A NUCLEATION MODEL OF HADRONS
BASED ON QCD STRING

Ikuo Senda

1990 MIRAMARE - TRIESTE
A NUCLEATION MODEL OF HADRONS BASED ON QCD STRING

Ikuo Senda
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

A dynamical model of hadrons is proposed. The dynamical equations for the system of hadrons are obtained in the analogy with the nucleation theory. The results of dual string theory are used to represent interactions between high resonances. At the energy density $\sim 0.5 \text{ GeV/fm}^3$, the models show a critical behavior, where hadrons in the system start coalescing to form highly excited resonances. The relation to the phase transition to quark-gluon plasma phase is discussed.

MIRAMARE-TRIESTE
November 1990

* Submitted for publication.
In this paper, we propose a nucleation model of hadrons, where the interactions between hadrons are given by the dual string model of QCD. The purpose of this model is to understand the behavior of hadronic matter at very high density and discuss the occurrence of phase transition to quark-gluon plasma (QGP).

Before discussing our model, it would be relevant to mention how the phase transition of the hadronic matter at high temperature/energy looks like in QCD string model and its relation to the other approaches to this phase transition. The phase transition of the hadronic matter at high temperature was first discussed by using the exponentially increasing mass level density, which was derived in the context of the statistical bootstrap model by Hagedorn.\cite{1} In the presence of the exponentially increasing level density, $\rho(m) \sim m^a e^{b m}$, the canonical partition function of the system diverges at the temperature $T_H = 1/b$. Although this temperature was first discussed as a limiting temperature, later it was noticed that it is actually a critical temperature where a phase transition to a new phase of hadrons is taken place.\cite{2} In order for the further understanding of the properties of the system having such mass level density, the investigations using the microcanonical method were performed.\cite{3} The important point of their results is that at high energy density region the system favors a very asymmetric configuration, namely one hadron in highly excited state, carrying most of the energy of the system, and the others in the lowest states. Therefore, at high energy region it is natural to expect that hadrons will coalesce to create excited states if the interactions are switched on. Recently, a model of interacting closed strings has been proposed by Lizzi and I.S. and indeed the model shows a criticality at a critical energy density, above which highly excited strings are created.\cite{4}

Another powerful method to approach the problem of the new phase of hadrons is the lattice QCD. Due to the asymptotic freedom of QCD, it is expected that the liberation of quarks and gluons will occur at high temperature.\cite{5} The results of the lattice simulation given by DeTar and Kogut suggest the following picture of QGP phase.\cite{6} At short range, the system looks like an ensemble of weakly interacting quarks and gluons, however, at long range it is strongly interacting and confined. This means that the excitations in QGP phase are still color singlet and non-perturbative effects play important roles. Therefore, even at the high temperature, if we look at the system of hadrons from a large distance, what we will find are again hadrons. But, probably this time they are
highly excited hadrons.

The picture the statistical bootstrap model and the lattice QCD have in common about the high temperature/energy phase of hadronic matter is the appearance of highly excited states. The QCD string model also shares this picture. In the string model, hadrons are described as quarks and antiquarks (or diquarks for baryons) connected by color flux.\(^7\) States of low excitation is referred as short strings and those of high excitation long strings because their color flux are long. Since the string model also has the exponentially increasing mass level density, what are true in the statistical bootstrap model apply in the same way to the string model. As for the connection to the lattice QCD, the picture of QGP phase given by lattice simulation is explained as follows in the string model. As we have mentioned above, in high energy phase, hadrons have coalesced to form highly excited states, or long strings. In such situation, quarks move freely because the color flux connecting quarks are very long and a quark at a end does not feel the existence of the other. Thus, locally quarks look as if they were free.

The nucleation model, which we use, is a standard technique to investigate phase transitions such as coagulation and evaporation of water.\(^8\) As we have mentioned before, in the string model of hadrons the signal of the phase transition is the appearance of high resonance states. In order to write down the nucleation equations, we need informations on the interactions between hadrons such as cross sections and decay width. For these quantities, we use the results of the string model. Although we will discuss the justification of using the string model to represent interactions between hadrons by comparing its results with the experimental data, there still remain parts which we assume that the string model works in reality. Our model has several features and we would like to stress two of them. The first one is that we can include the contributions of all resonances not only those on the leading Regge trajectories but also those on daughters. The second one is that this model has an ability to trace the time evolution of the system, which would be important to discuss the occurrence of QGP in experiment. In this paper, we report some of results obtained as first attempt towards this direction. We consider a system consisting of \((u,d)\)-quark mesons, thus the total baryon number of the system is zero. The interactions between hadrons are represented by the three open string interaction in the string model. The extension of the model in more complete form is mentioned at the end of the paper.
We will deal with an isolated system with finite total energy $E_{\text{tot}}$ and the volume $V$, which would be a relevant condition to discuss the QGP in ion collision and in the early universe. The quantities we are going to discuss are the number of pions in the system $N_\pi$, the energy carried by pions $E_\pi$ and the distribution function of resonances $N_r(E)$, which represents the number of resonances in the energy interval $E \sim E + dE$ by $N_r(E)dE$. These quantities are in general functions of time. One of the most important ingredients of our model is the exponentially increasing hadron mass level density,

$$
\rho(m) = \rho_0 \left( \frac{m_\pi}{m} \right)^{3} e^{m/m_\pi},
$$

(1)

where $\rho_0$ is a constant of order 1, $m_\pi$ is the mass of pion and $m$ is the mass of a resonance.\(^1\) This exponential behavior of mass level density is achieved only when resonances on the leading and all daughter trajectories are included. Using eq.(1), one can calculate the expectation value of the mass of a resonance with energy $E$ and obtain $<m>_E \approx E - m_\pi$.\(^4\) Therefore most of the energy given to resonances is carried as rest mass. Taking account that the temperature we are interested in is of order of the pion mass, we can build our model such that the system consists of two parts, namely the pion part and the other mesons, which we call as the resonance part although $\eta$ is included in it. For pions, one has to treat them as relativistic particles. In order to avoid the complexity of the elastic scattering between pions, we make an approximation such that within a typical time scale of strong interaction, $\sim 10^{-23}$s, the distribution of pions reaches the equilibrium given by the Boltzmann distribution with temperature $T_\pi$, which is obtained by the method of microcanonical ensemble as a function of $E_\pi$, $N_\pi$ and $V$, even though $E_\pi$ and $N_\pi$ change in time due to strong interactions between hadrons. For the resonances, one can treat them non-relativistically.

In the following discussion, we will use the results that the expected mass of the resonance with energy $E$ is $<m>_E = E - m_\pi$, which is not affected much by finite temperature of order $m_\pi$.

For the sake of the numerical calculation, we discretize the energy in unit of the pion mass, $E_i = i \cdot m_\pi$, $E_{\text{tot}} = I \cdot m_\pi$, $i, I \in \mathbb{Z}^+$, and let this discrete energy variable 'i' represents the energy in the interval $(i - 1)m_\pi < E_i \leq im_\pi$.

The energy distribution of resonances is divided in bins of energies and it is represented by $N_{r(i)}$, $i_0 \leq i \leq I$, where the continuous variable $E$ is replaced by

\(^1\)We assume that both isospin-one and -sero states are contained in this mass level density. In our model, we are neglecting the subtleties of isospin-sero states such as the mixing with other quantum numbers and the decay modes of $\eta$. 

4
discrete one \( i \) (the bin number) and \( i_0 \) is the lowest energy of the resonances in unit of the pion mass. Similarly, we introduce a notation \( \mathcal{N}_\pi(T_\pi,i) = \mathcal{N}_\pi(i) \), \( 2 \leq i \leq I \), to represent the number of pions in the \( i^{th} \) bin (having energy \( i \cdot m_\pi \)).

As we have mentioned above, \( \mathcal{N}_\pi \) is given by the Boltzmann distribution with temperature \( T_\pi \) which is given as a function of \( E_\pi, N_\pi \) and \( V \). The nucleation equations describe the time evolution of \( N_\pi \) and \( N_{r(i)} \). The time evolution of the number of pions is given by,

\[
\frac{\delta N_\pi}{\delta t} = 2 \sum_{i=i_0}^{I} \Gamma_i N_{r(i)} B_{i \rightarrow 2\pi} + \sum_{i=i_0+1}^{I} \sum_{j=i_0}^{i-2} \Gamma_i N_{r(i)} B_{i \rightarrow \pi+j} - \frac{1}{2} \sum_{i=2}^{I} \sum_{j=2}^{i-1} K_\pi(i,j) \frac{R_1}{V} \mathcal{N}_\pi(i) \mathcal{N}_\pi(j) - \sum_{i=2}^{I} \sum_{j=i_0}^{i-1} \frac{R_2}{V} \mathcal{N}_\pi(i) N_{r(j)}.
\]

(2)

The meaning of each term is easily understood. The first line is the decay terms, some resonances with energy \( im_\pi \) (mass \( \sim (i - 1)m_\pi \)) decay with decay rate \( \Gamma_i \) to two pions with branching ratio \( B_{i \rightarrow 2\pi} \) and decay to one pion and one lower resonance in \( j^{th} \) bin with the branching ratio \( B_{i \rightarrow \pi+j} \). The second line is the joining terms. The first term indicates that two pions of energies \( i \) and \( j \) combine to form some resonance with the interaction rate \( R_1 \) and the kinetic factor \( K_\pi(i,j) \). The interaction rate \( R_1 \) is defined without relative flux factor of interacting pions and it is included in \( K_\pi(i,j) \), which is given by averaging over the relative angle between colliding pions:

\[
K_\pi(i,j) = \frac{1}{3} \frac{\sqrt{(m_\pi^2 + k_i^2)(m_\pi^2 + k_j^2)}}{k_i k_j} \times \left[ |\frac{k_i}{\sqrt{m_\pi^2 + k_i^2}} + \frac{k_j}{\sqrt{m_\pi^2 + k_j^2}}|^3 - |\frac{k_i}{\sqrt{m_\pi^2 + k_i^2}} - \frac{k_j}{\sqrt{m_\pi^2 + k_j^2}}|^3 \right],
\]

where \( k_i \), \( k_j \) are the magnitudes of momenta of the pions with energy \( im_\pi \) and \( jm_\pi \). The last term in eq.(2) is the joining of pions and resonances to form higher resonances with an interaction rate \( R_2 \). The time evolution of the number of resonances becomes,
\[
\frac{\delta N_{r(i)}}{\delta t} = \\
- \Gamma_i N_{r(i)} + \sum_{j=i+2}^I \Gamma_{j} N_{r(j)} B_{j \rightarrow r+i} + \sum_{j=i+4}^I \Gamma_{j} N_{r(j)} B_{j \rightarrow r+i} \\
+ \frac{1}{2} \sum_{j=2}^{i-2} \frac{R_1}{V} K_\nu(j,i-j) N_{r(j)} N_{r(i-j)} \\
+ \sum_{j=2}^{i-4} \frac{R_2}{V} N_{r(j)} N_{r(i-j)} - \sum_{j=2}^{i-4} \frac{R_3}{V} N_{r(j)} N_{r(i)} \\
+ \frac{1}{2} \sum_{j=i+6}^{i-4} \frac{R_2}{V} N_{r(j)} N_{r(i-j)} - \sum_{j=i+6}^{i-4} \frac{R_3}{V} N_{r(j)} N_{r(i)} ,
\]

(3)

where terms are understood similarly to those in eq. (2). The first line on the right hand side is the decays: The first term is the decay of resonances in \(i^\text{th}\) bin thus goes out of the bin and the second and the third represent that the decay products of higher resonances come into \(i^\text{th}\) bin. The third line is the interaction terms between pions and resonances. The last line is the interactions between resonances: The first term represents that two lower resonances combine to produce the one in \(i^\text{th}\) bin and the second the resonances in \(i^\text{th}\) bin combine with \(j^\text{th}\) resonances to create higher resonances, thus going out of the bin. This set of non-linear equations is considered to be the analogue of the Boltzmann equation in the system of hadrons. In the above equations, conservations such as angular momenta and isospin are not treated explicitly, but we are assuming that there are always open channels with some probabilities since the states in each bin are mixture of states with various quantum numbers. In the rest of the paper, we concentrate on equilibriums of the equations which are given by the solutions to \(\delta N_{r}/\delta t = \delta N_{r(i)}/\delta t = 0\), under the constraint of the energy conservation \(E_T + \sum_i \hbar c_i n_{r(i)} = E_{r}^\text{max}\).

Let us discuss the quantities \(R_1, R_2, \Gamma_i\) and the branching ratio appearing in equations. \(R_1\) is the interaction rate that the two pions combine to form a resonance, which is defined without the relative flux factor. Compared with the pion mass, the interaction energy in this process is sufficiently large. It is a well-known experimental result that the total cross section in the high energy scattering of hadrons, such as pion-proton, shows a constant behavior. This behavior is reproduced in the string model (or dual resonance model) in the Regge limit. In this limit, the cross section \(\sigma_p(s)\) becomes,

\[
\sigma_p(s) \sim s^{\alpha_0} \sigma_0 ,
\]

(4)
where $\sigma_0$ is a constant. The rough estimate of $\sigma_0$ is obtained from the simple additive quark model. In the case of pion-pion scattering, one finds $\sigma_0 \sim 20m$ barn. Thus we use $R_1 = \sigma_0$. The second quantity $R_2$ is the interaction rate that two resonances combine to produce a higher resonance. Since resonances are considered to be moving nonrelativistically and they have large rest masses, eq.(4) is not available in this case. Instead, one has to consider large mass limit. Applying the method in ref. [10] to the open strings interacting at the ends, one obtains the cross section for two resonances with masses $m_1$ and $m_2$ to form a higher resonance,

$$
\sigma_r(m_1, m_2) \propto \frac{k_1 \cdot k_2}{E_1 E_2 v_{rel}} \sim \sigma_0 / v_{rel},
$$

where $\sigma_0$ is the same as the one in eq.(4), $k_1$ and $k_2$ are momenta of initial states and $v_{rel}$ is a relative velocity between them. This result is valid when either $m_1$ or $m_2$ is sufficiently large. Unfortunately, there is no scattering experiments of resonances to compare with this result. Therefore it is our assumption to use this formula for the interaction rates where resonances are involved, namely $R_2 = v_{rel} \cdot \sigma_r = \sigma_0$. As for the decay rates, the string model tells that the decay width is proportional to the mass of the state, which is the same prediction as the flux tube model of hadrons, and gives $\Gamma \propto M$. [11] Let us examine if this holds in the experimental data. Fig.1 is a plot of the full decay width versus the mass of mesons with isospin 1. [8] The straight line is the result of fitting the decay width of $a_0(980)$, $b_1(1235)$, $\rho(1450)$, $\pi_1(1670)$, $\rho_3(1690)$ and $\rho(1700)$ to the form $\Gamma = pM + q$, where $M$ is the mass and $p, q$ constants to be fitted. [4] The slope of this line is $p = 0.28$ with the scale factor of the fitting 0.11. We also did the same fitting for excited kaons and obtained the value of slope $\sim 0.2$. We should be careful to conclude the linear increase of the decay width from these fittings, because of large uncertainty of the experimental measurement. However, we interpret that Fig.1 supports the string model and we will use the following formula for the decay width $\Gamma$,

$$
\Gamma(M) = \gamma M,
$$

where $\gamma$ is a constant. This formula is considered to work better for higher resonances since they are suitable for string description. The branching ratio in equations are combinations of the momentum phase space volume and the mass level density. Since the resonances are treated non-relativistically, the effect of

\footnote{We did not include $\rho(770)$ in this fitting. For isospin-zero states, it is difficult to consider this sort of fitting because of mixings with other quantum numbers.}
Lorentz boost is neglected. For example, the branching ratio of the decay of a state with mass $M$ to states of masses $m_1$ and $m_2$ is given by,

$$B_{M \to m_1 + m_2} \propto \rho(m_1)\rho(m_2) \frac{\sqrt{(M^2 - (m_1 + m_2)^2)(M^2 - (m_1 - m_2)^2)}}{2M^2},$$

where $\rho(m)$ is the mass level density. We use $\rho(m_\pi) = 3$ for pions due to isospin degeneracy and eq.(1) for the resonance part. In this formula we substitute the expected values of masses of the resonances in each bin and normalize it.

We calculated equilibrium configurations of $\{N_\pi, N_{r(i)}\}$ with various total energies $E_{tot}$ and volumes $V$ numerically. The equilibrium have been obtained by the Powell method, where the accuracy is $10^{-10}$, that is the set $\{N_\pi, N_{r(i)}\}$ is considered to be a solution if all of $\delta N/\delta t$ are smaller than $10^{-10}$. In general, the numerical results for $\{N_\pi, N_{r(i)}\}$ are not integers and they are interpreted as the probable numbers of pions and resonances. In the results we will show below, we are using parameters $\sigma_0 = 20mb$, $\gamma = 0.1$, $\rho_0 = 8$ and $i_0 = 6$, which sets the lowest energy of resonance to $6m_\pi$. However, results are not altered much by changing parameters around these values. The Fig.2 shows the energies carried by pions and resonances and their numbers as functions of volume of the system for the total energies $50m_\pi$, $100m_\pi$ and $150m_\pi$. One can see that at low energy density there are lots of pions and most of the energy is carried by pions. Around the energy density $\sim 0.1GeV/fm^3$, resonances of relatively light masses are formed. If one compress the system adiabatically further, every object coalesce to one or few high resonances. In Fig.3, the equilibrium distributions of the number of resonances are shown for large and smaller volumes in the case of total energy $100m_\pi$. We find that at about the energy density of $\sim 3m_\pi/fm^3$, the formation of high resonances is started. In the sense of the nucleation theory, this is the critical point and what we have found so far is considered to be a phase transition.[8]

It is interesting to see how the temperature of pions behaves as a function of total energy and volume. In Fig.4, the temperature levels are shown on total energy-volume plane, where the temperature be read in unit of the pion mass. The important point is as follows. Suppose we have an isolated system with a fixed volume, for example $10 fm^3$. Let us start with a small total energy and increase the energy of the system. At first, the temperature of the pions increases. But, at one point, the temperature stops increasing and after that it decreases. The energy density at the turning point is about $3m_\pi/fm^3 \sim 0.5GeV/fm^3$, which agrees with the critical point at which the formation of high resonances are
started. It would be useful to compare this energy density with the one at the critical temperature in the lattice QCD. One can see that two different methods give similar values of critical energy densities. Another point one should notice is that the behavior of the system depends on the total energy. For example, consider a system of total energy $50m_\pi$. As we compress the system adiabatically, the temperature of pions increases. Although the increase slows down around the critical point, it continues increasing. However, if the system has large total energy, ($\gtrsim 60m_\pi$), the situation is different. As we compress the system, the temperature of pions increases. But at critical point, it stops increasing and starts decreasing. After a while of decreasing, it starts increasing again. This behavior is because the system with larger total energy has a stronger tendency to form large resonances due to the exponential increase of the mass level density.

Let us assume, for the moment, that the temperature of pions represents the temperature of the whole system well. Then one obtain the specific heat of the system. In Fig.4, the specific heat at the constant volume is shown. The specific heat diverges at the critical density and in high energy density region it is negative. In the thermodynamics, a negative specific heat means the instability of the system and its divergence first order phase transition. However, in our case, the equilibrium configuration is given as a solution to the nucleation equations and one can check its stability by examining the time evolution starting from various initial conditions. Thus they are stable and the negative specific heat in high energy region means that the thermodynamical description of the system as hadron gas becomes irrelevant due to the coagulation to high resonances. As for the order of the phase transition, we have to take the divergence of the specific heat cautiously because the temperature we are considering is not the one of the whole system but of pions. Although we are not able to say definite things, we suspect that it would be first order.

We would like to mention about the extension of the model presented in this paper. There are many ways to extend it. One might say that, in the view of string theory, the system of open strings (hadrons) with three-string interaction is not consistent by itself and one has to introduce four-string interaction and also closed strings (pomeron or glueball).\textsuperscript{12} Though the existence of glueballs is still controversial and introducing them into the model increases the free parameters, we can guess these parameters from results in dual resonance model, string theory, lattice simulations and existing data on glueball candidates. As far as we have seen, the presence of glueballs does not change the results much due to their heavy masses. As for the four-string interaction, or rearrangement
between strings, it gives important contributions in high energy phase where
hadrons are overlapping with each other. It has a tendency to accelerate the
creation of higher resonances. However, its effect in low energy density is small
and the critical density does not change.\textsuperscript{13}

An important thing is to discuss what sort of suggestion our model can make
to experiments. Since our model does not treat quarks and gluons directly,
it is not suitable to discuss strange particle enhancement or $J/\psi$ suppression.
The best thing we should discuss would be final hadron distributions. The
investigation in this direction is now in progress.

The author would like to thank Dr. R. Hagedorn and Dr. T. Hatsuda for
useful discussions and express special thanks to Dr. F. Lizzi for continuous
discussions. I am also grateful to ICTP, IAEA and UNESCO for support.
REFERENCES


Figure Captions

1. The plot of the full decay width versus the mass of mesons with isospin 1. For $a_2(1320)$, the decay width to $K^\pm K^0$-mode is shown.

2. (1) The energy carried by pions and resonances in unit of the pion mass and (2) The numbers of pions and resonances in the system as functions of the volume. The total energy $E_{tot}$ is written in unit of the pion mass. In figures, the solid curves are for resonances and dashed curves are for pions.

3. The equilibrium distributions of the number of resonances, $N_{r(i)}$, in the high (above) and the low (below) energy regions when the total energy of the system is $100m_\pi$. The numbers in the figure are the volumes of the system in unit fm$^3$ and the horizontal axis is the bins for energies in unit of the pion mass.

4. The temperature of pions, $T_\pi$, as a function of the total energy and the volume. The equal temperature levels are shown. The numbers are the temperatures in unit of the pion mass. The temperature difference between solid curves is 0.1$m_\pi$ and the difference between dashed curves is 0.01$m_\pi$.

5. The specific heat at constant volume calculated by using pion temperature, $\sim \partial \epsilon / \partial T_\pi |_V$, where $\epsilon$ is the energy density of the system in unit $m_\pi$/fm$^3$. In the figure, the results of the total energies $50m_\pi$, $100m_\pi$ and $150m_\pi$ are shown.
Figure 1
(2) The Number of Objects  (1) Energy Occupation

(2) The Number of Resonances

Figure 2

Figure 3