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ON THE WEAK APPROXIMATION
IN ALGEBRAIC GROUPS
AND RELATED QUESTIONS

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ABSTRACT

We prove here a theorem on the weak approximation in adjoint groups.

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1. INTRODUCTION

Let \( G \) be an algebraic group defined over a field \( k \). If \( k \) is a topological field, for example \( k = \mathbb{R} \) (real numbers) or \( k = \mathbb{C} \) (complex numbers), then the group \( G(k) \) of \( k \)-rational points of \( G \) can be considered as a topological group with the topology induced from \( k \). If \( k_v \) denotes the completion of \( k \) at a valuation \( v \) of \( k \) then we say that \( G \) satisfies the weak approximation with respect to a finite set \( S \) of valuations of \( k \), if, via the diagonal embedding, \( G(k) \) is dense in the product \( \prod_{v \in S} G(k_v) \) with respect to the product topology, and \( G \) satisfies the weak approximation over \( k \) if the above holds for any \( S \). We refer the reader to [P-R], [S] for more complete expostions of the question. There is a conjecture due to Platonov, stating that over any field \( k \), every adjoint semisimple \( k \)-group satisfies the weak approximation (see [P-R] for a stronger version of the conjecture), which generalizes a similar result of Harder [H] over global fields. (cf also [S]). In [Th1], [Th2] we have discussed this conjecture and related problems, showing that the conjecture holds for a large class of semisimple groups and also over a large class of fields.

In this paper we extend the class of adjoint semisimple groups, for which the conjecture holds. Namely we have the following

**Theorem** Let \( k \) be any field. Then every adjoint semisimple \( k \)-group, which contains neither anisotropic simple factors of exceptional type \( D_4, E_6 \), nor the types \( E_7^{13}, E_8^{13}, E_8^{133} \) (in Tits notation [T1]) satisfies the weak approximation over \( k \).

2. PRELIMINARIES

Throughout this paper \( k \) will denote a field, all algebraic groups are supposed to be linear, \( H^1(\cdot, \cdot) \) denotes the set of 1-Galois cohomology and \( \mu_2 \) denotes the group \((\pm 1)\). A set of valuations always means a set of nonequivalent valuations.

**Lemma 1** Let \( \text{char} \cdot k \neq 2 \) and \( S \) be any finite set of valuations of \( k \). Then the canonical map \( H^1(k, \mu_2) \to \prod_{v \in S} H^1(k_v, \mu_2) \) is surjective.

**Proof** We have to prove that for any \( x_v \in k_v^* \), \( v \in S \), there is \( x \in k^* \) s.t. \( x \equiv x_v \) (mod. \( k_v^2 \)). But this follows immediately from the well known facts that \( k_v^2 \) is an open subgroup in \( k_v^* \) with respect to the \( v \)-adic topology and that \( k_v^* \) is dense in the product \( \prod_{v \in S} k_v^* \).

The following lemma and its corollaries have been proved in [Th3], but for the convenience of reading, we give its proof here.

**Lemma 2** Let \( G \cdot H \) be an almost direct product of \( k \)-groups with \( F = G \cap H = (G \cap H)(k) \) (set-theoretic intersection). If \((\text{char} \cdot k, \text{card}(F)) = 1, \dim G \text{ (resp. } \dim H) > 0 \), then the \( k \)-projection \( \lambda_1 : G \cdot H \to G \cdot H/H \) (resp. \( \lambda_2 : G \cdot H \to G \cdot H/G ) \)
is surjective on \(k\)-points.

**Proof** Let \( \Gamma = \text{Gal}(\overline{k}/k) \), where \( \overline{k} \) denotes a fixed algebraic closure of \( k \). It is clear that

\[
\pi_1^{-1}( (G/F)(k) ) = \bigcup_{f \in \text{Hom}(\Gamma, F)} \{ g \in G : g^s = f(s) \cdot g, \forall s \in \Gamma \},
\]

where \( \pi_1 \) is the \( k \)-projection \( G \to G/F \) and \( \text{Hom}(\Gamma, F) \) denotes the set of all group homomorphisms from \( \Gamma \) to \( F \). By the way, the above set is mapped onto \( (G/F)(k) \) under \( \pi_1 \). We now prove that \( (G \cdot H)(k) \) is mapped onto the set of \( k \)-points of \( G \cdot H/H \). Let \( \text{pr}_1 : G \times H \to G \) be the projection onto \( G \). Then, from the following commutative diagram of \( k \)-groups

\[
\begin{array}{ccc}
G \times H & \xrightarrow{\Delta} & G/F \\
\pi \downarrow & \nearrow & \downarrow \tau \\
G \cdot H & \xrightarrow{\lambda_1} & G \cdot H/H
\end{array}
\]

and from the separability of \( \Delta = \pi_1 \circ \text{pr}_1 \), \( \lambda_1 \) and \( \pi \) (by assumption), it follows the separability of the map \( \tau \), i.e. \( G/F \) is \( k \)-isomorphic to \( G \cdot H/H \) canonically as \( k \)-groups (cf. [R]).

Now let \( g \cdot h \in (G \cdot H)(k) \). Hence \( (g \cdot h)^s = g \cdot h \) for any \( s \in \Gamma \), or equivalently, \( g^{-1} \cdot g^s = h \cdot h^{-s} = f(s) \in F \). Let \( \Gamma' \) be the subgroup of \( \Gamma \), consisting of all \( s \) fixing \( g \). It is clear that the map \( s \to f(s) \) is a homomorphism from \( \Gamma \) to \( F \), and \( \Gamma' = \text{Ker}(f) \). Let \( k' \) be the extension of \( k \) corresponding to \( \Gamma' \). Then \( g \in G(k'), h \in H(k') \) and denoting by the same \( f \) the induced homomorphism from \( \Gamma / \Gamma' \) to \( F \), we have \( g^s = f(s) \cdot g, h^s = f(s^{-1}) \cdot h \) for all \( s \in \Gamma / \Gamma' = \text{Gal}(k'/k) \). Hence we have the following equality

\[
(G \cdot H)(k) = G(k) \cdot H(k) \bigcup_{k' \in \text{Hom}(\Gamma', F)} \{ g \cdot h : g \in G(k'), h \in H(k'), s.t. g^s = f(s) \cdot g, h^s = f(s^{-1}) \cdot h, \forall s \in \Gamma = \text{Gal}(k'/k) \},
\]

where \( k' \) runs over all Galois extensions of \( k \) constrained in a fixed algebraic closure \( \overline{k} \) of \( k \). Now we consider the \( k \)-homomorphism \( \theta : G \cdot H \to G/F \), mapping \( g \cdot h \) to \( g \cdot F \). It is sufficient to prove that \( \theta \) is surjective on \( k \)-points since \( \tau \) is a \( k \)-isomorphism. Applying the map \( \theta \) to the above decomposition of \( (G \cdot H)(k) \) we see that \( \theta((G \cdot H)(k)) = \theta(\pi_1^{-1}( (G/F)(k) )) = (G/F)(k), \) since \( \theta \) coincides with \( \pi_1 \) on \( G \).

From this lemma we deduce the following

**Lemma 3** With the above notation, the induced map \( H^1(k, G) \to H^1(k, G \cdot H) \) (resp. \( H^1(k, H) \to H^1(k, G \cdot H) \)) is injective.

**Proof** It follows from the sequence of cohomology sets derived from the exact sequence of \( k \)-groups \( 1 \to G \to G \cdot H \to G \cdot H/G \to 1 \) (resp. the exact sequence \( 1 \to H \to G \cdot H \to G \cdot H/H \to 1 \)) and from lemma 2.
Let \( f \) be a nondegenerate hermitian form with respect to an involution of the first kind of a division algebra of finite dimension over its centre \( k \). Denote by \( \tilde{G} \) the special unitary \( k \)-group of the form \( f \) and by \( G \) the adjoint \( k \)-group of \( \tilde{G} \). An interesting characterization of the groups of \( k \)-points of these groups is given in the following

**Lemma 4** With the above notation, assume that \( \text{char} \cdot k \neq 2 \). Then we have the following exact sequence of groups

\[
1 \rightarrow \mu_2 \rightarrow \tilde{G}(k) \rightarrow G(k) \rightarrow H^1(k, \mu_2) \rightarrow 0 .
\]

**Proof** We can form an almost direct product \( \tilde{G} \cdot G_m \) with \( \tilde{G} \cap G_m = (\pm 1) \). Let \( H = \tilde{G} \cdot G_m \) and we have the following commutative diagram with exact lines

\[
\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \uparrow \\
1 & \rightarrow & \mu_2 \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \\
\downarrow & & \uparrow \\
& & H \\
\rightarrow \downarrow & & \\
G_m & & G_m \\
\uparrow & & \\
1 & & 1
\end{array}
\]

and from this the following commutative triangle

\[
H^1(k, \tilde{G}) \xrightarrow{\gamma} H^1(k, G) \\
\alpha \downarrow \beta \Rightarrow \gamma \downarrow \beta
\]

From the Lemma 3 (resp. from the exact sequence of \( k \)-groups \( 1 \rightarrow G_m \rightarrow H \rightarrow G \rightarrow 1 \)), it follows that \( \alpha \) (resp \( \beta \) is injective, hence \( \gamma \) is injective too, and from this the lemma follows.

**Lemma 5** (cf. [Th2]) Let \( Q \) be a connected reductive \( k \)-group, \( P \) be a parabolic subgroup of \( Q \) defined over \( k \), \( P = M \cdot R_u(P) \) be a Levi decomposition of \( P \), \( S \) be a finite set of valuations of \( k \). By \( A(S, \cdot) \) we denote the defect of the weak approximation in \( S \), i.e. \( A(S, Q) = \prod_{v \in S} Q(k_v)/Q(k) \) (the closure is taken in the product topology), etc. Then we have the following bijections

\[
A(S, Q) \leftrightarrow A(S, P) \leftrightarrow A(S, M)
\]

In particular, \( Q \) satisfies the weak approximation with respect to \( S \) if and only if \( P \) (and so \( M \)) satisfies.
3. **PROOF OF THE THEOREM**

From the results of [Th2] it follows that we have only to prove the weak approximation property for adjoint almost simple \( k \)-groups of type \( C_n \) and \( D_n \). Let \( G \) be such a group and let \( \check{G} \) be its unitary covering over \( k \), \( S \) be a finite set of valuations of \( k \). We may assume that \( \text{char} \cdot k \neq 2 \), since otherwise, the assertion follows from the pure inseparability and we may use the Cayley transform in any characteristic cf. [D]). Now we have the following commutative diagram with exact lines

\[
\begin{array}{c}
\check{G}(k) \rightarrow G(k) \rightarrow H^1(k, \mu_2) \rightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \sigma \\
\prod_{v \in S} \check{G}(k_v) \rightarrow \prod_{v \in S} G(k_v) \rightarrow \prod_{v \in S} H^1(k_v, \mu_2) \rightarrow 0
\end{array}
\]

due to Lemma 3 in the Sec.1. Now, by making use of the weak approximation in the group \( G \), combining with the surjectivity of the map \( \sigma \) (Lemma 1), by the same method of chasing on diagram as in [K], the weak approximation property of \( G \) follows.

4. **THE UNIVERSAL PROPERTY**

The following discussion can be given in a more general context, but for the sake of simplicity, we restrict ourselves to the case of algebraic groups. A \( k \)-group \( G \) is said to satisfy the weak approximation universally if for any extension \( k' \) of \( k \), \( G \) satisfies the weak approximation over \( k' \). In their important work [C-S] J.L. Colliot-Thélène and J. J. Sansuc have considered this notion in the case of algebraic tori and have shown that tori, which satisfy the weak approximation universally are the obvious ones; namely they are factors of rational \( k \)-varieties. One may ask if the same result is still true for other algebraic groups. We have the following

**Proposition** Any quasi-split semisimple \( k \)-groups, which satisfy the weak approximation universally, are direct factors of varieties rational over \( k \) (i.e. birationally isomorphic to affine space over \( k \)).

**Proof** Let \( G \) be a quasi-split semisimple \( k \)-group, \( S \) be its maximal \( k \)-split torus, contained in a maximal \( k \)-torus \( T \) of \( G \). Then as it is well known, \( T \) coincides with the centralizer of \( S \) in \( G \). Thus by Lemma 5, \( T \) satisfies the weak approximation universally, hence is a direct factor of a rational \( k \)-variety. Since \( G \) is birationally equivalent to the product of \( T \) with \( k \)-split unipotent groups by Bruhat decomposition, hence the proposition follows.

5. **REMARKS**

With respect to the case of anisotropic exceptional simple groups \( D_4 \), \( E_6 \) and groups of type \( E_7^{28} \), \( E_8^{28} \) and \( E_6^{133} \) the situation seems to be quite unexpected since we do not have sufficiently
clear models of these groups. However, one may hope that using the approach proposed by Tits [T2] we can study some of these groups (strongly inner, . . .) basing on the connection between the Brauer group $Br(k)$ and Jordan algebras over $k$.

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REFERENCES


