta e\mathbb{Z}, \quad (3.4)
$$

where $p$ is a polynomial of elements from $P$. Since for any $g$, defined in (3.4), and any $\nu\in\mathbb{Z}$

$$g_\nu=(e_{15}+e_{42})^0(e_{35}+e_{43})^0(e_{25}+e_{44})^0(e_{45})^\nu, \quad \eta_1, \eta_2, \eta_3=0,1, \eta e\mathbb{Z}, \nu e\mathbb{Z}, \quad (3.5)
$$

one concludes that

$$\tilde{W}=\text{lin.env.}(e_{15}+e_{42})^0(e_{35}+e_{43})^0(e_{25}+e_{44})^0(e_{45})^\nu, \eta_1, \eta_2, \eta_3=0,1, \eta e\mathbb{Z}, \nu e\mathbb{Z}, \quad (3.6)
$$

are linearly independent in $U$. Therefore if $\{V_Q|\eta eK\}$ is any basis
in $V_Q$ then the vectors

$$(e_{15}+e_{42})^0(e_{35}+e_{43})^0(e_{25}+e_{44})^0(e_{45})^\nu, \eta_1, \eta_2, \eta_3=0,1, \eta e\mathbb{Z}, \nu e\mathbb{Z}, \quad (3.7)
$$

constitute a basis in the induced module $\tilde{W}$. From now on we assume
that $\tilde{W}$ is induced from an irreducible module $V_q$ of the stability
algebra $G_q=so(3)\otimes gl(1)$. Then $V_q=V_1\otimes V_2$, where $V_1$ and $V_2$
are irreducible modules of $gl(1)$ and $so(3)$, respectively. Since
$gl(1)$ is one-dimensional (it is spanned on $H_0$), $V_1$ is also an
one-dimensional space. For each $q\in\mathbb{Z}$ define a $gl(1)$ fidirmod $V_q(q)$
spanned on $x_q, V\gamma(q)=x_q$:

$$H_{0,q}x_q=q x_q.\tag{3.8}
$$

As representation spaces $V_q$ for $so(3)$ we take finite-dimensional
irreducible modules. Let $V_\gamma(p)$ be an $so(3)$ fidirmod with a "spin"
p$e\mathbb{Z}/2$. Choose a basis of weight vectors $[p,i]$ in $V_\gamma(p)$,
vectors which transform as \( (a_i^+ f^i) \)
\[
as_i^+ p, i=1 \left\{ (p+1)!/(p+1) \right\}^{1/2}, \quad F_0 (p, i)=p+1, \quad F_0 (p, i)=p+1.
\]
Then we have \( N(x \ast p, p)=0 \), i.e., \( \{ x \ast p, p \}\) is a highest weight vector of the induced module \( \tilde{V} \). Since moreover \( \{ x \ast p, p \}\) is the signature (=the weight of \( \{ x \ast p, p \}\) of \( \tilde{W} \) in the Cartan basis \( F_0, H_0 \) is \([p, q] \). Therefore we set \( \tilde{W} = W([p, q]) \).

The assumption to consider only finite-dimensional so(3) modules \( \tilde{V} \) is inessential from a point of view of the construction to be carried out. It puts however strong restrictions on the osp(3/2) representations. In particular it means that we exclude from consideration all modules without highest weights. Nevertheless the class of the representations that are left is quite large. We shall see that it contains all star highest weight modules (typical and atypical), all Kac modules, all finite-dimensional modules and several other irreducible or indecomposable infinite-dimensional modules of osp(3/2).

As a basis in the induced module \( \tilde{W}([p, q]) \) we choose [see \( (3.7)-(3.9) \)] the vectors
\[
\{ \psi p, i \ast \gamma \epsilon, \theta \ast \gamma, \theta \ast \gamma, n \ast i \},
\]
where
\[
\theta \ast \gamma, \theta \ast \gamma = 0,1, n \in \mathbb{Z}, \quad i=p, p+1, \ldots, p
\]
and we refer to this basis as to induced basis. The transformations of the induced basis under the action of the Chevalley generators are easily derived:
\[
a_0 \sqrt{(1-\theta_j)} \ast \gamma \epsilon, \theta \ast \gamma, \theta \ast \gamma, n \ast i, \quad (3.13)
\]
\[
a_0 \sqrt{(1-\theta_j)} \ast \gamma \epsilon, \theta \ast \gamma, \theta \ast \gamma, n \ast i, \quad (3.14)
\]
From these relations and from eqs.\( (2.22) \) one can derive the transformation of the induced basis under the action of any other generator. In particular the expressions for the action of the \( sp(2) \) root vectors read:
\[
\epsilon_{\gamma} \ast \gamma \epsilon, \theta \ast \gamma, \theta \ast \gamma, n \ast i, \quad (3.17)
\]
\[
\epsilon_{\gamma} \ast \gamma \epsilon, \theta \ast \gamma, \theta \ast \gamma, n \ast i, \quad (3.18)
\]
The eqs.\( (3.13)-(3.16) \) define already a class of representations of osp(3/2), labeled with any two numbers \( 2p, q \). Each module \( \tilde{W}([p, q]) \) is either an irreducible or an indecomposable infinite-dimensional module. The structure of \( \tilde{W}([p, q]) \) however and in particular its invariant subspaces are difficult to be described in terms of the induced basis \( (3.11) \). We now proceed to introduce a new, reduced with respect to the even subalgebra basis, which will make more transparent the structure of the induced modules.
B. Reduced so(3)®sp(2) basis

In order to decompose \( \tilde{\mathcal{W}}(p,q) \) into a direct sum of (generally indecomposable) modules of the even subalgebra we have found a set of so(3)®sp(2) singular weight vectors \( |(p,q);I,J) \) in \( \tilde{\mathcal{W}}(p,q) \), i.e., vectors from \( \tilde{\mathcal{W}}(p,q) \) with the property

\[
F^* |(p,q);I,J) = 0, \quad F_0 |(p,q);I,J) = I |(p,q);I,J),
\]

\[
H^* |(p,q);I,J) = 0, \quad H_0 |(p,q);I,J) = J |(p,q);I,J).
\]

These singular vectors read (in the relations below \( a,b,b_l,b_i,c_i, c_2,c_3,d \) are arbitrary nonzero constants):

\[
|\langle p,q; s\rangle = a |\langle p,q,0,0,0,p> s=0,2q+1;
\]

\[
|\langle p,q; s\rangle = b |\langle p,q,0,0,0,p> s=0,2q;
\]

\[
|\langle p,q; s\rangle = c |\langle p,q,0,0,0,p> s=0,2q-1;
\]

\[
|\langle p,q; s\rangle = d |\langle p,q,0,0,0,p> s=0,2q-2.
\]

The latter follows from the circumstance that \( e_{15} \) commutes with 

\[
(e_{15} = e_{15})^\theta, (e_{25} = e_{25})^\theta, (e_{25} = e_{45}),
\]

The undefined vectors \( (3.22),(3.23),(3.25),(3.26) \) for \( p=0,1/2 \) will be called redundant vectors.

Each nonredundant vector \( |\langle p,q; I,J) \) \) in \((3.20)-(3.27)\) corresponding to \( s=0 \) is a highest weight vector of an so(3)®sp(2) module \( V(p,q;I,J) \), which is characterized with its so(3) "spin" \( I \) and sp(2) "spin" \( J \). Acting with the lowering operators \( H^- \) and \( F^- \) on any such vector one generates a basis

\[
|\langle p,q; I,J; i,j) = (F^-)^{I-i} (H^-)^{J-j} |\langle p,q; I,J), \quad i=-I,-I+1,\ldots, I,
\]

\[
\quad \quad \quad \quad j=-J,-J+1,\ldots, J.
\]

of the infinite-dimensional module \( V(p,q;I,J) \). All vectors

\[
|\langle 0,q; 0,0-1/2;i,j), \quad |\langle 0,q; 0,1-1/2;i,j), \quad |\langle 0,q; 1-1/2;i,j), \quad |\langle 0,q; 1;i,j)
\]

\[
|\langle 1/2,q; 0,0-1/2;i,j), |\langle 1/2,q; 0,1-1/2;i,j), \quad |\langle 1/2,q; 1;i,j)
\]

generated from redundant vectors are equal to zero; we call them also redundant vectors.

**Proposition 1:** If \( q=0,1/2 \) then each induced module \( \tilde{W}(p,q) \) is a direct sum of the so(3)®sp(2) modules \( V(p,q;I,J) \), generated from the \( s=0 \) singular vectors \((3.20)-(3.27)\), i.e.,

\[
\tilde{W}(p,q) = V(p,q;0,0) \oplus V(p,q;0,1) \oplus V(p,q;1,0) \oplus V(p,q;1,1).
\]

**Proof:** The different so(3)®sp(2) modules in \( (3.31) \) correspond to different highest weights and therefore they are linearly independent. Hence the sum in \( (3.31) \) is a direct sum. Consequently the collection of the basis vectors \( |\langle p,q; n+1,0,0; I,J) \) \) of all nonzero spaces \( V(p,q; n+1,0,0) \) in the r.h.s. of \( (3.31) \), i.e.,

\[
\sum_n V(p,q;n,0,0) \oplus V(p,q;n,1,1) \bigoplus \cdots
\]

constitute a basis in \( V \) which we call a reduced basis. The
equality $V = \hat{W}(p,q)$ follows now from the observation that each induced basis vector can be expressed as a linear combination of the reduced basis. The explicit expressions of the reduced basis vectors in terms of the induced ones and the inverse relations are given in the Appendix.

From (3.15)-(3.18) one obtains easily the transformations of the reduced basis under the action of the $so(3)$ generators,

$$a^\pm_i(p,q;I,J;i,J) = \{I\pm i\}(Ii\pm i+1)^{1/2}(p,q;I,J;i,J),$$

$$F_0(p,q;I,J;i,J) = i(p,q;I,J;i,J),$$

and under the action of the $sp(2)$ generators,

$$e_{\pm i}(p,q;I,J;i,J) = (p,q;I,J;i,J),$$

$$e_{\pm i}(p,q;I,J;i,J) = (J+J+1)(J-J)(p,q;I,J;i,J),$$

$$H_0(p,q;I,J;i,J) = J(p,q;I,J;i,J).$$

These relations indicate that each $V(p,q;I,J)$ is in fact a tensor product of a $(2I+1)$-dimensional $so(3)$ module $V_{so}(p,q;I)$ and an infinite-dimensional $sp(2)$ module $V_{sp}(p,q;J)$,

$$V(p,q;I,J) = V_{so}(p,q;I) \otimes V_{sp}(p,q;J).$$

We recall that the $so(3) \otimes sp(2)$ modules $V(p,q;I,J)$ are generated from the singular vectors (3.20)-(3.27), corresponding to $s=0$. Therefore $J$ takes at most the values $q, q-1/2, q-3/2, \ldots$, where $q$ is an arbitrary complex number. The coefficients $(J+J+1)(J-J)$ in the r.h.s. of (3.36) could vanish for certain values of $J$ only if $2q \in \mathbb{Z}$. Thus we come to the following conclusion.

Proposition 2: For $2q \notin \mathbb{Z}$, the $osp(3/2)$ induced module $\hat{W}(p,q)$ is a direct sum of irreducible infinite-dimensional $so(3) \otimes sp(2)$ modules $V(p,q;I,J)$.

The above considerations indicate that each $V(p,q;I,J)$ in the sum (3.31), corresponding to $2J \notin \mathbb{Z}$, is an infinite-dimensional indecomposable $so(3) \otimes sp(2)$ module. Its maximal $so(3) \otimes sp(2)$ invariant subspace $V_{inv}(p,q;I,J)$ is (see (3.35)-(3.36))

$$V_{inv}(p,q;I,J) = \text{lin.env.} \{ (p,q;I,J;i,J) | i = I, I+1, \ldots, I, J = J-n-1, n \notin \mathbb{Z} \}$$

and it has a signature $[I, J-1]$. The factor space

$$W(p,q;I,J) = V(p,q;I,J) / V_{inv}(p,q;I,J)$$

is a finite-dimensional irreducible $so(3) \otimes sp(2)$ module with an $so(3)$ "spin" $I$ and an $sp(2)$ "spin" $J$.

The transformations of the reduced basis under the action of $a^\pm_i$ are obtained from (3.13)-(3.14) using the basis transformation (A.2)-(A.17). We write the final result of these relatively long calculations setting for simplicity

$$|(p,q;I,J;i,J) = \sum_{i=-J,-J+1,\ldots} \sum_{i=-J+n, n \mathbb{Z}}$$

In the formulae to follow the relations (3.44),(3.45),(3.47),(3.48) should be omitted if $p=0$; (3.45),(3.48) have to be skipped if $p=1/2$ (this is indicated in the r.h.s. of the corresponding equations; all other relations hold for any $p \in 0, 1, 1/2, 3/2, \ldots$).

$$a^\pm_i(p,q;i,j) = \delta(p-j)^{1/2} \delta(p+i)^{1/2} \delta(p+i+1)^{1/2} \delta(p+i+2)^{1/2} |(p+1, q-1/2, i+1, j+1/2)$$

$$b^\pm_i(p+1) = \frac{1}{2} |(p+1)(p+i+1)| p, q-1/2, i+1, j+1/2)$$

$$b^\pm_i(p+1) = \frac{1}{2} |(p+1)(p+i+1)| p, q-1/2, i+1, j+1/2)$$

$$a^\pm_i(p+1, q-1/2, i,j)$$

$$= \frac{b((p+1)q)(q+j+2)^{1/2} (p+1)^{1/2} (p+i+1)^{1/2} (p+i+2)^{1/2}}{2(p+1)(p+i+1)(p+i+2)} |p, q, i+1, j+1/2)$$

$$+ \frac{b_{-i}(p+1)(q+j-1)^{1/2} (p+1)^{1/2} (p+i+1)^{1/2} (p+i+2)^{1/2}}{2(p+1)(p+i+1)(p+i+2)} |p, q, i-1, j+1/2)$$

$$+ \frac{b^\pm_i(p+1)(q-j-1)^{1/2} (p+i)^{1/2} (p+i+1)^{1/2} (p+i+2)^{1/2}}{2(p+1)(p+i+1)(p+i+2)} |p, q, i+1, j-1/2)$$

$$+ \frac{b_{-i}(p+1)(q-j+1)^{1/2} (p+i)^{1/2} (p+i+1)^{1/2} (p+i+2)^{1/2}}{2(p+1)(p+i+1)(p+i+2)} |p, q, i-1, j-1/2)$$

$$+ \frac{b^\pm_i(p+1)(q-j+1)^{1/2} (p+i)^{1/2} (p+i+1)^{1/2} (p+i+2)^{1/2}}{2(p+1)(p+i+1)(p+i+2)} |p, q, i+1, j+1/2)$$

$$+ \frac{b_{-i}(p+1)(q-j-1)^{1/2} (p+i)^{1/2} (p+i+1)^{1/2} (p+i+2)^{1/2}}{2(p+1)(p+i+1)(p+i+2)} |p, q, i-1, j-1/2)$$
\[ a_i^+|p-1, q-1/2, i, j, j \rangle = \frac{1}{2a_p} (q+j+1/2)^{1/2} \left[ \frac{(p+i+1)(p+i+2)}{2(p-1)} \right]^{1/2} |p, q-1/2, i+1, j+1/2 \rangle \]

\[ a_i^-|p+1, q-1/2, i, j, j \rangle = \frac{c_i}{b_i(p+2q+2)} (q+j)^{1/2} \left[ \frac{(p+i+1)(p+i+2)}{2(p-1)} \right]^{1/2} |p+1, q-1/2, i+1, j+1/2 \rangle \]

\[ b_i^+|p-2q+1, j, j, j \rangle = \frac{c_i}{b_i(p-2q+1)} (q+j+1/2)^{1/2} \left[ \frac{(p+i+1)(p+i+2)}{2(p-1)} \right]^{1/2} |p, q-1/2, i+1, j+1/2 \rangle \]

\[ b_i^-|p+2q, j, j, j \rangle = \frac{c_i}{b_i(p+2q+2)} (q+j)^{1/2} \left[ \frac{(p+i+1)(p+i+2)}{2(p-1)} \right]^{1/2} |p+1, q-1/2, i+1, j+1/2 \rangle \]

\[ b_i^+|p-2q+1, j, j, j \rangle = \frac{c_i}{b_i(p-2q+1)} (q+j)^{1/2} \left[ \frac{(p+i+1)(p+i+2)}{2(p-1)} \right]^{1/2} |p, q-1/2, i+1, j+1/2 \rangle \]

\[ b_i^-|p+2q, j, j, j \rangle = \frac{c_i}{b_i(p+2q+2)} (q+j)^{1/2} \left[ \frac{(p+i+1)(p+i+2)}{2(p-1)} \right]^{1/2} |p+1, q-1/2, i+1, j+1/2 \rangle \]

\[ a_i^+|p+2q, j, j, j \rangle = \frac{c_i}{b_i(p+2q+2)} (q+j)^{1/2} \left[ \frac{(p+i+1)(p+i+2)}{2(p-1)} \right]^{1/2} |p+1, q-1/2, i+1, j+1/2 \rangle \]

\[ a_i^-|p-2q+1, j, j, j \rangle = \frac{c_i}{b_i(p-2q+1)} (q+j)^{1/2} \left[ \frac{(p+i+1)(p+i+2)}{2(p-1)} \right]^{1/2} |p, q-1/2, i+1, j+1/2 \rangle \]

\[ \text{For the meaning of the underlined vectors - see the end of the next Subsect. C.} \]

\[
\begin{align*}
\text{C. Infinite-dimensional typical, atypical and indecomposable } & \text{osp}(3/2) \text{ representations} \\
\text{Denote by } & \text{Typ}(2q \in \mathbb{Z}, p+2q \neq 0) \text{ all induced modules } \hat{W}(p,q), \\
\text{corresponding to } & 2q \in \mathbb{Z} \text{ and } p+2q \neq 0, \\
\text{Typ}(2q \in \mathbb{Z}, p+2q = 0) = & \{ \hat{W}(p,q) | 2p \in \mathbb{Z}, 2q \in \mathbb{Z}, 2p-2q = 0 \}.  
\end{align*}
\]

\[ \text{Proposition 3: The induced modules } \hat{W}(p,q) \text{ from Typ}(2q \in \mathbb{Z}, p+2q = 0) \text{ are infinite-dimensional irreducible } \text{osp}(3/2) \text{ modules.} \]

\[ \text{Proof: We have to show that for any two nonzero vectors } \]

\[ x, y \in \hat{W}(p,q) \text{ there exists } Q \in U \text{ such that } y = Qx. \]

\[ \text{To this end observe that according to (3.31) } x \text{ is a sum of vectors, which are form } \]

\[ \text{spaces } V(p,q;I,J) \text{ with different signatures. Consequently there always exists a polynomial } Q \text{ of the even generators such that } Qx = \]

\[ \text{for certain } I \text{ and } J. \text{ From (3.42)-(3.49) one easily concludes (using essentially that } p+2q \neq 0 \text{) that there exists } Q \in U \]

\[ \text{which maps } |(p,q);I,J;I,J) \text{ onto the } \hat{W}(p,q) \text{ highest weight vector, } \]

\[ Qx = |(p,q);p,q;p,q;p,q;1e(x)_{p,p} \rangle. \]

\[ \text{From the very construction of the induced module } \hat{W}(p,q) \text{ it is clear that there exists } Q \in U \text{ such that } y = Qx. \]
The modules from 

\[ \text{Typ}(2q\mathbb{Z}, p+2q=0) \]

could be viewed as infinite-dimension analogs of the (finite-dimensional) typical modules in the terminology of Kac. For this reason we call them (infinite-dimensional) typical modules. The transformations of all typical modules \( \tilde{W}(p,q) \) are completely described both with eqs. (3.13)-(3.16) of the induced basis or with eqs. (3.33), (3.42)-(3.49) of the reduced basis. This class of representations is a large one. In particular it contains, as we shall see, all typical star modules from the class \( V^* \) in the classification of Van der Jeugt. 

**Proposition 4:** Any induced module \( \tilde{W}(p,q) \) corresponding to \( p+2q=0, q \neq 0 \) is an infinite-dimensional indecomposable \( \text{osp}(3/2) \) module with a maximal invariant subspace

\[
\tilde{W}(p+1,q-1/2) = \text{V}(p,q;p+1,q-1/2) \text{V}(p,q;p,q-1/2); \tag{3.51}
\]

\( \text{W}(p,q) \) is an irreducible infinite-dimensional highest weight \( \text{so}(3/2) \) module with a signature \( \{p+1,q-1/2\} \) and a highest weight vector \( |p+1,q-1/2; p+1,q-1/2 \rangle \). The factor space

\[
\text{V}(p,q) = \frac{\tilde{W}(p,q)}{\text{W}(p+1,q-1/2) \mid V}; \tag{3.52}
\]

is also an infinite-dimensional irreducible \( \text{osp}(3/2) \) module with a signature \( \{p,q\} \).

**Proof:** Recall that we exclude from consideration the modules, corresponding to \( q=0,1/2 \). The invariance of \( \text{W}(p+1,q-1/2) \) under the action of the \( \text{osp}(3/2) \) generators follows directly from the transformation relations (3.33), (3.42)-(3.49). The irreducibility of both \( \text{W}(p+1,q-1/2) \) and \( \text{W}(p,q) \) is proved in the same way as in the previous proposition.

All irreducible representations carried by \( \tilde{W}(p,q) \)

\[ \text{Typ}(2q\mathbb{Z}, p+2q=0) \]

as well as by \( \text{W}(p,q) \) have been predicted by Van der Jeugt. Here we obtain in addition indecomposable modules, constructed out of \( \text{V}(p,q) \) and \( \text{W}(p,q) \). The modules \( \text{W}(p,q) \) are infinite-dimensional analogs of the finite-dimensional atypical modules. We denote all such modules as

\[ \text{Atyp}(p+2q=0) = \{\text{W}(p,q) | 2p \in \mathbb{Z}, 2q \in \mathbb{Z}, p+2q=0\}. \tag{3.53} \]

and refer to them as to (finite-dimensional) atypical modules. The transformations of the indecomposable modules \( \text{W}(p,q) \) as well as of their submodules \( \text{W}(p+1,q-1/2) \) are described with eqs. (3.33) and (3.42)-(3.49) of the reduced basis. In order to obtain the transformation relations for the \( p+2q=0 \) atypical modules one has to omit the relations (3.43), (3.46), (3.47), (3.49) and replace by zero the terms with the basis vectors from \( \text{W}(p+1,q-1/2) \) in the r.h.s. of eqs. (3.42), (3.44), (3.45), (3.48) (these vectors are underlined).

D. Star representations

The concept of star and grade star operations and of star and grade star representations of a Lie superalgebra was introduced by Scheunert, Nahm and Rittenberg (see also Ref.49). The star operation \( \ast \) in a LS \( L = L^+ \oplus L^- \) is a homogeneous map \( \mathbb{R}: L \rightarrow L \) of degree zero, which is an antilinear antivolution, i.e., for all \( a, b \in L \) and all \( \alpha, \beta \in \mathbb{C} \) (\( \alpha \) denotes complex conjugate)

\[
(a + \beta b) \ast = a^{\ast} + \beta b^{\ast}, \quad [a, b] = [b^{\ast}, a^{\ast}], \quad (a^{\ast})^{\ast} = a. \tag{3.54}
\]

Let \( V \) be a Hilbert space with a scalar product \( \langle \cdot, \cdot \rangle \). If \( A \) is an operator in \( V \), denote by \( A^{\ast} \) the conjugate to \( A \) operator. The representation \( n \) of \( L \) in \( V \) is said to be a star representation if for each \( a \in L \) and all \( \alpha, \beta \in \mathbb{C} \)

\[
(a + \beta b) \ast = a^{\ast} + \beta b^{\ast}, \quad [a, b] = [b^{\ast}, a^{\ast}], \quad (a^{\ast})^{\ast} = a. \tag{3.55}
\]

The corresponding classes of representations are denoted as \( D^c \). This notion is to indicate that the representation of \( \text{sp}(2) \), generated from the highest weight vector of \( \tilde{W}(p,q) \) if \( c=1 \) (resp. from the lowest weight vector if \( c=-1 \) - a case which we do not consider), is from the discreet series \( D^c \) of unitarizable representations of the noncompact real form \( \text{su}(1,1) \) of \( \text{sp}(2) \) (see Ref.50).

We proceed now to write down explicit expressions for all
highest weight star induced representations, i.e. those from $D^\cdot$. To this end we first observe, following Van der Jeugt, that the modules $\bar{W}(p,q)$ corresponding $q \in R$ or to $p+2q > 0$ are not star modules. Consequently it remains to consider the subclass of typical modules

$$\text{Typ}(p+2q < 0) = (\bar{W}(p,q) | W(p,q) \in \text{Typ}(2q \in R, p+2q = 0), p+2q < 0, q \in R)$$

(3.56)

and the class (3.53) of the infinite-dimensional atypical modules. In both cases $q$ is a negative real number and in both cases we introduce a new basis $|p,q;i,j;I,J>$ within each $so(3) \otimes sp(2)$ module $V(p,q;I,J)$ with the relations

$$|p,q;I,J;i,j⟩ = \left[\frac{(-2q-1)!}{(p+q-1)!} \right]^{1/2} |p,q;i,j⟩$$

(3.57)

where $z! = z(z+1)$ is the factorial function. We introduce a scalar product in $\bar{W}(p,q)$ and in $W(p,q)$ postulating that the new basis (we refer to it also as to a star basis) is orthonormed:

$$\langle |p,q;i,j⟩ | p,q;i',j'⟩ = \delta_{ij}$$

(3.58)

D.1 Typical infinite-dimensional star representations ($p+2q < 0$)

Let $\bar{W}(p,q) \in \text{Typ}(p+2q < 0)$. The requirements (3.55) with $c=-1$, namely

$$\langle a'x, y⟩ = (a'x, y) \quad (a'x, y) = (x, a'y) \quad \forall x, y \in \bar{W}(p,q)$$

(3.59)

lead to a set of equations for the constants $a, b_1, c_1, d$ [see (3.42)-(3.49)] with the following solutions:

$$|b_1| = |a| \left[\frac{-2q}{2p+1}\right]^{1/2}, \quad |c_1| = |a| \left[\frac{1}{2p+1} \left(1 - \frac{2q}{2p+1}\right) \right]^{1/2}$$

$$|c_2| = |a| \left[\frac{2p+1}{2q} \right]^{1/2}, \quad |c_3| = |a| \left[\frac{2p+1}{2q} \left(1 - \frac{2q}{2p+1}\right) \right]^{1/2}$$

$$|d| = |a| \left[\frac{1}{2(2p+1) + 1} \right]^{1/2}$$

(3.60)

We assume for simplicity that $a=1$ and that all other constants are positive real numbers. Then from (3.42)-(3.49), (3.57) and (3.60) we obtain the following explicit transformations for the infinite-dimensional typical star modules in the orthonormed basis (3.57) (the the relations (3.63), (3.64), (3.66), (3.67) should be omitted if $p=0$; (3.64), (3.67) have to be skipped if $p=1/2$):

$$a_1^p |p,q;1;j⟩ = \left[\frac{2p+2q}{4q(p+1)} \right]^{1/2} |p+1,q-1/2;i,j⟩$$

$$\sigma_0(p-1/2) \left[\frac{2p+2q}{4q(p+1)} \right]^{1/2} |p,q;1/2;i,j⟩, \quad (3.61)$$

$$a_1^p |p+1,q-1/2;i,j⟩ = \left[\frac{2p+2q}{4q(p+1)} \right]^{1/2} |p+1,q-1/2;i,j⟩$$

$$\tau_p \left[\frac{2p+2q}{4q(p+1)} \right]^{1/2} |p,q;1/2;i,j⟩$$

(3.62)

$$a_1^p \left[\frac{p+1}{p} \right]^{1/2} \left[\frac{2p+2q}{4q(p+1)} \right]^{1/2} \left[\frac{2p+1}{2q} \right]^{1/2} |p,q;1/2;i,j⟩$$

$$\left[\frac{2p+2q}{4q(p+1)} \right]^{1/2} |p,q;1/2;i,j⟩$$

$$\left[\frac{2p+2q}{4q(p+1)} \right]^{1/2} |p,q;1/2;i,j⟩$$

(3.63)

$$a_1^p |p,q;1/2;i,j⟩$$

$$\left[\frac{2p+2q}{4q(p+1)} \right]^{1/2} |p,q;1/2;i,j⟩$$

$$\left[\frac{2p+2q}{4q(p+1)} \right]^{1/2} |p,q;1/2;i,j⟩$$

(3.64)
\[
\frac{1}{(p+1)} \left| \frac{2(p+q)(p+i+1)}{(2g-1)} \right|^{1/2} |p,q-1/2;i+1,j+1/2>
\]

\[
\frac{1}{(p+1)} \left| \frac{2(p+q)(p+i+1)}{(2g-1)} \right|^{1/2} |p,q-3/2;i+1,j+1/2>, \quad (3.65)
\]

\[
\frac{1}{2} \left| \frac{2(p+q)(p+i+1)}{(2g-1)} \right|^{1/2} |p,q-1/2;i+1,j+1/2>, \quad (3.66)
\]

\[
\left| \frac{2(p+q)}{(2g-1)} \right|^{1/2} |p,q-3/2;i+1,j+1/2>, \quad (3.67)
\]

\[
\left| \frac{2(p+q)}{(2g-1)} \right|^{1/2} |p,q-3/2;i+1,j+1/2>, \quad (3.68)
\]

The approach is essentially the same as in the previous subsection, however one has first to replace all vectors from the maximal invariant subspace \(W(p+1,q-1/2)\) [see (3.51)] by zero. We write the final result for any \(W(p,q)\) for \(p+2q=0\)

\[
\frac{1}{2} \left| \frac{2(p+q)}{(2g-1)} \right|^{1/2} |p,q-1/2;i+1,j+1/2>, \quad (3.69)
\]

\[
\left| \frac{2(p+q)}{(2g-1)} \right|^{1/2} |p,q-3/2;i+1,j+1/2>, \quad (3.70)
\]

\[
\left| \frac{2(p+q)}{(2g-1)} \right|^{1/2} |p,q-3/2;i+1,j+1/2>, \quad (3.71)
\]

\[
\left| \frac{2(p+q)}{(2g-1)} \right|^{1/2} |p,q-3/2;i+1,j+1/2>, \quad (3.72)
\]

\[
\left| \frac{2(p+q)}{(2g-1)} \right|^{1/2} |p,q-3/2;i+1,j+1/2>, \quad (3.73)
\]

If in the formulae above a \(\delta\)-function vanishes, then the corresponding term should be replaced by zero independently of the fact that some other multiples in the same term are undefined (at \(p=0\) one has multiples \(0/0\)).

The transformations under the action of \(a^+\) remain the same [see (3.33)] also in the star basis (3.57):

\[
a^+ |p,q;i,j,j> = (I+i)(I+j+1)^{1/2} |p,q;i,j;j>. \quad (3.69)
\]
considers only vectors from the corresponding atypical module \( W(p,q) \) [see (3.52)].

E. Finite-dimensional representations.

The \( \mathfrak{so}(3) \otimes \mathfrak{sp}(2) \) structure of the finite-dimensional irreducible \( \mathfrak{osp}(3/2) \) modules has been described by Van der Jeugt. He has computed also the reduced matrix elements of the odd generators for all representations with \( p>3/2 \) and \( q>3/2 \). The reduced matrix elements for the \( \mathfrak{osp}(3/2) \) irreps have been found also by Le Blanc and Rowe. Therefore up to some technical details the matrix elements of the generators are known. In the present section we shall study the Kac modules which are either irreducible (typical representations) or indecomposable. We write down explicit relations for the transformations of the Kac modules under the action of \( a_1^+ \) and \( a_1^- \). For completeness we write also the relations for the typical and atypical representations. Our contribution here is only in the technical part: we succeed to write the transformation relations in a relatively compact form.

E1. Kac modules, typical modules, indecomposable modules

It remains to consider the \( \mathfrak{osp}(3/2) \) induced modules \( \tilde{W}(p,q) \) with \( 2p \in \mathbb{Z} \) and \( q=1,3/2,2, \ldots, n/2, \ldots \) (3.74)

In this case each \( \mathfrak{so}(3) \otimes \mathfrak{sp}(2) \) module \( V(p,q;I,J) \), \( 2q \in \mathbb{Z} \), appearing in the sum (3.31) carries an indecomposable representation of the even subalgebra with a maximal invariant subspace \( V_{\text{inv}}^*(p,q;p-l,q-l/2) \) [see (3.39)]. The only exception appears at \( q=1 \). Then one of the \( \mathfrak{so}(3) \otimes \mathfrak{sp}(2) \) modules \( V(p,q;I,J) \), namely \( V(1,1;1,1) \), is irreducible. By a straightforward computation one checks that the subspace

\[
V_{\text{inv}}(p,q) = V_{\text{inv}}(p,q;p-q-1) \oplus V_{\text{inv}}(p,q;p-1,q-1/2) \\
\oplus V_{\text{inv}}(p,q;p-1,q-1/2) \oplus V_{\text{inv}}(p,q;p-q-1/2) \\
\oplus V_{\text{inv}}(p,q;p+1,q-1/2) \\
\oplus V_{\text{inv}}(p,q;p+1,q-1/2)
\]

is an invariant and in fact even an irreducible \( \mathfrak{so}(3/2) \) module with a highest weight \( [p,-q-1] \). The factor space

\[
W_{\text{inv}}(p,q) = \tilde{W}(p,q)/V_{\text{inv}}(p,q)
\]

is a finite-dimensional \( \mathfrak{osp}(3/2) \) module, the Kac module in the terminology of Ref.32. One obtains the transformations of \( W_{\text{inv}}(p,q) \) replacing everywhere in eqs.(3.42)-(3.49) the basis vectors from \( V_{\text{inv}}(p,q) \) by zero. In order to avoid this 'cancellation' procedure it is convenient to imbed isomorphically \( W_{\text{inv}}(p,q) \) into \( \tilde{W}(p,q) \) through the natural identification

\[
W_{\text{inv}}(p,q) \cong \mathfrak{osp}(3/2) \quad \text{and} \quad \tilde{W}(p,q) \cong \mathfrak{osp}(3/2)
\]

where \( \Xi([p,q];I,J;i,j) \) is the equivalence class of the vector \([p,q];I,J;i,j] \). Thus [see also (3.40)] for the \( \mathfrak{so}(3) \otimes \mathfrak{sp}(2) \) structure of \( W_{\text{inv}}(p,q) \) we obtain:

\[
W_{\text{inv}}(p,q) = \mathfrak{osp}(p,p-q-1) \oplus \mathfrak{osp}(p,p-q-1/2) \oplus \mathfrak{osp}(p,p-q-1/2) \oplus \mathfrak{osp}(p,p-q-1/2) \\
\oplus \mathfrak{osp}(p,p+1,q-1) \oplus \mathfrak{osp}(p,p+1,q-1/2) \oplus \mathfrak{osp}(p,p+1,q-1/2) \\
\oplus \mathfrak{osp}(p,p+1,q-1/2) \oplus \mathfrak{osp}(p,p+1,q-1/2) \\
\oplus \mathfrak{osp}(p,p+1,q-1/2)
\]

Introduce a new basis \([p,q];I,J;i,j] \) in \( W_{\text{inv}}(p,q) \) setting for each \( \mathfrak{so}(3) \otimes \mathfrak{sp}(2) \) module \( V(p,q;I,J) \), which by definition has a nonnegative \( J \) [see (3.40)],

\[
\Xi([p,q];I,J;i,j]) = \left( \frac{J+1}{2} \right)^{1/2} [p,q];I,J;i,j]
\]

where

\[
i=J,J+1, \ldots, I+1, I, J, J-1, J-1, \ldots, J.
\]

In the basis (3.78) we have the usual relations for the finite-dimensional \( \mathfrak{sp}(2) \) modules:

\[
\epsilon_{\text{st}}(p,q;I,J;i,j] = \left( \frac{J+1}{2} \right)^{1/2} [p,q];I,J;i,j]
\]

\[
\epsilon_{\text{st}}(p,q;I,J;i,j] = \left( \frac{J+1}{2} \right)^{1/2} [p,q];I,J;i,j]
\]

The transformations of \( \mathfrak{osp}(3/2) \) under the action of \( a_1^+ \) remain the same as in (3.33) also in the new basis; for \( a_1^- \) we obtain (the relations (3.83), (3.84), (3.86), (3.87) should be omitted if \( p=0 \); (3.84), (3.87) have to be skipped if \( p=1/2 \); the last equation (3.88) should be omitted at \( q=1 \), since according to (3.75)
It is known\(^7\)\(^{40}\) (and one can check it directly from eqs. (3.33), (3.81)-(3.88)) that the Kac module \(W_{i\lambda}(p,q)\) is irreducible if and only if \(p-2q+1 = 0\). The modules corresponding to \(p-2q+1 \neq 0\) are the typical \(osp(3/2)\) modules in the terminology of Kac.\(^7\) If \(p-2q+1 = 0\), then \(\mathcal{W}(p,q)\) carries an indecomposable representation.
of \(osp(3/2)\). The maximal invariant subspace, as one can easily derive it from eqns. (3.81)-(3.88), reads [see (3.40)]:

\[
W(p-1,q-1/2) = W(p,q;p-1,q-1/2) \otimes W(p,q;p-1,q-1) \\
= W(p,q;p-1,q-1) \otimes W(q-3/2) W(p,q;p,q-3/2).
\]  

(3.89)

The above conclusion should not be applied for \(p=0\) since in view of \(p-2g+l=0\) it corresponds to \(q=1/2\) - a case, which we exclude from consideration [see (3.74)]. Thus the relations (3.33) and (3.81)-(3.88) describe explicitly the transformations of all irreducible (= typical) and all indecomposable Kac modules, except those corresponding to \(q=0, 1/2\).

E2. Atypical modules, irreducible modules

The atypical modules correspond to the case \(p-2g+l=0\). In order to obtain them one has to factorize the Kac module \(W_{\text{typ}}(p,q)\) with respect to the maximal invariant subspace (3.89):

\[
W(p,q) = W(p,q;p-1,q-1/2) W(p,q;p,q-3/2).
\]

(3.90)

We write the result in a form which incorporates both the typical modules \((p-2q+1=0)\) and the atypical modules \((p-2q+1=0)\):

\[
W(p,q) = W(p,q;p,q-3/2) \otimes W(p,q;p,q-3/2).
\]

(3.91)

In the atypical case \((p-2q+1=0)\) the last four terms in the decomposition (3.91) vanish so that \(W(p,q)\) is a sum of no more than four \(so(3) \otimes \mathfrak{sp}(2)\) irreducible modules. The decomposition (3.91) holds for all possible values of \(p\) and \(q\), including the forbidden so far \(q=0, 1/2\). It describes in an unified form all cases in the classification of Van der Jeugt\(^40\) (in the case \(A3\), p. 3340 of Ref. 40 \((3/2,1)\) should be replaced by \((3/2,1/2)\)).

Setting at \(p-2q+1=0\)

\[
|p-1,q-1/2;i,j\rangle = |p-1,q-1;i,j\rangle = |p,q-3/2;i,j\rangle = 0
\]

(3.92)
in (3.81)-(3.88) one obtains the transformations of all atypical modules. Observe that according to (3.91) several other vectors in (3.81)-(3.88) have to be replaced by zero for small values of \(p\) and \(q\) (in both the typical and the atypical cases):

1. for \(q=0,1/2\) \(|p,q-3/2;i,j\rangle = 0\) (3.93)
2. for \(q=0,1/2\) \(|p,q-1;i,j\rangle = |p+1,q-1;i,j\rangle = 0\) (3.94)
3. for \(q=0\) all basis vectors except \(|0,0\rangle, |0,0\rangle, |0,0\rangle\) are zero; (3.95)
4. for \(p=0\) \(|p-1,q-1/2;i,j\rangle = 0\) (3.96)
5. for \(p=1/2\) \(|p-1,q-1/2;i,j\rangle = 0\) (3.97)

We have derived eqs. (3.62)-(3.68) as transformation relations of the infinite-dimensional typical star modules. It is however straightforward to check that the these relations describe also the transformations of all finite-dimensional irreducible typical modules of \(osp(3/2)\), if for small \(p\) and \(q\) one is replacing by zero all vectors indicated in (3.93)-(3.97). Taking into account all this and (3.92) we obtain the following transformations of the \((p-2q+1=0)\) atypical modules:

\[
a_{1}^{\pm}(p,q;1;i,j) = \frac{(p+2)(2p+1)(p+1)(p+2)}{2q(p+1)} \left| p+1,q-1/2;i,j+1/2 \right> (3.98)
\]

\[
a_{2}^{\pm}(p,q;1;i,j) = \frac{(q+1)(q+2)(p+1)(p+2)}{2q(p+1)} \left| p+1,q-1/2;i,j-1/2 \right> (3.99)
\]

\[
a_{3}^{\pm}(p,q-1/2;i,j) = \frac{(p+2)(2p+1)(p+1)(p+2)}{2q(p+1)} \left| p+1,q;i+1,j+1 \right> (3.100)
\]

\[
+ \frac{1}{p+1} \frac{(p+2)(2p+1)(p+1)}{2(p+1)} \left| p+1,q-1/2;i,j+1/2 \right> (3.101)
\]

\[
+ \frac{1}{q+1} \frac{(q+1)(q+2)(p+1)}{2q(p+1)} \left| p+1,q;i+1,j+1 \right> (3.102)
\]
(3.101) The transformations (3.98)-(3.101) are identical with the transformations (3.61)-(3.66) if the latter are applied only to the vectors from the irreducible modules (3.91).

Define a nondegenerate Hermitian form $(\cdot,\cdot)$ in the typical module $\mathcal{W}(p,q)$ or in the atypical module $\mathcal{W}^{(p,q)}$ setting

\[ (\Omega(p,q);\Omega'(p,q)) = g(I,J) \delta_{i,i'} \delta_{j,j'} \delta_{,j}^i \delta_{J,j'}^j, \]

with $g(I,J) = \tau$ as given in Table I of Ref.40. By a straightforward computation one shows that

\[ (a^*_x,y) = (-1)^{\text{deg}(x)}(x,a^*_y), \quad (a^*_x,y) = (x,a^*_y). \]

Therefore with respect to the form (3.103) the representation of $osp(3/2)$ is a grade star representation; the orthogonal and normed up to a sign basis (3.78) is a grade star basis. As it was shown by Van der Jeugt (apart from some exceptional cases) the finite dimensional representations cannot be turned into grade star representations in a space with positive definite nondegenerate Hermitian form (i.e., in a space with a proper scalar product).

IV. CONCLUDING REMARKS

The motivation for the present investigation is of a physical origin. Elsewhere we shall show that one can define a three-dimensional noncanonical harmonic oscillator (in the sense of Ref.51) in such a way that the position operators $q_i$ and the momentum operators $p_i$, $i=1,2,3$, of the oscillator are the odd generators of the $osp(3/2)$ Lie superalgebra. The requirement that the energy spectrum is bounded from below leads to consideration of highest weight representation. The state space of the oscillator has to be proper Hilbert space. It turns out that the condition $q_i$ and $p_i$ to be selfadjoint operators is equivalent to the condition the representation space to be a star module of $osp(3/2)$. This was the reason we to consider here only the induced modules, generated from finite-dimensional representations of the stability subalgebra $so(3)\mathfrak{sl}(1)$. As we have seen these modules carry all highest weight star representations. From a mathematical point of view it is certainly of interest to write down the transformation relations for the other infinite-dimensional modules, classified in Ref.40. This should be relatively easy for the rest of the star modules, namely those from the class $D'$, and the related to them indecomposable modules. To this end one should consider lowest weight modules induced from finite-dimensional irreps of $so(3)\mathfrak{sl}(1)$. More generally, one has to consider modules, induced from infinite-dimensional representations of the stability subalgebra and more precisely of $so(3)$.

We hope that the approach used in the present investigation can be applied also to the highest rank orthosymplectic algebras. The reason for this stems from the observation that our results have been derived without using the Clebsch-Gordon coefficients of the even subalgebra. For the higher rank algebras this could be a real advantage since the Clebsch-Gordon coefficients in those cases are in general unknown.

Coming back to our results we underline that all typical and atypical star representations (infinite-dimensional and finite-dimensional) are described with one and the same transformation relations, namely (3.33) and (3.61)-(3.68). To this end however one has to use the relations only to vectors properly belonging to the corresponding module, setting all other vectors equal to zero (see (3.93)-(3.97)). Let us mention also that the results related to the indecomposable representations do not cover the case $q=0$ and $q=1/2$ (although we cover these values for the irreducible finite-dimensional representations). The peculiarity of these modules stems from the observation that they cannot be represented as direct sums of only highest weight modules of the even subalgebra.

Finally we add that we have done an independent check of the validity of our results: the operators $a^*_x$ (defined with eqs.(3.42)-(3.49) or with (3.61)-(3.68) or with (3.81)-(3.88)) together with th operators $a^*_{x,x}$ always satisfy the (anti)commutation relations (2.20).
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First we express the reduced basis in terms of the induced one. Throughout we skip \((p, q)\) in the notion of the induced and the reduced basis vectors, setting

\[
|p, q; 1, j\rangle = a|0, 0, 0, q-j; 1\rangle, \quad \text{(A.2)}
\]

\[
|p+1, q-1/2; 1, j\rangle = \frac{b_1}{2(p+1)(2p+1)^{1/2}} \left\{ |(p+i)(p+i+1)|^{1/2} \left| 1, 0, 0, q-j-1/2; i=1 > + |2(p-i+1)(p+i+1)|^{1/2} |0, 1, 0, q-j-1/2; i\rangle \right. \right.
\]

\[
- |(p-1)(p+1)|^{1/2} |0, 0, 1, q-j-1/2; i=1 > \}, \quad \text{(A.3)}
\]

\[
|p, q-1/2; 1, j\rangle = \frac{b_2}{p} \theta(p-1/2) \left\{ \left| \frac{1}{2} \right| |p-i+1)(p+i)|^{1/2} |1, 0, 0, q-j-1/2; i=1 > + |1/2 - 1/2 |0, 1, 0, q-j-1/2; i=1 > \}, \quad \text{(A.4)}
\]

\[
|p-1, q-1/2; 1, j\rangle = \frac{b_3}{p} \theta(p-1) \left\{ \left| \frac{1}{2} \right| |p-i+1)(p+i)|^{1/2} |1, 0, 0, q-j-1/2; i=1 > + |2(p-i)(p+i+1)|^{1/2} |0, 1, 0, q-j-1/2; i\rangle \right. \right.
\]

\[
\left. + |(p+i)(p+i+1)|^{1/2} |0, 0, 1, q-j-1/2; i=1 > \}, \quad \text{(A.5)}
\]

\[
|p+1, q-1/2; 1, j\rangle = \frac{c_1}{2(p+1)(2p+1)^{1/2}} \left\{ |(p+i)(p+i+1)|^{1/2} |1, 1, 0, q-j-1/2; i=1 > - |2(p-i+1)(p+i+1)|^{1/2} |1, 0, 1, q-j-1/2; i\rangle \right. \right.
\]

\[
\left. - |(p-1)(p+i+1)|^{1/2} |1, 1, 0, q-j-1/2; i=1 > + |2(p-i+1)(p+i+1)|^{1/2} |0, 0, 0, q-j; i\rangle \}, \quad \text{(A.6)}
\]

\[
|p, q-1; 1, j\rangle = \frac{c_2}{p} \theta(p-1/2)(1-\delta_{q,0}) \left\{ \left| \frac{1}{2} \right| |p-i+1)(p+i)|^{1/2} |1, 1, 0, q-j-1/2; i=1 > \right.
\]
We recall that the \( \Theta \)-functions in the above expressions are to indicate that for \( p=0 \) only the basis vectors defined with eqs. (A.2), (A.3), (A.6) and (A.9) remain; similar for \( p=1/2 \) one has to skip all vectors (A.5) and (A.8).

In the inverse to (A.2)-(A.9) relations (see below) one has to replace with zero all redundant vectors (3.30) (which happens only if \( p=0 \) or \( 1/2 \)).

\[
|0,0,0;n;i> = \frac{1}{a} |p,q;i,q-n> \\
|1,0,0;n;i> = \frac{1}{B_1} \left| \frac{(p+1)(p+i+2)}{2(p+1)(2p+1)} \right|^{1/2} |p+1,q-1/2;i+1,q-n-1/2> \\
- \frac{1}{B_2(2p+1)} \left| \frac{(p-i)(p+i+1)}{2} \right|^{1/2} |p,q-1/2;i+1,q-n-1/2> \\
- \frac{1}{B_3(2p+1)} \left| \frac{(2p-1)(p-i)(p+i-1)}{2} \right|^{1/2} |p-1,q-1/2;i+1,q-n-1/2>, (A.11)
\]

\[
|0,1,0;n;i> = \frac{1}{B_1} \left| \frac{(p+1)(p+i+1)}{(p+1)(2p+1)} \right|^{1/2} |p+1,q-1/2;i,q-n-1/2> \\
+ \frac{1}{B_2(2p+1)} |p,q-1/2;i,q-n-1/2> \\
+ \frac{1}{B_3(2p+1)} \left| \frac{(2p-1)(p-1)(p+i)}{2} \right|^{1/2} |p-1,q-1/2;i,q-n-1/2>, (A.12)
\]

\[
|0,0,1;n;i> = \frac{1}{B_1} \left| \frac{(p+i+1)(p+i+2)}{2(p+i)(2p+1)} \right|^{1/2} |p+1,q-1/2;i,q-1/2,q-n-1/2> \\
- \frac{1}{B_2(2p+1)} \left| \frac{(p+i)(p+i+1)}{2} \right|^{1/2} |p,q-1/2;i,q-1/2,q-n-1/2> \\
+ \frac{1}{B_3(2p+1)} \left| \frac{(2p-1)(p+i)(p+i-1)}{2} \right|^{1/2} |p-1,q-1/2;i,q-1/2,q-n-1/2>, (A.13)
\]

\[
|1,0,1;n;i> = \frac{1}{c_1} \left| \frac{(p+i)(p+i+1)}{2(p+i)(2p+1)} \right|^{1/2} |p+1,q-1;i+1,q-n-1> \\
+ \frac{1}{c_2(2p+1)} \left| \frac{(2p-1)(p+i)(p+i-1)}{2} \right|^{1/2} |p-1,q-1;i+1,q-n-1> \\
- \frac{1}{c_3(2p+1)} \left| \frac{(2p-1)(p-i)(p-i-1)}{2} \right|^{1/2} |p-1,q-1;i+1,q-n-1>., (A.14)
\]

\[
|1,0,1;n;i> = \frac{1}{c_1} \left| \frac{(p+i)(p+i+1)}{2(p+i)(2p+1)} \right|^{1/2} |p+1,q-1;i,q-n-1> \\
+ \frac{1}{c_2(2p+1)} \left| \frac{(2p-1)(p-i)(p-i-1)}{2} \right|^{1/2} |p-1,q-1;i,q-n-1> \\
- \frac{1}{c_3(2p+1)} \left| \frac{(2p-1)(p-i)(p-i-1)}{2} \right|^{1/2} |p-1,q-1;i,q-n-1>, (A.15)
\]

\[
|0,1,1;n;i> = \frac{1}{c_1} \left| \frac{(p+i)(p+i+2)}{2(p+i)(2p+1)} \right|^{1/2} |p+1,q-1;i-1,q-n-1> \\
+ \frac{1}{c_2(2p+1)} \left| \frac{(2p-1)(p-i)(p+i)}{2} \right|^{1/2} |p-1,q-1;i-1,q-n-1> \\
+ \frac{1}{c_3(2p+1)} \left| \frac{(2p-1)(p-i)(p+i-1)}{2} \right|^{1/2} |p-1,q-1;i-1,q-n-1>, (A.16)
\]

\[
|1,1,1;n;i> = \frac{1}{a} |p,q-3/2;i,q-n-3/2> \\
- \frac{1}{B_1(2p+1)} \left| \frac{(p+i)(p+i+1)}{2} \right|^{1/2} |p+1,q-1/2;i,q-n-3/2> \\
- \frac{1}{B_2(2p+1)} \left| \frac{(2p-1)(p+i)(p+i-1)}{2} \right|^{1/2} |p-1,q-1/2;i,q-n-3/2> \\
- \frac{1}{B_3(2p+1)} \left| \frac{(2p-1)(p-i)(p-i-1)}{2} \right|^{1/2} |p-1,q-1/2;i,q-n-3/2>, (A.17)
\]
For further information along this line see T.D.Palev, "Lie superalgebras, infinite-dimensional algebras and quantum statistics", Preprint YITF/K-888, Kyoto (1990).


