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FERMION INDUCED SINGULARITIES IN A NON-RELATIVISTIC EFFECTIVE LAGRANGIAN FOR ELECTROMAGNETISM

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ABSTRACT

Effective field theory methods are applied to the study of non–relativistic quantum electrodynamics in a slowly varying electromagnetic background. It is shown that the one–loop effective action has singularities at those values of the background fields which correspond to complete fillings of Landau levels. One immediate result of our work is a simple derivation of the oscillatory behaviour of the energy density , magnetization and permittivities, at zero temperature.

1. INTRODUCTION

In relativistic field theories it is nowadays common to express the contributions of quantum corrections in the form of an effective Lagrangian. It is characteristic of the formalism that the fields are slowly varying in space and time but not necessarily weak *). Examples are provided by the Euler Heisenberg [6] formulation of electrodynamics in which the electron variables are eliminated, their virtual motions giving rise to non–linear corrections to the Maxwell equations and the Coleman–Weinberg [5] formulation of gauge theories and its many extensions. This approach can be usefully applied also in the case of non–relativistic theory [9]. Such theories have their own peculiarities including, in the case of fermions, a ground state in which the density of particles is non–vanishing. In this note we undertake the task of constructing the leading terms of the effective Lagrangian for the electromagnetic field in the presence of charged fermions. Although this system has been analyzed quite thoroughly over the years, we believe that it may be instructive to consider this old problem from a somewhat more modern point of view. In keeping with the general philosophy of the method, we shall treat the magnetic field as a finite quantity although its derivatives are small relative to the microscopic scales of the system. It will turn out that the effective field equations are highly nonlinear. Indeed, there are singularities at those values of the fields which correspond to thresholds at which new Landau levels [10] are opened to occupation **). The result of the computation to one–loop order is expressed in the form

$$\mathcal{L} = \partial_j \chi (E_j + \partial_j \phi) - U(\phi, B) + \frac{1}{2}\, \varepsilon_{ij}(\phi, B) E_i E_j + \ldots \tag{1.1}$$

where χ and ϕ are a pair of auxiliary scalar fields. Both E_j and all derivatives, $\partial_j \phi$, $\partial_j B_k$, etc. are considered to be small quantities, powers of which are neglected. The potential $U(\phi, B)$ is given

*) The effective action is also known as the generating functional for irreducible vertices. Starting from the functional methods introduced by Schwinger [1] the properties of this functional were developed by many authors including particularly Jona–Lasinio [2] and De Witt [3]. When restricted to slowly varying fields the effective action can be expressed as the spacetime integral of the effective Lagrangian, a local function of the fields and their derivatives, up to any desired order. The leading term in the effective Lagrangian is the effective potential, which involves no derivatives. The latter function was first shown by Goldstone, Salam and Weinberg [4] to be of central importance in the study of spontaneous symmetry breaking. In this context, the systematic expansion in powers of derivatives was developed by Coleman and Weinberg [5]. Historically, the first calculation of the leading term in an effective Lagrangian was due to Heisenberg and Euler [6] who considered the contribution of charged electrons in a constant electromagnetic field. Generalizations of the effective action to functionals in which the Green's functions as well as the fields are treated as independent variables were developed by, for example, De Domenicis [7] and Cornwall, Jackiw and Tomboulis [8].

**) Analogous singularities in the effective theory of charged fermions in 2+1 dimensional spacetime were the subject of a recent study in induced Chern–Simons terms [11].

by a finite sum,

$$U(\phi, B) = \frac{B^2}{2e^2} - \frac{B}{6\pi^2 m} \sum_{n=0}^{N-1} (2m\phi - (2n+1)B)^{3/2} \tag{1.2}$$

where $B = \sqrt{B_j B_j}$, e^2 is the coupling strength, m is the fermion mass and $N = N(\phi, B)$ is a positive integer defined by the inequalities

$$\left(N + \frac{1}{2}\right)\frac{B}{m} > \phi > \left(N - \frac{1}{2}\right)\frac{B}{m}. \tag{1.3}$$

This means that the sum in (1.2) contains just those terms for which the quantity, $2m\phi - (2n+1)B$, is positive. The permittivity tensor, ε_{ij}, is defined by similar sums, given in the text below.

Although the potential U and its first derivatives are continuous functions of ϕ and B, its second derivatives are not. They, like the components of ε_{ij} have singularities at odd integer values of the ratio, $2m\phi/B$, where the function $N(\phi, B)$ is discontinuous.

The auxiliary fields ϕ and χ have been introduced in order to express the effective Lagrangian in a gauge invariant form. They can be eliminated quite simply in the Coulomb gauge, $\partial_j A_j = 0$, by substituting $\phi = A_0$. This follows because $\underline{\partial}^2 \chi$ acts as a Lagrange multiplier. An interpretation of the auxiliary fields is given in Section 2.

The one–loop contributions to U and ε_{ij} are obtained in Section 2 where also there is some discussion of the weak field approximation. Because of the above mentioned singularities the effective Lagrangian is not analytic in B. For small values of the ratio, $B/2m\phi$, the potential, for example, exhibits the well–known de Haas–van Alphen oscillations [12]. Section 3 is devoted to the case of weak fluctuations on a uniform background. It is shown there that the uniform background is stable and the spectrum of photon–like excitations is obtained

2. ONE–LOOP CONTRIBUTION

We consider the contributions due to a charged one–component non–relativistic fermion, ψ. Its equation of motion is given by

$$\left(i\nabla_0 + \frac{\nabla_j^2}{2m}\right)\psi = 0 \tag{2.1}$$

where, in the case of a uniform background the covariant derivatives are

$$\nabla_0 = \partial_0 - iA_0, \quad \nabla_1 = \partial_1 + iBx_2, \quad \nabla_2 = \partial_2, \quad \nabla_3 = \partial_3 \tag{2.2}$$

with A_0 and B constant. We have here a constant electrostatic potential and a constant magnetic field directed along the 3 axis. The general solution of (2.1) can be expressed in the form

$$\psi = \sum_n \int \frac{d^2 k}{(2\pi)^2}\, e^{-i(\varepsilon_{nk} - A_0)x^0} < \underline{x}|nk > \psi_{nk} \tag{2.3}$$

where $k = (k_1, k_3)$. The functions $< \underline{x}|nk >$ satisfy the equation

$$< \underline{x}|H|nk > = -\frac{1}{2m} \left\{ (\partial_1 + iBx_2)^2 + \partial_2^2 + \partial_3^2 \right\} < \underline{x}|nk > = \varepsilon_{nk} < \underline{x}|nk > \tag{2.4}$$

which can be separated. Its solution is given by

$$< \underline{x}|nk > = e^{ik_1x_1 + ik_3x_3}\, B^{1/4}\, v_n \left(\sqrt{B}\, x_2 + \frac{k_1}{\sqrt{B}} \right) , \tag{2.5}$$

$n = 0, 1, 2, \ldots, \quad -\infty < k_1, k_3 < \infty$. The functions v_n represent normalized oscillator wave functions,

$$v_n(Q) = (\sqrt{\pi}\, 2^n n!)^{-1/2} H_n(Q) e^{-Q^2/2}$$

and the eigenvalues are

$$\varepsilon_{nk} = \left(n + \frac{1}{2} \right) \frac{B}{m} + \frac{k_3^2}{2m} \tag{2.6}$$

which are seen to be independent of k_1. These are the well-known Landau levels characterizing the motion of a charged particle in a uniform magnetic field [10].

The equal time anticommutors are given as usual by

$$\{\psi(x), \psi(x')\} = 0 = \{\psi^+(x), \psi^+(x')\}$$
$$\{\psi(x), \psi^+(x')\} = \delta_3(\underline{x} - \underline{x}')$$

and, with the normalizations adopted here this implies

$$\{\psi_{nk}, \psi_{n'k'}\} = 0 = \{\psi_{nk}^+, \psi_{n'k'}^+\}$$
$$\{\psi_{nk}, \psi_{n'k'}^+\} = \delta_{nn'}\, 2\pi\, \delta(k_1 - k_1')\, 2\pi\, \delta(k_3 - k_3') \,.$$

The fermion ground state, $|F>$, is defined by

$$\begin{aligned} \psi_{nk}^+ |F> &= 0 , \qquad \varepsilon_{nk} < A_0 \\ \psi_{nk} |F> &= 0 , \qquad \varepsilon_{nk} > A_0 \,. \end{aligned} \tag{2.7}$$

The Fermi energy is thereby identified with A_0. The fermion Green's function is defined by *)

$$\begin{aligned} < F|T\psi(x)\psi^+(x')|F > &= \frac{1}{i}\, G_0(x, x') \\ &= \int \frac{d\omega}{2\pi i}\, e^{-i(\omega - A_0)(t-t')} < \underline{x}| \frac{1}{H - \omega} |\underline{x}' > \end{aligned} \tag{2.8}$$

where the Hamiltonian H is given by (2.4). The integration contour in (2.8) must be routed such that the poles corresponding to occupied states lie above the contour, while the rest are below it.

*) The conventions adopted here are as in Ref.11.

This can be achieved by taking the contour along the real axis and displacing the poles according to the prescription

$$\frac{1}{H - \omega} \rightarrow \frac{1}{H - \omega + i\eta(H)}$$

where $\eta(H)$ is an infinitesimal real operator defined such that

$$sgn\, \eta(\varepsilon_{nk}) = \begin{cases} 1, & \varepsilon_{nk} < A_0 \\ -1, & \varepsilon_{nk} > A_0 \end{cases}$$

The ground state expectation values of the number and energy densities can be evaluated by elementary methods,

$$\begin{aligned} < F|\psi^+(x)\psi(x)|F > &= i \lim_{\delta \to +0} G_0(x, x') \Big|_{\substack{\underline{x}' = \underline{x} \\ t' = t + \delta}} \\ &= \lim \sum_n \int \frac{d^2k}{(2\pi)^2}\, e^{i(\varepsilon_{nk} - A_0)\delta} \theta(A_0 - \varepsilon_{nk}) | < \underline{x}|nk > |^2 \\ &= \frac{B}{2\pi^2} \sum_0^{N-1} (2mA_0 - (2n+1)B)^{1/2} \end{aligned} \tag{2.9}$$

$$\begin{aligned} < F|\psi^+(x) i\partial_0 \psi(x)|F > &= i \lim_{\delta \to +0} \left(-\frac{1}{2m} \nabla_j^2 - A_0 \right) G_0(x, x') \Big|_{\substack{\underline{x}' = \underline{x} \\ t' = t + \delta}} \\ &= \lim \sum_n \int \frac{d^2k}{(2\pi)^2}\, e^{i(\varepsilon_{nk} - A_0)\delta} (\varepsilon_{nk} - A_0) \cdot \\ &\quad \cdot \theta(A_0 - \varepsilon_{nk}) | < \underline{x}|nk > |^2 \\ &= -\frac{B}{6\pi^2 m} \sum_0^{N-1} (2mA_0 - (2n+1)B)^{3/2} \end{aligned} \tag{2.10}$$

where $N = N(A_0, B)$ is an integer defined by the inequalities

$$\left(N + \frac{1}{2} \right) \frac{B}{m} > A_0 > \left(N - \frac{1}{2} \right) \frac{B}{m} \,. \tag{2.11}$$

It is, of course, a discontinuous function of A_0 and B.

Our main purpose here is to develop the leading terms in a expansion of the one-loop contribution to the effective action for the electromagnetic field. This takes the form

$$\begin{aligned} \Gamma^{(1)} &= i\, \ell n\, Det\, G(A) \\ &= \int d^4x \left[-U^{(1)} + \gamma^{(1)} A_i B_i + \frac{1}{2}\, \varepsilon_{ij}^{(1)}\, E_i E_j + \ldots \right] \end{aligned} \tag{2.12}$$

where $G(A) = G_0 + \ldots$ denotes the fermion Green's function in the presence of a slowly varying background, A_μ. The function $U^{(1)}$ is given by the expression on the right-hand side of (2.10) with B understood to represent the positive scalar $\sqrt{B_j B_j}$. The pseudoscalar $\gamma^{(1)} A_i B_i$ will not be present in the model treated here because the electrodynamics of charged fermions is parity conserving, $\gamma^{(1)} = 0$. The permittivity tensor, because of rotation invariance, has two independent components,

$$\varepsilon^{(1)}_{ij} = \varepsilon^{(1)}_1\ \delta_{ij} + \varepsilon^{(1)}_2\ B_i B_j \tag{2.13}$$

where ε_1 and ε_2 are functions of A_0 and B. These functions can be expressed in terms of the propagator, G_0, defined above. If needed, it would be possible to compute higher order terms, $(\partial B)^2, E^4$, etc., in the expansion (2.12). We illustrate the method here by explicitly computing the quantities $U^{(1)}$ and $\varepsilon^{(1)}_{ij}$.

The functional derivatives of $\Gamma^{(1)}$ with respect to $A_0(x)$ take the form,

$$\frac{\delta\Gamma^{(1)}}{\delta A_0} = -\frac{\partial U^{(1)}}{\partial A_0} + \frac{1}{2}\ \frac{\partial \varepsilon^{(1)}_{ij}}{\partial A_0}\ E_i E_j + \partial_i\left(\varepsilon^{(1)}_{ij} E_j\right) + \ldots \tag{2.14a}$$

$$\frac{\delta^2\Gamma^{(1)}}{\delta A_0(x)\delta A_0(x')} = -\frac{\partial^2 U^{(1)}}{\partial A_0^2}\ \delta_4(x-x') - \varepsilon^{(1)}_{ij}\ \partial_i\partial_j\ \delta_4(x-x') + \ldots \tag{2.14b}$$

where, in the latter formula, we are evaluating at $E_i = 0, A_0, B = \text{const}$. On the other hand, from the definition (2.8) it follows that

$$\frac{\delta}{\delta A_0(x)}\ G_0^{-1}(y, y') = -\delta_4(x-y)\delta_4(x-y') \ .$$

This implies that the functional derivatives of $\Gamma^{(1)} = i\ \ell n\ Det\ G$ evaluated in the uniform background are given by

$$\begin{aligned}\frac{\delta\Gamma^{(1)}}{\delta A_0(x)} &= i\ G_0(x,x) \\ &= -\frac{\partial U^{(1)}}{\partial A_0} \end{aligned}\tag{2.15}$$

$$\begin{aligned}\frac{\delta^2\Gamma^{(1)}}{\delta A_0(x)\delta A_0(x')} &= i\ G_0(x,x')G_0(x',x) \\ &= -\frac{\partial^2 U^{(1)}}{\partial A_0^2}\ \delta_4(x-x') - \varepsilon^{(1)}_{ij}\ \partial_i\partial_j\ \delta_4(x-x') + \ldots \end{aligned}\tag{2.16}$$

Equation (2.15) is equivalent to (2.9) and serves to determine the function $U^{(1)}$ up to a term which is independent of A_0. Eq.(2.16) can be used to determine $\varepsilon^{(1)}_{ij}$. It should be clear that higher terms in (2.12) could be determined in a similar fashion by examining various functional derivatives of $\Gamma^{(1)}$, which can all be expressed in terms of G_0 when evaluated in the uniform background, $E_j = 0, A_0, B = \text{const}$.

To continue with the evaluation of $\varepsilon^{(1)}_{ij}$, take the Fourier transform with repect to x and x' of Eq.(2.16),

$$\begin{aligned}(2\pi)^4\delta_4(k-k')\left\{-\frac{\partial^2 U^{(1)}}{\partial A_0^2} + \varepsilon^{(1)}_{ij}\ k_i k_j + \ldots\right\} &= \\ &= \int d^4x\ d^4x'\ e^{-ikx+ik'x'}\ i\ G_0(x,x')G_0(x',x) \\ &= i\ 2\pi\ \delta(k_0-k_0')\int\frac{d\omega}{2\pi}\ Tr\left(e^{-i\underline{kx}}\ G(\omega)\ e^{i\underline{k'x}}\ G(\omega+k_0)\right)\end{aligned}\tag{2.17}$$

where $G(\omega) = (H + i\eta(H) - \omega)^{-1}$ and the trace is over the one–fermion Hilbert space. Evaluation of the trace is made somewhat complicated by the fact that the fermions are moving in a uniform magnetic background, $B_i = B\ \delta_{i3}$. However, it can be handled by purely algebraic means.

To simplify the trace introduce the operator notation x_i, π_i with $[x_i, \pi_j] = i\ \delta_{ij}$, etc. The Hamiltonian (2.4) is then given by

$$\begin{aligned}H &= \frac{1}{2m}\ \left\{(\pi_1 + Bx_2)^2 + \pi_2^2 + \pi_3^2\right\} \\ &= \frac{1}{2m}\left\{P^2 + B^2Q^2 + \pi_3^2\right\}\end{aligned}\tag{2.18}$$

where

$$P = \pi_1 + Bx_2, \qquad Q = -\pi_2/B\ . \tag{2.19}$$

Notice that the components P and Q are canonical conjugates, $[Q, P] = i$. An independent canonical pair, which commutes with H, is given by

$$p = \pi_1, \qquad q = x_1 + \pi_2/B\ . \tag{2.20}$$

It is necessary to express x_1 and x_2 in terms of the new variables so that

$$\underline{kx} = k_1(q+Q) + \frac{k_2}{B}(P-p) + k_3x_3\ .$$

The trace in (2.17) contains the factor

$$Tr\left[\exp\frac{1}{i}\left(k_1q - \frac{k_2}{B}p\right)\exp\frac{1}{i}\left(-k_1'q + \frac{k_2'}{B}p\right)\right] = 2\pi B\ \delta(k_1-k_1')\delta(k_2-k_2')\ . \tag{2.21}$$

The other factor is

$$\begin{aligned}&Tr\left[\exp\frac{1}{i}\left(k_1Q + \frac{k_2}{B}\ P + k_3x_3\right)G(\omega)\exp i\left(k_1Q + \frac{k_2}{B}\ P + k_3'x_3\right)G(\omega+k_0)\right] = \\ &= 2\pi\ \delta(k_3-k_3')\sum_{nn'}\int\frac{dp_3}{2\pi}\ \frac{|<n|\exp\frac{1}{i}\left(k_1Q + \frac{k_2}{B}\ P\right)|n'>|^2}{\left(\varepsilon_{np_3} - \omega\right)\left(\varepsilon_{n'p_3+k_3} - \omega - k_0\right)}\end{aligned}\tag{2.22}$$

where we have introduced a basis of oscillator eigenstates,

$$|n> = \frac{1}{\sqrt{n!}}\ \left(\sqrt{\frac{B}{2}}\ Q - i\ \frac{P}{\sqrt{2B}}\right)^n |0>, \quad n = 0, 1, 2, \ldots$$

Substituting the factors (2.21) and (2.22) into the right–hand side of (2.17) and integrating over ω we obtain

$$(2\pi)^4\delta_4(k-k')\,\frac{B}{2\pi}\sum_{nn'}\int\frac{dp}{2\pi}\,\frac{|<n|\exp\frac{1}{i}\left(k_1Q+\frac{k_2}{B}P\right)|n'>|^2}{\varepsilon_{n'p+k_3}-\varepsilon_{np}-k_0}$$

$$\cdot\left\{\theta\left(\varepsilon_{n'p+k_3}-A_0\right)\theta\left(A_0-\varepsilon_{np}\right)-\theta\left(\varepsilon_{np}-A_0\right)\theta\left(A_0-\varepsilon_{n'p+k_3}\right)\right\} \quad (2.23)$$

which is to be expanded in powers of k_μ and compared with the left–hand side of (2.17). Consider firstly the case $k_0 = k_1 = k_2 = 0, k_3 > 0$. Apart from the momentum conservation factor, the expression (2.23) reduces to

$$\frac{B}{2\pi}\sum_n\int\frac{dp}{2\pi}\frac{2m}{(p+k_3)^2-p^2}\,\{\theta(\varepsilon_{np+k_3}-A_0)\theta(A_0-\varepsilon_{np})-\theta(\varepsilon_{np}-A_0)\theta(A_0-\varepsilon_{np+k_3})\}=$$

$$=\frac{B}{2\pi}\frac{4m}{2\pi}\sum_n\int dp\,\frac{\theta(|p+k_3|-k_F)\theta(k_F-|p|)}{(p+k_3)^2-p^2}$$

$$=\frac{Bm}{\pi^2}\sum_n\int_{k_F-k_3}^{k_F}dp\,\frac{1}{(p+k_3)^2-p^2}$$

$$=\frac{Bm}{\pi^2}\sum_n\left[\frac{1}{2k_F}+\frac{1}{24}\frac{k_3^2}{k_F^3}+\ldots\right]$$

where

$$k_F^2 = 2m\,A_0-(2n+1)B, \qquad n=0,1,\ldots,N-1\,.$$

On comparison with (2.17) this gives

$$-\frac{\partial^2U^{(1)}}{\partial A_0^2}=\frac{Bm}{2\pi^2}\sum_0^{N-1}(2mA_0-(2n+1)B)^{-1/2} \quad (2.24a)$$

$$\varepsilon_{33}^{(1)}=\frac{Bm}{24\pi^2}\sum_0^{N-1}(2mA_0-(2n+1)B)^{-3/2}\,. \quad (2.24b)$$

Note that (2.24a) is consistent with the expression (2.10) for $U^{(1)}$. To obtain $\varepsilon_{11}^{(1)}$ we should examine the configuration $k_0 = k_2 = k_3 = 0, k_1 > 0$, picking out the coefficient of k_1^2 in the expansion of (2.23). For this we need the oscillator matrix elements

$$|<n|e^{-ik_1Q}|n+1>|^2 = (n+1)\frac{k_1^2}{B}$$

$$|<n|e^{-ik_1Q}|n-1>|^2 = n\frac{k_1^2}{B}$$

so that (2.23) gives

$$\frac{B}{2\pi}\sum_n\int\frac{dp}{2\pi}\left[\frac{(n+1)k_1^2/B}{B/m}\left\{\theta\left(\left(n+\frac{3}{2}\right)\frac{B}{m}+\frac{p^2}{2m}-A_0\right)\theta\left(A_0-\left(n+\frac{1}{2}\right)\frac{B}{m}-\frac{p^2}{2m}\right)-\right.\right.$$

$$\left.-\theta\left(\left(n+\frac{1}{2}\right)\frac{B}{m}+\frac{p^2}{2m}-A_0\right)\theta\left(A_0-\left(n+\frac{3}{2}\right)\frac{B}{m}-\frac{p^2}{2m}\right)\right\}$$

$$-\frac{nk_1^2/B}{B/m}\left\{\theta\left(\left(n-\frac{1}{2}\right)\frac{B}{m}+\frac{p^2}{2m}-A_0\right)\theta\left(A_0-\left(n+\frac{1}{2}\right)\frac{B}{m}-\frac{p^2}{2m}\right)-\right.$$

$$\left.\left.-\theta\left(\left(n+\frac{1}{2}\right)\frac{B}{m}+\frac{p^2}{2m}-A_0\right)\theta\left(A_0-\left(n-\frac{1}{2}\right)\frac{B}{m}-\frac{p^2}{2m}\right)\right\}\right]$$

$$=2\,\frac{B}{2\pi}\frac{mk_1^2}{B^2}\sum_n(n+1)\int\frac{dp}{2\pi}\,\theta(p^2-2mA_0+(2n+3)B)\theta(2mA_0-(2n+1)B-p^2)$$

$$=\frac{mk_1^2}{2\pi^2B}\sum_n(n+1)\left\{(2mA_0-(2n+1)B)^{1/2}-(2mA_0-(2n+3)B)^{1/2}\right\}$$

$$=\frac{mk_1^2}{2\pi^2B}\sum_0^{N-1}(2mA_0-(2n+1)B)^{1/2}$$

which implies

$$\varepsilon_{11}^{(1)}=\frac{m}{2\pi^2B}\sum_0^{N-1}(2mA_0-(2n+1)B)^{1/2}\,. \quad (2.25)$$

This expression, together with (2.24b), determines the invariant components of the permittivity tensor as expressed in the formula (2.13), viz.

$$\varepsilon_1^{(1)}(A_0,B)=\frac{m}{2\pi^2B}\sum_0^{N-1}(2mA_0-(2n+1)B)^{1/2}$$

$$B^2\varepsilon_2^{(1)}(A_0,B)=\frac{m}{2\pi^2B}\sum_0^{N-1}\left\{-(2mA_0-(2n+1)B)^{1/2}+\frac{B^2}{12}(2mA_0-(2n+1)B)^{-3/2}\right\}\,. \quad (2.26)$$

The energy density, determined in the course of this calculation, is given by

$$U^{(1)}(A_0,B)=-\frac{B}{6\pi^2m}\sum_0^{N-1}(2mA_0-(2n+1)B)^{3/2}\,. \quad (2.27)$$

The results (2.26) and (2.27) are to be adjoined to the zeroeth order expressions representing the free Maxwell Lagrangian,

$$\varepsilon_1^{(0)}=\frac{1}{e^2},\quad \varepsilon_2^{(0)}=0,\quad U^{(0)}=\frac{B^2}{2e^2} \quad (2.28)$$

where e^2 denotes the electric charge. In principle one could compute higher order corrections to these quantities, contributions of order $(e^2)^{L-1}$ corresponding to graphs with L loops. The final result of these calculations is the effective Lagrangian

$$\mathcal{L} = -U(A_0, B) + \frac{1}{2}\,\varepsilon_{ij}(A_0, B)E_iE_j + \dots \tag{2.29}$$

The presence of the electrostatic potential A_0 in the expression (2.29) is a little irritating because it seems to suggest a lack of gauge invariance. This is an illusion, however, because the constant part of A_0 is gauge invariant: it is to be interpreted as the Fermi energy. To avoid even the suggestion of non–invariance we can regard (2.29) as a Coulomb gauge formula. In the Coulomb gauge, $\partial_j A_j = 0$ so that we have

$$A_0 = -\frac{1}{\underline{\partial}^2}\,\partial_j\,E_j\,. \tag{2.30}$$

Let us therefore introduce a Lagrange multiplier $\underline{\partial}^2\chi$ to enforce the cosntraint (2.30), replacing (2.9) by

$$\mathcal{L} = \partial_j\chi(E_j + \partial_j\phi) - U(\phi, B) + \frac{1}{2}\,\varepsilon_{ij}(\phi, B)E_iE_j + \dots \tag{2.31}$$

regarding ϕ and χ as auxiliary gauge invariant scalars. In this arrangement the Fermi level is represented by the scalar field, ϕ. The other scalar, χ, can be related to the fermion charge density. Thus, in the presence of an external charge density, j_0^{ext}, the Gauss law derived from (2.31) takes the form

$$\partial_i(\varepsilon_{ij}E_j) = j_0^{ext} - \underline{\partial}^2\chi$$

so that χ can be interpreted as the electrostatic potential due to the fermions.

The effective "Maxwell" equations derived from (2.31) are nonlinear in B and ϕ. We shall not attempt to discuss their properties in this note apart from an analysis, to be given in Section 3, of weak perturbations on a static uniform background. Probably the most interesting feature of (2.31) is the fact that the functions U and ε_{ij} are singular. They have branch points at the odd integer values of the ratio, $2m\phi/B$. Although U and its first derivatives are continuous at these points, its second derivatives – which occur in the field equations – are divergent. The longitudinal permittivity ε_2 and the first derivatives of the transverse permittivity ε_1 are also divergent. These singularities are associated with thresholds at which $N(\phi, B)$ changes discontinuously, and they seem to imply the existence of spatial surfaces across which some of the field components are discontinuous.

To conclude this section we consider the behaviour of the effective Lagrangian for small values of the ratio $B/2m\phi$. Quantities such as the potential, U, and the permittivities, ε_{ij}, exhibit branch point singularities associated with discontinuities of the function $N(\phi, B)$ and these become dense at the origin, $B/2m\phi = 0$. It is possible to develop asymptotic formulas for the neighbourhood of the origin where the quantities of interest show the oscillatory behaviour characteristic of

the de Haas–van Alphen effect. We begin with the potential (2.27),

$$\begin{aligned} U^{(1)}(\phi, B) &= -\frac{B}{6\pi^2 m}\sum_0^{N-1}(2m\phi - (2n+1)B)^{3/2} \\ &= -\frac{(2m\phi)^{5/2}}{6\pi^2 m}\,x\sum_0^{N-1}(1-(2n+1)x)^{3/2} \end{aligned} \tag{2.32}$$

where $x = B/2m\phi$. To manipulate sums of this kind it is useful to replace the power, $3/2$, by an arbitrary number. Define the function

$$S(x) = x\sum_0^{N-1}(1-(2n+1)x)^{-s}, \quad \frac{1}{2N+1} < x < \frac{1}{2N-1}\,. \tag{2.33}$$

An asymptotic expression for the neighbourhood of $x = 0$ can be obtained by the method explained in, for example, the text of Landau and Lifshitz [12]. It is based on the Fourier identity

$$\sum_n \delta(y-n) = \sum_k e^{2\pi iky} \tag{2.34}$$

where the sums range over all integers. From (2.34) we have also,

$$\begin{aligned} \sum_n \delta(y-(2n+1)) &= \frac{1}{2}\sum_n \delta\left(\frac{y-1}{2} - n\right) \\ &= \frac{1}{2}\sum_k (-)^k e^{i\pi ky}\,. \end{aligned}$$

This can be used to rearrange the sum (2.33) as follows,

$$\begin{aligned} S(x) &= x\int_0^{1/x} dy\,(1-xy)^{-s}\sum_n \delta(y-(2n+1)) \\ &= \frac{x}{2}\sum_k (-)^k\int_0^{1/x} dy\,(1-xy)^{-s}\,e^{i\pi ky} \\ &= \frac{1}{2}\sum_k (-)^k\int_0^1 dt\,(1-t)^{-s}\,e^{i\pi kt/x}\,. \end{aligned}$$

The restrictions on the range of y serves to pick out the delta functions with $n = 0, 1, \dots, N-1$. The interchange of the order of summation and integration is justifiable if x does not coincide with any of the singularities. The integral over t converges for $Re\ s < 1$. For $k = 0$ the integral is elementary. For $k \neq 0$ it can be expressed in terms of incomplete gamma functions [13]. Thus,

$$\begin{aligned} S(x) &= \frac{1}{2(1-s)} + \sum_{k>0}(-)^k\int_0^1 dt\,(1-t)^{-s}\cos\left(\frac{\pi kt}{x}\right) \\ &= \frac{1}{2(1-s)} + \sum_{k>0}(-)^k\,Re\left[e^{i\pi k/x}\left(e^{i\pi/2}\frac{\pi k}{x}\right)^{s-1}\gamma\left(1-s, e^{i\pi/2}\frac{\pi k}{x}\right)\right]. \end{aligned} \tag{2.35}$$

This result is presumably exact but, if $x \ll 1$, we can replace γ by its asymptotic expansion,

$$\gamma(1-s,u) \simeq \Gamma(1-s) - u^{-s}\, e^{-u}(1+O(1/u))$$

which is valid for $|u| \to \infty, -3\pi/2 < arg\ u < 3\pi/2$. We obtain thereby, the asymptotic formula

$$S(x) \simeq \frac{1}{2(1-s)} + \Gamma(1-s) \sum_{k>0} (-)^k \left(\frac{\pi k}{x}\right)^{s-1} \cos\left(\frac{\pi k}{x} + (s-1)\frac{\pi}{2}\right) \qquad (2.36)$$

up to terms of order $\exp(-1/x)$. The convergence of the sum is rather delicate but the oscillations in $1/x$ with period 2 are clear. Singularities at $x = (2N+1)^{-1}$ can appear when s is not sufficiently negative. With $s = -3/2$ we obtain

$$U^{(1)}(\phi, B) \simeq -\frac{(2m\phi)^{5/2}}{30\pi^2 m} + \frac{B^{5/2}}{8\pi^4 m} \sum_{k>0} (-)^k k^{-5/2} \cos\left(\frac{2\pi k m \phi}{B} - \frac{\pi}{4}\right).$$

Analogous formulae for $\varepsilon_{11}^{(1)}$ and $\varepsilon_{33}^{(1)}$ could be obtained from (2.36) by taking $s = -1/2$ and $s = 3/2$. In the latter case the sum over k diverges and ceases to be useful. The same is true for the derivatives of $U^{(1)}$. We shall not take this any further since our purpose has been only to highlight the singular aspects of the effective Lagrangian.

That the effective Lagrangian is not analytic in B is clear from expressions (2.24)–(2.27). This may come as a surprise to those who are familiar with the development of relativistic field theories where power series methods are commonly used. Relativistic theories are usually concerned with perturbations on a Lorentz invariant ground state, the vacuum. In contrast, we are dealing here with a non–invariant ground state, one which is characterized by the occupation of some specified one–fermion states. The specification of these states depends in a non–analytic way on the field B, and is reflected in the fact that the fermion Green's function, G_0, is not analytic in B. The various terms in the effective Lagrangian which we have computed are basically functionals of G_0 and hence they too cannot be expanded in power series.

3. UNIFORM BACKGROUND

To test the consistency of the effective theory derived in Section 2 we consider the structure of weak perturbations on a uniform background. We suppose that the system is coupled to a constant external charge density, $j_0^{ext} = n_e$. This is effected by adding to the Lagrangian (2.31), the source term,

$$\mathcal{L}_{source} = -n_e\, A_0(x)\ . \qquad (3.1)$$

It is straightforward to find a uniform solution of the effective field equations,

$$E_i = 0\,, \quad B_i = \overline{B}\,\delta_{i3}\,, \quad \phi = \bar{\phi} \qquad (3.2)$$

where $\overline{B}$ and $\bar{\phi}$ are constants. With this ansatz the field equations reduce to the pair,

$$-\frac{\partial U(\bar{\phi}, \overline{B})}{\partial \bar{\phi}} = n_e \qquad (3.3a)$$

$$\underline{\partial}^2\, \overline{\chi} = n_e\,. \qquad (3.3b)$$

Equation (3.3a) serves to determine $\bar{\phi}$ as a function of $\overline{B}$, the latter being an arbitrary (positive) constant. Although $\overline{\chi}$ is not well behaved in the case of infinite volume, this problem is an artifact of the infinite volume limit and will not affect any question of substance.

We wish to study the weak perturbations of the uniform solution. To this end we write

$$\begin{aligned} \phi &= \bar{\phi} + \phi', \qquad & \chi &= \overline{\chi} + \chi' \\ B_i &= \overline{B}\,\delta_{i3} + B_i', \qquad & E_i &= E_i' \end{aligned} \qquad (3.4)$$

where the primed quantities are small. When (3.4) is substituted into the Lagrangian, $\mathcal{L} + \mathcal{L}_{source}$, the first order terms vanish while the second order terms take the form

$$\mathcal{L}_2 = \frac{1}{2}\,\varepsilon_{ij}\, E_i' E_j' - \frac{1}{2}\,\alpha_{ij}\, B_i' B_j' - \beta\, B_3' \phi' + \frac{\mu^2}{2}\,\phi'^2 + \partial_j \chi'(E_j' + \partial_j \phi') \qquad (3.5)$$

in which the coefficients are defined by

$$\begin{aligned} \alpha_{11} = \alpha_{22} &= \frac{1}{B}\frac{\partial U}{\partial B} \\ \alpha_{33} &= \frac{\partial^2 U}{\partial B^2} \\ \beta &= \frac{\partial^2 U}{\partial \phi \partial B} \\ \mu^2 &= -\frac{\partial^2 U}{\partial \phi^2} \end{aligned} \qquad (3.6)$$

evaluated in the constant background, $\phi = \bar{\phi}, B = \overline{B}$. The quantities (3.6), together with ε_{ij}, are given up to one–loop accuracy by the formulae of Section 2, (2.28), (2.27), (2.24) and (2.25). The determination of $\bar{\phi}$ in terms of $\overline{B}$ is a non–trivial algebraic problem: from (2.27) the one–loop approximation to the equation of motion (3.3a) takes the form,

$$\frac{\overline{B}}{2\pi^2} \sum_{0}^{N-1} (2m\bar{\phi} - (2n+1)\overline{B})^{1/2} = n_e\,. \qquad (3.7)$$

The linearized equations for the weak fields, E_i', etc., are obtained from the expression (3.5). To test their stability it is useful to construct a Hamiltonian density

$$H_2 = \Pi_i'\, \partial_0 A_i' - \mathcal{L}_2 \qquad (3.8)$$

where the canonical momenta Π_i' are defined in the usual way,

$$\Pi_i' = \frac{\partial \mathcal{L}_2}{\partial(\partial_0 A_0')} = \varepsilon_{ij} E_j' + \partial_i \chi' . \quad (3.9)$$

One finds

$$H_2 = \frac{1}{2}\varepsilon_{ij} E_i' E_j' + \frac{1}{2}\alpha_{ij} B_i' B_j' + \beta \phi' B_3' - \frac{\mu^2}{2}\phi'^2 - \partial_j \chi' \partial_j \phi' + \Pi_i' \partial_i A_0' \quad (3.10)$$

where E_i' is to be read as a function of Π_i' and $\partial_i \chi'$ given by (3.9). The variables ϕ', χ' and A_0' are not dynamical and should be eliminated. Notice firstly that the variational derivative with respect to A_0 gives the Gauss law constraint,

$$\partial_i \Pi_i' = 0 . \quad (3.11)$$

Secondly, variation of ϕ' gives the equation

$$\phi' = \frac{1}{\mu^2}(\beta B_3' + \underline{\partial}^2 \chi') \quad (3.12)$$

which can be used to eliminate ϕ' from (3.10). When this is done the Hamiltonian reduces to

$$H_2 = \frac{1}{2}\varepsilon_{ij} E_{ij}' E_j' + \frac{1}{2}\alpha_{ij} B_i' B_j' + \frac{1}{2\mu^2}(\beta B_3' + \underline{\partial}^2 \chi')^2 . \quad (3.13)$$

Finally, the auxiliary variable χ' could be eliminated by solving the equation

$$0 = \frac{\partial H_2}{\partial \chi'} = \varepsilon_{ij}^{-1}(\partial_i \Pi_j' - \partial_i \partial_j \chi') + \frac{1}{\mu^2}\underline{\partial}^2(\beta B_3' + \underline{\partial}^2 \chi') \quad (3.14)$$

but this would lead to a non–local expression. In any event, the expression (3.13) for the energy density associated with weak fluctuations is positive if the matrices ε_{ij} and α_{ij} are positive definite and if $\mu^2 > 0$. If these conditions are met then the uniform solution is stable against weak perturbations. We list the relevant formulae,

$$\varepsilon_{11} = \varepsilon_{22} = \frac{1}{e^2} + \frac{m}{2\pi^2 \overline{B}} \sum_0^{N-1} (2m\bar{\phi} - (2n+1)\overline{B})^{1/2}$$

$$\varepsilon_{33} = \frac{1}{e^2} + \frac{m\overline{B}}{24\pi^2} \sum_0^{N-1} (2m\bar{\phi} - (2n+1)\overline{B})^{-3/2}$$

$$\alpha_{11} = \alpha_{22} = \frac{1}{e^2} + \frac{1}{12\pi^2 m\overline{B}} \sum_0^{N-1} (2m\bar{\phi} - (2n+1)\overline{B})^{1/2}(5(2n+1)\overline{B} - 4m\bar{\phi})$$

$$\alpha_{33} = \frac{1}{e^2} + \frac{1}{8\pi^2 m} \sum_0^{N-1} (2m\bar{\phi} - (2n+1)\overline{B})^{-1/2}(2n+1)(-5(2n+1)\overline{B} + 8m\bar{\phi})$$

$$\mu^2 = \frac{m\overline{B}}{2\pi^2} \sum_0^{N-1} (2m\bar{\phi} - (2n+1)\overline{B})^{-1/2} . \quad (3.15)$$

In each of these sums the upper limit $N = N(\bar{\phi}, \overline{B})$ is determined by the inequalities

$$(2N+1)\overline{B} > 2m\bar{\phi} > (2N-1)\overline{B} . \quad (3.16)$$

Observe that μ^2 is represented by a sum of positive quantities. It is clear also that the ε_{ij} and α_{ij} can be positive for some range of $\bar{\phi}$ and $\overline{B}$ because the coupling parameter e^2 is an independent positive quantity. For this range stability is thus ensured.

To conclude this analysis we compute the excitation spectrum. The most efficient way to do this is to eliminate the auxiliary fields ϕ' and χ' from the Lagrangian (3.5), reducing it to the non–local form,

$$\mathcal{L}_2' = \frac{1}{2}\varepsilon_{ij} E_i' E_j' - \frac{1}{2}\alpha_{ij} B_i' B_j' + \beta B_3' \frac{1}{\underline{\partial}^2}\partial_j E_j' + \frac{\mu^2}{2}\partial_i E_i' \left(\frac{1}{\underline{\partial}^2}\right)^2 \partial_j E_j' . \quad (3.17)$$

Add to this the gauge–fixing and source terms,

$$\mathcal{L}'' = C' \partial_j A_j' + J^\mu A_\mu' \quad (3.18)$$

where C' is a Lagrange multiplier and J^μ is an external current. The linear, inhomogeneous equations of motion which result are

$$\left\{-\varepsilon_{ij}\partial_0^2 - \beta \frac{\partial_0 \partial_\ell \partial_j}{\underline{\partial}^2}\varepsilon_{i3\ell} + \beta \frac{\partial_0 \partial_\ell \partial_i}{\underline{\partial}^2}\varepsilon_{j3\ell} + \mu^2 \frac{\partial_0^2 \partial_i \partial_j}{\underline{\partial}^4} + \alpha_{mn}\varepsilon_{mki}\varepsilon_{n\ell j}\partial_k \partial_\ell\right\} A_j' + \left\{\varepsilon_{ij}\partial_j \partial_0 + \beta \varepsilon_{i3\ell}\partial_\ell - \mu^2 \frac{\partial_i \partial_0}{\underline{\partial}^2}\right\} A_0' - \partial_i C' = -J_i$$

$$\left\{\varepsilon_{ij}\partial_j \partial_0 - \beta \varepsilon_{i3\ell}\partial_\ell - \mu^2 \frac{\partial_i \partial_0}{\underline{\partial}^2}\right\} A_0' + \left\{-\varepsilon_{ij}\partial_i \partial_j + \mu^2\right\} A_0' = J_0$$

$$\partial_i A_i' = 0 . \quad (3.19)$$

The equation for C' is easily isolated from these,

$$\underline{\partial}^2 C' = \partial_\mu J^\mu ,$$

so that C' can be eliminated. The resulting four equations for A_μ' can be solved algebraically in momentum space. For determining the spectrum the relevant quantity is the determinant of the coefficients, $\Delta(k)$, which is a polynomial in the frequency, k_0. It takes the form

$$\Delta(k) = \frac{k_0}{\underline{k}^2}(a k_0^4 - b k_0^2 + c) \quad (3.20)$$

where the coefficients, a, b, c are rather complicated functions of the wave vector k_j. The frequencies of the propagating waves are given by the zeroes of of Δ, viz.

$$\omega_\pm^2 = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} . \quad (3.21)$$

The result of a lengthy computation is

$$
\begin{aligned}
a &= \varepsilon_\ell \varepsilon_t^2 (k_t^2 + k_\ell^2)^2 + \varepsilon_t(\varepsilon_\ell k_t^2 + \varepsilon_t k_\ell^2)\mu^2 \\
b &= (\alpha_\ell \varepsilon_\ell + \alpha_t \varepsilon_t)\varepsilon_t k_t^6 + 2(\alpha_\ell \varepsilon_\ell + \alpha_t \varepsilon_\ell + \alpha_t \varepsilon_t)\varepsilon_t k_t^4 k_\ell^2 + \\
&\quad + (\alpha_\ell \varepsilon_\ell + 4\alpha_t \varepsilon_\ell + \alpha_t \varepsilon_t)\varepsilon_t k_t^2 k_\ell^4 + 2\alpha_t \varepsilon_\ell \varepsilon_t k_\ell^6 \\
&\quad + \beta^2(\varepsilon_\ell k_t^2 + \varepsilon_t k_\ell^2)k_t^2 + \left((\alpha_\ell \varepsilon_\ell + \alpha_t \varepsilon_t)k_t^4 + (\alpha_t \varepsilon_\ell + \alpha_\ell \varepsilon_t + 2\alpha_t \varepsilon_t)k_t^2 k_\ell^2 + 2\alpha_t \varepsilon_t k_\ell^4\right) \\
c &= \alpha_t(k_t^2 + k_\ell^2)^2 \left\{\alpha_\ell \varepsilon_t k_t^4 + (\alpha_\ell \varepsilon_\ell + \alpha_t \varepsilon_t)k_t^2 k_\ell^2 + \alpha_t \varepsilon_\ell k_\ell^4 + \beta^2 k_t^2 + \mu^2(\alpha_\ell k_t^2 + \alpha_t k_\ell^2)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{k_1^4}(b^2 - 4ac) = &\Big\{(-\alpha_\ell \varepsilon_\ell + \alpha_t \varepsilon_t)\left[\varepsilon_t(k_t^2 + k_\ell^2)^2 + k_t^2 \mu^2\right] + (-\alpha_t \varepsilon_\ell - \alpha_\ell \varepsilon_t + 2\alpha_t \varepsilon_t)k_\ell^2 \mu^2 \\
&- \beta^2(\varepsilon_\ell k_t^2 + \varepsilon_t k_\ell^2)\Big\}^2 + 4\alpha_t \varepsilon_t(\varepsilon_t - \varepsilon_\ell)^2 \beta^2 k_\ell^2 (k_t^2 + k_\ell^2)^2 .
\end{aligned}
$$

In these formulae we distinguish transverse and longitudinal components:

$$
\alpha_{11} = \alpha_{22} = \alpha_t, \quad \alpha_{33} = \alpha_\ell, \quad k_1^2 + k_2^2 = k_t^2, \quad k_3^2 = k_\ell^2 ,
$$

etc. These quantities are all positive. It follows that a, b, c and $b^2 - 4ac$ are also positive. Hence the frequencies ω_+ and ω_- are both real. This confirms the stability argument given above. (It may also be remarked that $\omega_\pm(\underline{k}) \sim O(\underline{k}^2)$ as $\underline{k} \to 0$.)

4. SUMMARY AND FINAL REMARKS

In this paper we used well developed methods of relativistic quantum field theory to obtain an effective nonlinear electrodynamic in non–relativistics QED. The nonlinearities are produced by the excitation spectrum of electron motion in a uniform magnetic field. A one–loop formula which is valid for weak as well as strong background fields is obtained and is shown to correctly reproduce the well–known oscillatory behaviour in the appropriate limit. We have also calculated various permittivities in the one–loop approximation and by proving the positivity of the Hamiltonian for weak pertubations we have shown that the uniform background is stable. We have then confirmed this result by showing that the frequencies of the slowly varying electromagnetic perturbations (photons) are real.

The one–loop effective Lagrangian has singularities at odd integer values of $2m\phi/B$, where B is the magnetic field and ϕ is the Fermi energy. These singularities correspond to thresholds at which new Landau levels are opened to occupation. The nonlinear field equations derived from our one–loop effective action may admit interesting solutions. However, in this paper we have not investigated such possibilities.

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