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ABSTRACT

The integer quantum Hall effect and associated magnetic phenomena are reconsidered in a 2–dimensional system with a flat boundary. The electromagnetic properties of this system are governed by an effective Lagrangian which includes an induced Chern–Simons term. The effective Lagrangian is relevant for the description of fields which are slowly varying about a uniform magnetic background associated with a fermionic ground state in which a whole number of Landau levels is filled. It is singular for field values that correspond to partially filled levels. The underlying assumption of translation invariance of the fermionic ground state fails in the vicinity of boundaries where the effective field theory is essentially non–local. The width of the boundary layer and the current flowing in it are estimated.

1. INTRODUCTION

The behaviour of non–relativistic charged fermions, minimally coupled to the electromagnetic field in 2+1 dimensions is reconsidered. It is believed that the fermions are able to aggregate in metastable configurations which sustain a non–vanishing but uniform magnetic field. These metastable states are characterized by the filling of a whole number of Landau levels. Because the magnetic field strength is a pseudoscalar with respect to reflections in two space dimensions, these states are not parity eigenstates. The most characteristic feature of the parity violation is the emergence of an induced Chern–Simons term, with quantized coefficient, in the effective action. Because of this term, gradients in the magnetic field generate electric fields and vice versa. The magnetic gradient can be attributed to a fermion current flowing in a direction orthogonal to the associated electric field. It is a Hall current and, since it does not dissipate energy, it may, perhaps usefully, be regarded as a supercurrent. The current is generally concentrated near the boundaries, where the magnetic field is variable, and it satisfies a 2–dimensional London equation. It can be argued that the magnetic field exhibits a kind of Meissner effect.

The existence of metastable states for systems of charged 2–dimensional fermions is made plausible by energy considerations. The energy density associated with motion of a fixed number (per unit area) of particles in the background of a uniform magnetic field exhibits local minima at those values of the field strength which correspond to the filling of a Landau level. It has been conjectured that these states can be stabilized, in a finite system, by applying an external field at the boundary *). The idea is that, with the magnetic field maintained at some fixed value on the boundary, it will tend to relax in the bulk of the system towards one of the metastable values associated with filled Landau levels and spatial uniformity. (It should perhaps be emphasized that we are considering a strictly 2+1–dimensional electrodynamics: the field components are E_1, E_2 and $B = B_3$. Missing are the components E_3, B_1 and B_2 which would be needed to fill out the usual electrodynamics. This means that, from the 3+1–dimensional viewpoint, we are dealing with a cylinder, i.e. a block which is infinitely extended in the x_3 direction, not a thin film. Hence, the field in the interior is dynamically generated, not freely adjustable.) We believe that the most stable arrangement would be for the bulk field to choose the value nearest to the externally imposed one. This would minimize gradient contributions to the energy. But the validity of this conjecture has yet to be established.

Much of this is well known but scattered [3]. Our purpose in reconsidering the integer quantum Hall effect in cylindrical geometries is to attempt some clarification, particularly with regard to edge effects. We shall argue in Section 3 that the status of the Chern–Simons term becomes uncertain in the vicinity of boundaries where a question of gauge invariance arises. To settle this question and, in particular, to find plausible matching conditions for field strength components at a boundary, we consider in some detail the problem of computing current density near a plane boundary. The outcome of this is that the boundary layer, where the effective local theory with its Chern–Simons term is expected to break down, is narrow relative to the length scale determined by the effective theory, provided the number of filled Landau levels is not too large. (Or, equivalently, provided the equilibrium field in the interior is not too small relative to the number density of charged particles.) The same condition will ensure that the discontinuity of magnetic field strength, across the boundary layer, is small relative to the interior field. On the other hand, it turns out that the width of the boundary layer is comparable to the penetration depth if the magnetic field is relatively weak, i.e., if the number of filled Landau levels is large. We will show that the latter circumstance is also required for the existence of metastable configurations.

*) This conjecture was originally expressed, as far as we know, by R.S. Markiewicz [1]. We learned of it from A. Cabo who had independently arrived at the same idea [2] and to whom we are grateful for telling us of his work prior to publication.

We should emphasize at once that we are not dealing with a theory of anyons [4]. We do not introduce an independent statistical field [5] of the Chern–Simons type; our charged particles are fermions. However, because the Maxwell equations acquire a Chern–Simons contribution as a result of ground state fluctuations, it could be argued that, at least for large separations, the charged fermions behave effectively like free anyons. We shall not pursue this argument here *).

Our purpose is two–fold. Firstly, in Section 2 we review the derivation of the Peierls formula for the energy density of charged fermions in a uniform magnetic background; neglecting their mutual interactions. Although our main interest is with the zero temperature system, we find it very helpful to employ a finite temperature expression for the density of free energy. This detour serves to clarify the dependence of the energy on a constant electrostatic potential, in effect the chemical potential, or Fermi energy. The energy density at zero temperature is a continuous but not everywhere differentiable function of the magnetic field with local minima that define the metastable states.

Secondly, in Section 3 we consider the somewhat less uniform case in which the 2–dimensional space is divided into two half–planes, one occupied by the charged fermions and the other empty. With such a simple geometry it is possible to understand the edge effects in some detail. We estimate the surface current and hence the magnetic discontinuity at the boundary (or rather the change across a boundary layer whose width we also estimate). For weakly varying electric and magnetic fields the effective classical boundary problem is easily solved. The Meissner–like behaviour of the magnetic field, and the related Hall current can be seen clearly in this configuration and it is possible to show that, under plausible circumstances the penetration depth is large in comparison with the thickness of the boundary layer.

*) Much of the work done in connection with anyon models of superconductivity can also be used in discussions of the integer quantum Hall effect [6),7),8)]. Both are based on the properties of Landau orbitals. We found the paper, "Magnetic and Thermal Properties of the Anyon Superconductor" by Hetrick, Hosotani and Lee [9], particularly useful in its treatment of the free energy density.

2. ENERGY DENSITY

To demonstrate the existence of metastable, translation invariant states it is necessary to construct an energy density function and show that it has local minima. The construction which we shall describe is not fully consistent since we shall treat the electromagnetic field as a fixed and uniform background in which the fermionic motion takes place. In such a background it is easy to compute the fermionic energy density. We can then adjust the background so as to minimize this energy. It turns out that the minima correspond to electron configurations in which a whole number of Landau levels is filled. This much is consistent, because it can be shown that a filled Landau level is a translation invariant, non–degenerate state and is therefore not incompatible with a uniform magnetic field. However, at values of the magnetic field for which there is a partially filled level, i.e. away from the minima, it is not clear that translation invariance can be maintained. Such a state may imply non–uniform charge and current distributions which would not be consistent with the assumed uniform magnetic field.

To describe the fermion contribution to the energy density it is necessary to draw on some properties of the Landau orbitals, eigenfunctions describing charged particles in a uniform magnetic field. In order to make the discussion self contained we give a brief account.

The system comprising the Maxwell field, A_μ, and a 1–component (non–relativistic) fermion, ψ, is governed by the action

$$S = \int d^3x \left[-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \psi^+ \left(i\nabla_0 + \frac{\nabla_k^2}{2m} \right) \psi - n_e A_0 \right] \tag{2.1}$$

where $\mu, \nu = 0, 1, 2$ and $k, \ell = 1, 2$. The covariant derivatives are defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{and} \quad \nabla_\mu \psi = (\partial_\mu - iA_\mu)\psi .$$

The coupling parameter, e^2, which has the dimensions of mass, is associated with the Maxwell kinetic term in (2.1) rather than the fermion term. This is a matter of convenience. The parameter, n_e, is included in (2.1) to account for a neutralizing but non–dynamical background of electric charge *).

The Landau orbitals provide a basis for the solutions of the fermion equation of motion,

$$\left(i\nabla_0 + \frac{\nabla_k^2}{2m} \right) \psi = 0$$

*) Although this model makes no reference to any crystal structure we can imagine that (2.1) is describing the remnant of a (3+1)–dimensional theory of electrons in a crystal with a layered structure that inhibits motion in the 3–direction. With this in mind we can give a rough estimate of the parameters e^2, m and n_e. If the thickness of one layer is δ then we should have $e^2\delta \sim 1/137$. The electron density should also have atomic dimensions, $n_e \sim \delta^{-2}$. There are two dimensionless ratios [6),7)], $e^2/m \sim 10^{-5}$ and $n_e/m^2 \sim 10^{-6}$.

in which the gauge potentials describe a uniform background. We choose

$$\nabla_0 = \partial_0 - i A_0 , \quad \nabla_1 = \partial_1 + i Bx_2 , \quad \nabla_2 = \partial_2 \tag{2.2}$$

with A_0 and B constant. Since B is a pseudoscalar we shall assume that the coordinate axes are oriented so as to make it positive. With the choice (2.2) we have translation invariance in the x_0 and x_1 directions. It is therefore convenient to expand as follows,

$$\psi = \sum_n \int \frac{dk}{2\pi} \psi_{nk}\, u_{nk}(x_2) e^{ikx_1 + i(\varepsilon_{nk} - A_0)x_0} \tag{2.3}$$

where $u_{nk}(x_2)$ satisfies the eigenvalue equation

$$\frac{1}{2m} \left(-\partial_2^2 + (k + Bx_2)^2 \right) u_{nk} = \varepsilon_{nk}\, u_{nk} . \tag{2.4}$$

This is just the problem of the simple harmonic oscillator and is solved by

$$u_{nk}(x_2) = B^{1/4}\, v_n \left(\sqrt{B}\, x_2 + \frac{k}{\sqrt{B}} \right)$$
$$\varepsilon_{nk} = \left(n + \frac{1}{2} \right) \frac{B}{m} , \tag{2.5}$$

where $n = 0, 1, 2, \ldots ; -\infty < k < \infty$ and the functions v_n are the orthonormalized oscillator wave functions,

$$v_n(Q) = (\sqrt{\pi}\, 2^n\, n!)^{-1/2} H_n(Q) e^{-Q^2/2} .$$

The eigenvalues (2.5) are independent of k, indicating a degeneracy of the Landau levels. In the next section we shall reconsider the eigenvalue problem (2.4) with particles restricted to the half–plane, $x_2 > 0$. In that case there is no degeneracy, the energies depend on k in a rather complicated way.

The coefficients ψ_{nk} in the expansion (2.3) satisfy the anticommutation relations,

$$\{ \psi_{nk}, \psi^+_{n'k'} \} = \delta_{nn'}\, 2\pi\, \delta(k - k')$$

and are realized in the usual way in Fock space. The fermion ground state $|F>$ is defined by,

$$\psi_{nk}\, |F> = 0 , \quad \varepsilon_{nk} > A_0$$
$$<F|\, \psi_{nk} = 0 , \quad \varepsilon_{nk} < A_0 \tag{2.6}$$

where A_0 is to be identified with the Fermi energy. Using this rule we can compute the ground state expectation value of the density operator,

$$<\psi^+\psi> = \sum_n \int \frac{dk}{2\pi} \theta(A_0 - \varepsilon_{nk}) |u_{nk}(x_2)|^2 . \tag{2.7}$$

In the simple case where ε_{nk} is independent of k, the definition (2.6) implies that the ground state comprises an integer number, N, of occupied Landau levels. This integer is defined as a function of A_0 and B by the inequalities

$$\varepsilon_{N-1} < A_0 < \varepsilon_N \; . \qquad (2.8)$$

If the Fermi energy actually coincides with one of the Landau levels then (2.6) becomes ambiguous and some refinement of the ground state definition is needed. This is the problem of partial filling which we shall come back to. However, if the inequalities (2.8) are strict then (2.7) can be evaluated using the normalized wave functions (2.5) and the result is a constant,

$$< \psi^+ \psi > = NB/2\pi \; . \qquad (2.9)$$

We have reproduced here the well–known fact that the degeneracy of states is given by $B/2\pi$ per unit area, per Landau level. (In the case of the half–plane, to be considered in Section 3, this simple rule fails near the edge where the number density is not uniform.) In the same way we compute the expectation value of the energy density,

$$\begin{aligned} \mathcal{E} &= < \frac{1}{2m} \nabla_j \psi^+ \nabla_j \psi - \psi^+ A_0 \psi > \\ &= \sum_n \int \frac{dk}{2\pi} \theta(A_0 - \varepsilon_{nk})(\varepsilon_{nk} - A_0) |u_{nk}(x_2)|^2 \\ &= \sum_{n=0}^{N-1} \frac{B}{2\pi} \left[\left(n + \frac{1}{2} \right) \frac{B}{m} - A_0 \right] \\ &= \frac{N^2 B^2}{4\pi m} - \frac{NB}{2\pi} A_0 \end{aligned} \qquad (2.10)$$

where N is regarded as a (discontinuous) function of A_0 and B, defined by (2.8).

The problem of partially filled Landau levels is specific to two dimensions where the spectrum of fermion energies is discrete. In three dimensions, where the spectrum is continuous, it is possible to construct a non–degenerate ground state for arbitrary values of the magnetic field and particle density. It is instructive to compare the formulae (2.9) and (2.10) with their three–dimensional analogues. In three dimensions the spectrum is given by

$$\varepsilon_{n\underline{k}} = \left(n + \frac{1}{2} \right) \frac{B}{m} + \frac{k_3^2}{2m} \, , \quad -\infty < k_3 < \infty$$

where k_3 denotes the momentum component parallel to the magnetic field. The eigenfunctions are

$$u_{n\underline{k}} = e^{i(k_1 x_1 + k_3 x_3)} B^{1/4} v_n \left(\sqrt{B}\, x_2 + \frac{k_1}{\sqrt{B}} \right)$$

The densities are given by

$$< \psi^+ \psi > = \sum_{n=0}^{N-1} \frac{B}{2\pi^2} (2mA_0 - (2n+1)B)^{1/2} \qquad (2.9')$$

$$\mathcal{E} = - \sum_{n=0}^{N-1} \frac{B}{6\pi^2 m} (2mA_0 - (2n+1)B)^{3/2} \qquad (2.10')$$

where the integer N is determined by the inequalities

$$\left(N - \frac{1}{2} \right) \frac{B}{m} < A_0 < \left(N + \frac{1}{2} \right) \frac{B}{m}$$

exactly as in (2.8). In contrast with (2.10), the energy density (2.10′) is an explicitly non–linear function of the Fermi energy, A_0. This implies that the equation $-\partial\mathcal{E}/\partial A_0 = n_e$ can be solved for A_0 as a function of B and n_e with the latter quantities freely adjustable. In other words, for arbitrary B and n_e it is possible to determine the Fermi level such that the one–fermion states of given energy are either all occupied or all empty. There is no degeneracy. In two dimensions, where A_0 acts as a Lagrange multiplier, the corresponding equation constrains the quantity $2\pi n_e/B$ to be an integer. If this constraint is not satisfied then one of the Landau levels must be partially occupied and the ground state is degenerate. It should be emphasized that this qualitative distinction between the two cases is due to the continuum nature of the three–dimensional spectrum.

The usual role of the electrostatic potential $A_0(x)$ in electrodynamics is to act as a Lagrange multiplier in enforcing the Gauss law. In the end it should drop out of physically meaningful, gauge independent quantities such as energy density. However, there is a qualification. The constant part of A_0, which we have identified with the Fermi energy, is not affected by the allowable gauge transformations, $A_0 \to A_0 + \partial_0 \Lambda$, which must satisfy the restriction, $\Lambda(x_0 = +\infty) = \Lambda(x_0 = -\infty)$. The Fermi energy is indeed gauge independent and physical quantities must, of course, depend on it.

To compute the total energy density in a non–degenerate ground state with specified number density, n_e, it is necessary to add the fermion contribution (2.10) to the background contribution, $n_e A_0$, and minimize with respect to A_0,

$$U = \frac{N^2 B^2}{4\pi m} - \frac{NB}{2\pi} A_0 + n_e A_0 \; . \qquad (2.11)$$

Minimization of U with respect to A_0 gives the constraint

$$B = \frac{2\pi n_e}{N} \equiv B_N \; . \qquad (2.12)$$

Comparing this with (2.9) we find $< \psi^+ \psi > = n_e$ which confirms the interpretation of n_e as fermion number. The resulting energy density is independent of N,

$$< F|U|F > = \frac{\pi n_e^2}{m} \qquad (2.13)$$

It is generally believed that these states are in some sense stable: that charged fermions in a uniform magnetic field will spread themselves in such a way as to satisfy (2.12), i.e., fill an integer number of Landau levels. An old argument, due to Peierls, [10] which suggests that these configurations correspond to local minima of an energy functional $U(B)$, is intuitively appealing. It goes as follows. Suppose that the Landau levels $n = 0, 1, \ldots, N-1$ are full, with number density $B/2\pi$, while the level $n = N$ is partially filled, with (uniform) number density $\delta n_e < \frac{B}{2\pi}$. Then the total number density is

$$n_e = \frac{B}{2\pi} N + \delta n_e ,$$

and the total energy density is

$$\begin{aligned} U(B) &= \frac{B}{2\pi} \sum_0^{N-1} \left(n+\frac{1}{2}\right) \frac{B}{m} + \delta n_e \left(N+\frac{1}{2}\right) \frac{B}{m} \\ &= \frac{N^2 B^2}{4\pi m} + \left(n_e - \frac{NB}{2\pi}\right) \left(N+\frac{1}{2}\right) \frac{B}{m} \\ &= \frac{\pi n_e^2}{m} \left[-N(N+1) \left(\frac{B}{2\pi n_e}\right)^2 + (2N+1) \frac{B}{2\pi n_e} \right] . \end{aligned} \quad (2.14)$$

This function, which is pictured in Fig.1, is continuous in B but not everywhere differentiable. Since the inequality, $\delta n_e < B/2\pi$, is assumed in the derivation we must regard N as a discontinuous function of B defined to be the integer part of $2\pi n_e/B$. The Peierls formula (2.14) exhibits local minima at $B = B_N$. At these points, where it takes the value (2.13), the magnetization, $-\partial U/\partial B$, is discontinuous (de Haas-van Alphen effect, Fig.2).

The Peierls formula can be obtained directly from (2.11) by making the substitution

$$A_0 = \left(N+\frac{1}{2}\right) \frac{B}{m}, \quad B_{N+1} < B < B_N \quad (2.15)$$

i.e., by choosing the Fermi level equal to the energy of the partially filled level. This is, of course, a reasonable choice but it should be borne in mind that the locus (2.15) is discontinuous (Fig.3). It is perhaps useful to see how this emerges from an examination of the equilibrium distribution at finite temperature by going to the limit of zero temperature. The thermodynamic potential which generalizes (2.11) is given by

$$\Omega = -\frac{B}{2\pi\beta} \sum_n \ell n \left(1 + e^{\beta(A_0 - \varepsilon_n)}\right) + n_e A_0 \quad (2.16)$$

where A_0 is to be understood as a chemical potential. At finite temperature the dependence of Ω on A_0 is not linear. Hence, the extremum condition, $\partial\Omega/\partial A_0 = 0$, can be solved for A_0 as a function of B and β, at least in principle. The equation takes the form

$$\frac{2\pi n_e}{B} = \sum_n \frac{1}{e^{\beta(\varepsilon_n - A_0)} + 1} \quad (2.17)$$

and it clearly reduces to (2.12) in the zero temperature limit if A_0 satisfies the inequalities (2.8). However, if A_0 approaches ε_N such that

$$A_0 = \varepsilon_N - \frac{1}{\beta} \ell n \xi \quad (2.18)$$

where ξ is real and positive and remains finite as $\beta \to \infty$, then (2.17) becomes

$$\frac{2\pi n_e}{B} = N + \frac{1}{\xi + 1} \quad (2.19)$$

in the limit. According to this scheme, the limiting value of A_0 is ε_N where N denotes the integer part of $2\pi n_e/B$. The locus of points in the $A_0 B$ plane given by $\partial\Omega/\partial A_0 = 0$ is a smooth curve (Fig.4) at finite temperature but it should be emphasized that its limiting form (2.15) does not correspond to an extremum of the function U defined in (2.11). Indeed, the equation $\partial U/\partial A_0 = 0$ gives only $B = B_N$.

To go beyond these considerations and establish an effective electrodynamics for this system it will be necessary to construct an action functional for slowly varying fields. The full contribution of the fermionic degrees of freedom is expressed, formally, as a functional determinant,

$$\Gamma_{fermi} = i\, \ell n \,\mathrm{Det}\; G(A) \quad (2.20)$$

where $G(A)$ represents a 1–fermion Green's function in the background of a given gauge field, A_μ. If this background is uniform, (constant A_0 and B) and if the fermionic ground state is translation invariant, then the functional (2.20) must reduce to the integral over spacetime of minus the energy density, $\mathcal{E}$, given by (2.10). When the background is slowly varying we can expand in powers of the derivatives. The result of this computation, accurate up to two derivatives, is given by [6]

$$\Gamma_{fermi} = \int d^3x \left[\frac{Nm}{4\pi} \frac{E_1^2 + E_2^2}{|B|} - \frac{N^2 B^2}{4\pi m} + \frac{N}{4\pi}\, sgnB (A_0 B - A_1 E_2 + A_2 E_1) + \ldots \right] \quad (2.21)$$

where $E_j = \partial_0 A_j - \partial_j A_0$, $B = \partial_1 A_2 - \partial_2 A_1$ and $sgnB = B/|B|$. The integer N is a function of A_0 and $|B|$ defined by the inequalities (2.8). In counting derivatives in the effective Lagrangian we should recognize that B contains a piece of order zero, The third term in (2.21) is the Chern–Simons action *). It should perhaps be emphasized that although the de Haas–van Alphen effect is well–known in a 3+1–dimensional gas of electrons in a magnetic background, the emergence of the Chern–Simons term in (2.21) with its topological implications is specific to 2+1–dimensional systems.

The action functional (2.21) is meaningful in regions of spacetime where the potentials are slowly varying and the magnetic field does not vanish and, more particularly, where the Fermi energy lies in a gap between Landau levels. Its derivation assumes that the fermion ground state is not degenerate.

Near points where the function $N(A_0, B)$ is discontinuous the functional (2.21) cannot be used. These discontinuities are associated with partial filling or, in other words, with ground state

*) Note that the coefficient of $A_0 B$ in (2.10) is $-N/2\pi$ whereas in (2.21) it is $N/4\pi$. The factor 2 is made up by the contributions of the term $-AE_2 = -x_2 B \partial_2 A_0 + \ldots = BA_0 + \ldots$ after integration by parts.

degeneracy. In cases where the ground state is degenerate, i.e., where A_0 takes one of the values (2.15), it is necessary to specify which states of the partially filled level are occupied. Presumably this must be decided on the basis of a self-consistent calculation involving interactions between the fermions. It is an unsolved problem.

To the extent that interactions can be neglected the Peierls formula should give an accurate representation of the fermion contribution to the energy density in a uniform magnetic field. The total energy density is then obtained by adding the purely magnetic contribution, $B^2/2e^2$, where e^2 is the coupling parameter introduced in (2.1),

$$U_{tot}(B) = \frac{1}{2e^2}B^2 + \frac{\pi n_e^2}{m}\left[-N(N+1)\left(\frac{B}{2\pi n_e}\right)^2 + (2N+1)\frac{B}{2\pi n_e}\right] \tag{2.22}$$

where N now denotes the integer part of $2\pi n_e/B$. This function has local "minima", in the sense that $\partial U_{tot}/\partial B$ changes sign, at the discontinuities of $N(B)$. Thus, at $B = 2\pi n_e/k, k = 1, 2, \ldots,$ we find

$$\left(\frac{\partial U_{tot}}{\partial B}\right)_{B=B_k\pm 0} = \frac{2\pi n_e}{k\,e^2} \pm \frac{n_e}{2m}.$$

There is a change of sign if

$$k > 4\pi m/e^2. \tag{2.23}$$

These minima are not degenerate and it is therefore plausible that they represent metastable configurations of the system. They should decay by tunneling towards the stable minimum at $k = \infty$.

If this picture is correct, it should be possible in principle to treat the decay of these metastable configurations as a nucleation process, i.e., set up a Euclidean version of the effective field theory and search for instanton-like solutions. Unfortunately, before this can be implemented it will probably be necessary to develop a more complete version of the effective theory in which the discontinuities of N are resolved. This of course means a theory in which the degeneracy of the partially filled Landau level is lifted.

3. EDGE EFFECTS

The conclusions of the previous section, leading up to the effective action (2.21), are based on the assumed translation invariance of the fermionic state. Indeed, the main point of the discussion was that, in the absence of obvious symmetry breaking features such as boundaries, the magnetic field would adjust itself to one of a discrete set of values such that a finite number of Landau levels is filled. Such states would be translation invariant. Of course there may be other homogeneous states or it may turn out that, when interactions are taken into account, the fermions choose to crystallize, breaking the translation symmetry down to a discrete subgroup. But our purpose here is only to consider the obvious kind of symmetry breaking associated with a boundary.

We shall suppose that the fermions are confined to the half-plane, $x_2 > 0$, the other half being vacuum. The behaviour of the electromagnetic field is then governed by the vacuum Maxwell equation for $x_2 < 0$ and by the effective equations derived on the assumption of translation invariance in the interior, $x_2 > 0$. Near the boundary, $x_2 = 0$, it will be necessary to take account of the symmetry breaking and include the effects of surface currents.

That some supplementary considerations are needed at the boundary can be seen by a superficial examination of the effective action which is to be used in the interior. It consists of the approximate expression (2.21) combined with the free Maxwell action and the neutralizing term,

$$S_{int} = \int_{x_2>0} d^3x \left[\frac{\varepsilon}{2}E_i^2 - \frac{1}{2\mu}B^2 + \frac{N}{8\pi}\varepsilon^{\lambda\mu\nu}A_\lambda F_{\mu\nu} - n_e A_0\right] \tag{3.1}$$

where the permittivities are given by

$$\varepsilon = \frac{1}{e^2} + \frac{N}{2\pi}\frac{m}{|B|}$$
$$\frac{1}{\mu} = \frac{1}{e^2} + \frac{N^2}{2\pi m}. \tag{3.2}$$

The exterior action is

$$S_{ext} = \int_{x_2<0} d^3x\,\frac{1}{2e^2}(E_i^2 - B^2). \tag{3.3}$$

The problem with (3.1) is that it is not quite gauge invariant. The Chern–Simons term changes by a total derivative in response to a gauge transformation and this will generally give a non-vanishing boundary contribution. In the usual applications of the Chern–Simons theory, to manifolds with no boundary, this problem does not arise. Here it is troublesome because, although the actual field equations derived from (3.1) and (3.3) are gauge invariant, the matching conditions on the boundary are not. The field equations deriving from (3.1) are

$$\partial_1(\varepsilon E_1) + \partial_2(\varepsilon E_2) = n_e - \frac{N}{2\pi}B$$
$$\partial_0(\varepsilon E_1) + \partial_2(B/\mu) = -\frac{N}{2\pi}E_2$$
$$\partial_0(\varepsilon E_2) - \partial_1(B/\mu) = \frac{N}{2\pi}E_1. \tag{3.4}$$

If we assume that these equations are satisfied for $x_2 > 0$ then the variation of S_{int} reduces to a boundary term,

$$\delta S_{int} = \int_{x_2>0} d^3x\,\partial_2\left[-\delta A_0\left(\varepsilon E_2 - \frac{N}{4\pi}A_1\right) + \delta A_1\left(\frac{B}{\mu} - \frac{N}{4\pi}A_0\right)\right].$$

(We ignore the ∂_0 and ∂_1 contributions which are associated with infinitely remote boundaries.) Assuming that the vacuum Maxwell equations are satisfied for $x_2 < 0$, the variation of (3.3) will reduce to the boundary term

$$\delta S_{ext} = \int_{x_2<0} d^3x\,\partial_2\left[-\delta A_0\left(\frac{1}{e^2}E_2\right) + \delta A_1\left(\frac{1}{e^2}B\right)\right].$$

In the normal boundary problem one would develop consistent matching conditions by requiring $\delta S_{int} + \delta S_{ext} = 0$ for arbitrary variations δA_0 and δA_1 on the boundary, i.e.

$$\left(\varepsilon E_2 - \frac{N}{4\pi} A_1\right)_{int} = \left(\frac{1}{e^2} E_2\right)_{ext}$$
$$\left(\frac{B}{\mu} - \frac{N}{4\pi} A_0\right)_{int} = \left(\frac{1}{e^2} B\right)_{ext}$$

But these equations are not gauge invariant, and we must conclude that the simple effective action given by the sum of S_{int} and S_{ext} is not adequate in the boundary region.

It is not difficult to spot the source of this inadequacy. In trying to evaluate the fermionic contribution, $i\,\ell n \,\mathrm{Det}\, G(A)$, for slowly varying fields, it is necessary to separate A_μ into the sum of a uniform background piece $\bar{A}_\mu$ and a small variable piece, a_μ. The effective action can be expanded in powers of a_μ,

$$\Gamma(A) = \Gamma(\bar{A}) + \int d^3x\, \Gamma^\mu(x, \bar{A}) a_\mu(x) + \frac{1}{2}\int d^3x d^3x'\, \Gamma^{\mu\nu}(x, x', \bar{A}) a_\mu(x) a_\nu(x') + \ldots \qquad (3.5)$$

where the integrals are restricted to $x_2 > 0$. The coefficients, $\Gamma^\mu, \Gamma^{\mu\nu}, \ldots$ are built from correlation functions of the fermion currents and, since these currents must vanish at the boundary, they cannot be translation invariant. Thus, Γ^μ will depend on x_2, $\Gamma^{\mu\nu}$ on x_2 and x_2' (not simply $x_2 - x_2'$), etc. The sort of approximations that led to the local form (2.21), involving in particular the long wavelength term,

$$\Gamma^{\mu\nu}(x, x') = \frac{N}{4\pi} \varepsilon^{\mu\nu\lambda} \partial_\lambda \delta_3(x - x') + \ldots$$

are not credible near the boundary. There must be a boundary layer in which the functional (3.5) is truly non-local. Our aim here is to study this boundary layer, estimate its thickness, and the current that flows in it. But first we consider, briefly, the behaviour of electric and magnetic field strengths in the interior, away from the boundary, where (3.1) is presumably trustworthy.

A solution of the equations, (3.4) which depends only on x_2, is easily extracted,

$$E_1 = 0$$
$$E_2 = E^0\, e^{-x_2/\lambda}$$
$$B = B_N + B^0\, e^{-x_2/\lambda} \qquad (3.6)$$

where E^0 and B^0 are integration constants and $B_N = 2\pi n_e/N$. The electric and magnetic fields decay with distance from the boundary, the one towards zero and the other to a uniform background value. The scale of this decay, the relaxation length, λ, is given by

$$\lambda^2 = \left(\frac{2\pi}{N}\right)^2 \frac{\varepsilon}{\mu} = \frac{m}{e^2 n_e}\left(1 + \left(\frac{2\pi}{N}\right)^2 \frac{n_e}{me^2}\right)\left(1 + \frac{N^2}{2\pi}\frac{e^2}{m}\right) \qquad (3.7)$$

where we have substituted $B = B_N$ in the formula (3.2) for ε.

Eqs.(3.4) can be read as 2+1–dimensional Maxwell equations for the electric displacement, $D_i = \varepsilon E_i$ and magnetic intensity, $H = B/\mu$. The right-hand sides are then interpreted as real charge and current densities. The current defined in this way,

$$j_i = -\frac{N}{2\pi} \varepsilon_{ij} E_j$$

is evidently a Hall current and the associated conductivity, $-N/2\pi$, is quantized. Moreover, since the solution (3.6) is time independent and not dissipative, the current $j_1 = -(N/2\pi) E_2$, can be interpreted as a supercurrent. The exponential fall off in the magnetic field can likewise be interpreted as a Meissner–type effect. There is a penetration region, of order λ, in which the supercurrent flows and the magnetic field decays. However, in contrast with the usual superconductor, there is also an electric field in this region and the magnetic field does not fall to zero but only to its background value *).

If the boundary layer is very thin in comparison with the relaxation length, λ, then its contribution to the development of electric and magnetic field strengths can be represented approximately as discontinuities. Such discontinuities should be ascribed in the usual way to surface currents, and these currents must be given, to a first approximation by the coefficient Γ^μ in (3.5). We turn now to the estimation of these currents as well as the width of the boundary layer. Unfortunately, we shall find that the "boundary layer" is not always thin. Indeed, it turns out that values of N which are large enough to meet the metastability criterion (2.18), $N > 4\pi m/e^2$, will be associated with boundary layers whose width is comparable to λ. Thin boundary layers arise only when N, the number of filled levels, is relatively small.

The fermion current operators, $j^\mu = \delta S/\delta A_\mu$, are defined by the action functional (2.1),

$$j_0 = -\psi^+\psi, \quad \text{and} \quad j_k = -\frac{i}{2m}\psi^+\nabla_k\psi + h.c.,$$

where the covariant derivatives refer to the uniform background, (2.2). The ground state expectation values of these currents are obtained, in the 1–loop approximation, by substituting the mode expansions (2.3) for ψ and ψ^+ and using the defining property (2.6) of the ground state, $|F>$. One finds

$$\Gamma_0 = -<F|\psi^+\psi|F> = -\sum_n \int \frac{dk}{2\pi}\, \theta(A_0 - \varepsilon_{nk})\, u_{nk}(x_2)^2 \qquad (3.8)$$

*) It may even *increase* towards the background value, depending on the history of the system. We believe that for a given field strength at the boundary, *one* of the values B_N for the interior will define a stable configuration but this needs to be proved.

$$\Gamma_1 = -\frac{i}{2m} < F|\psi^+\nabla_1\psi|F> + c.c.$$
$$= \sum_n \int \frac{dk}{2\pi}\, \theta(A_0 - \varepsilon_{nk})\, \frac{k + Bx_2}{m}\, u_{nk}(x_2)^2 \tag{3.9}$$
$$\Gamma_2 = 0$$

where A_0 is the Fermi energy and $u_{nk}(x_2)$ is an eigenfunction of energy ε_{nk}. The eigenvalue problem is based on the differential equation (2.4) but the domain is restricted to the semiaxis, $x_2 \geq 0$, and the eigenfunctions are required to vanish at $x_2 = 0$. They are normalized such that

$$\int_0^\infty dx_2\, u_{nk}(x_2)\, u_{n'k}(x_2) = \delta_{nn'} . \tag{3.10}$$

It turns out that these eigenfunctions are real, and it is for this reason that Γ_2 vanishes. The current flow is parallel to the boundary.

It is not surprising that the determination of u_{nk} and ε_{nk} is less elementary than the calculation of Section 2 where the domain of x_2 was unrestricted. The boundary condition, $u_{nk}(0) = 0$, distinguishes this from the problem of the simple harmonic oscillator. It is still soluble, but the familiar Hermite polynomials are replaced by parabolic cylinder functions. The qualitative features of the spectrum are easily extracted. We give a brief account.

It is convenient to replace x_2 by the new variable

$$Q = \sqrt{B}\, x_2 + \frac{k}{\sqrt{B}}, \quad \frac{k}{\sqrt{B}} \leq Q < \infty \tag{3.11}$$

and define a new wave function,

$$u_n\left(Q, \frac{k}{\sqrt{B}}\right) = B^{-1/4}\, u_{nk}(x_2) . \tag{3.12}$$

It satisfies the equations

$$(-\partial_Q^2 + Q^2)u_n = \frac{2m\varepsilon_{nk}}{B}\, u_n \tag{3.13a}$$
$$u_n\left(\frac{k}{\sqrt{B}}, \frac{k}{\sqrt{B}}\right) = 0 \tag{3.13b}$$

and it is normalized such that

$$\int_{k/\sqrt{B}}^\infty dQ\, u_n\left(Q, \frac{k}{\sqrt{B}}\right)\, u_{n'}\left(Q, \frac{k}{\sqrt{B}}\right) = \delta_{nn'} . \tag{3.14}$$

The general solution of (3.13a) is a linear combination of two parabolic cylinder functions [11], $D_{\pm\nu}(\sqrt{2}Q)$, where the index ν is defined by

$$\varepsilon_{nk} = \left(\nu + \frac{1}{2}\right) \frac{B}{m} .$$

The allowed values of ν are to be determined by the boundary condition (3.13b). This means searching for the zeroes of the D_ν.

Being associated with a second order differential equation with only one singular point, an irregular one at $z = \infty$, the parabolic cylinder function $D_\nu(z)$ is relatively straightforward. It is an entire function in the complex z plane. It vanishes like $\exp(-z^2/4)$ for $|z| \to \infty, |arg\ z| < \pi/4$. Along the positive real axis it is very well behaved but, along the negative axis it blows up unless ν is a non–negative integer. (It is the latter fact which makes the simple harmonic oscillator indeed simple.) The other important property of $D_\nu(z)$ is that, for real ν, it has $[\nu + 1]$ real zeroes [11].

We have not been able to find much information about these zeroes in the mathematical literature but it is not difficult to guess the main features. These are most easily comprehended by representing them as trajectories in the νz–plane (Fig.5). The zero trajectories, $\nu_n(z), n = 0, 1, 2, \ldots$ are monotonically increasing functions of z (real). The value of z at which the trajectory ν_n crosses the integer $m > n$ is given by one of the zeroes of the hermite polynomial $H_m(z/\sqrt{2})$: the furthest one to the left for $n = m - 1$, the next left for $n = m - 2$, etc. In general,

$$\begin{aligned} &\nu_n(z) \to n \quad \text{for} \quad z \to -\infty \\ &\nu_n(0) = 2n + 1 \\ &\nu_n(z) \simeq \frac{z^2}{4} \quad \text{for} \quad z \to +\infty \end{aligned} \tag{3.15}$$

The behaviour at large positive z can be deduced from the well studied large n behaviour of the zeroes *) of H_n.

The solutions of the eigenvalue problem (3.13) are given by

$$\begin{aligned} u_n(Q, t) &= N_n(t)\, D_{\nu_n(\sqrt{2}t)}(\sqrt{2}Q), \\ \varepsilon_{nk} &= \left(\nu_n(\sqrt{2}t) + \frac{1}{2}\right) \frac{B}{m}, \end{aligned} \tag{3.16}$$

where $t = k/\sqrt{B}$ and $n = 0, 1, 2, \ldots$; N_n is a normalization factor. The eigenvalues can be read directly from Fig.5. Their most important feature is the tendency to increase with k. This means that, for finite A_0, the integrals (3.8) and (3.9) have an upper cutoff.

Since $u_n(Q, t)$ is supported mainly in the vicinity of $Q = 0$, it follows from the definition (3.11) that if x_2 is sufficiently positive then k will have to be negative to the extent that $\nu_n(k\sqrt{2/B}) \sim n$. In this case the wave function is Gaussian in k, peaked at a negative value, and the upper cutoff is not significant. This means that the translation invariant regime of Section 2 is effectively restored when x_2 is sufficiently positive. What is the relevant scale? Since

*) For example, the largest positive zero of $H_n(z)$ is given for large n by Szego [12] in the formula, $z \simeq \sqrt{4n} + O(n^{-1/6})$.

$Q = \sqrt{B}\, x_2 + k/\sqrt{B}$, and the simple oscillator wave function $v_n(Q)$ has width $\Delta Q \sim \sqrt{n + 1/2}$, it appears that translation invariance must be restored for

$$x_2 > \sqrt{\left(N + \frac{1}{2}\right)/B}$$

where N labels the highest "filled" level. Substituting the background value, $B = 2\pi n_e/N$, we arrive at an estimate for the width of the boundary layer,

$$\delta \sim \frac{N}{\sqrt{2\pi n_e}} . \qquad (3.17)$$

This length is to be compared with the relaxation length of Eq.(3.7),

$$\begin{aligned} \left(\frac{\delta}{\lambda}\right)^2 &= \frac{e^2}{m}\frac{N^2}{2\pi}\left(1 + \left(\frac{2\pi}{N}\right)^2 \frac{n_e}{me^2}\right)^{-1}\left(1 + \frac{N^2}{2\pi}\frac{e^2}{m}\right)^{-1} \\ &= \left(1 + \frac{4\pi m}{Ne^2}\frac{\pi n_e}{Nm^2}\right)^{-1}\left(1 + \frac{4\pi m}{Ne^2}\frac{1}{2N}\right)^{-1} . \end{aligned} \qquad (3.18)$$

To satisfy the metastability criterion (2.18) we must have $4\pi m/Ne^2 < 1$ and this implies $\delta \sim \lambda$ (unless $n_e \gg Nm^2$, which does not seem realistic). We must therefore conclude that the simple translation invariant equations (3.4) are not trustworthy in the metastable cases. However, in the strong field configurations where N is relatively small, it is reasonable to use (3.4) and in these cases the boundary layer is thin.

If the boundary layer is indeed narrow then it makes sense to estimate the total current flowing through it,

$$\begin{aligned} I_1 &= \int_0^\delta dx_2\, \Gamma_1 , \\ &\simeq \int_0^\infty dx_2 \sum_n \frac{dk}{2\pi}\,\theta(A_0 - \varepsilon_{nk})\,\frac{k + Bx_2}{m}\, u_{nk}(x_2)^2 \\ &= \sum_n \int \frac{dk}{2\pi}\,\theta(A_0 - \varepsilon_{nk})\,\frac{\sqrt{B}}{m}\, W_n\!\left(\frac{k}{\sqrt{B}}\right) \end{aligned} \qquad (3.19)$$

where

$$W_n(t) = \int_t^\infty dQ\, Q\, u_n(Q,t)^2 \qquad (3.20)$$

If t is sufficiently negative then $u_n(Q,t) \sim v_n(Q)$ and the integral (3.20) will vanish. On the other hand, if t is positive then $W_n(t) > t$. Its actual value will reflect the spread, ΔQ, in the wave function but, to a first approximation we shall ignore this and write

$$W_n(t) = \begin{cases} 0, & t < 0 \\ t, & t > 0 \end{cases} \qquad (3.21)$$

which makes the integral (3.19) easy to evaluate. The result is

$$I_1 \simeq \sum_n \frac{k_n^2}{4\pi m} \qquad (3.22)$$

where k_n represents the upper limit, determined by $\varepsilon_{nk} = A_0$, and the sum is restricted to those levels for which k_n is positive. The largest contribution is from the $n = 0$ level which we estimate by means of the asymptotic expression (3.15),

$$\begin{aligned} \varepsilon_{0k} &= \left(\nu_0 + \frac{1}{2}\right)\frac{B}{m} \\ &\simeq \frac{1}{4}\left(k\sqrt{\frac{2}{B}}\right)^2 \frac{B}{m} \\ &= \frac{k^2}{2m} . \end{aligned}$$

On setting this equal to the Fermi energy, $A_0 = 2\pi n_e/m$, obtained in Section 2, we arrive at the estimate

$$k_0 \simeq \sqrt{4\pi n_e} .$$

Since the leading asymptotic part of ν_n is independent of n, this estimate can probably be used for all the more or less full levels, $n \le N - 1$. We thereby obtain the total surface current,

$$I_1 \simeq N\, n_e/m . \qquad (3.23)$$

By an application of Ampere's law the discontinuity in the magnetic field, due to the current (3.23), is given by

$$\frac{1}{e^2} B_{ext} - \frac{1}{\mu} B_{int} = N\,\frac{n_e}{m} \qquad (3.24)$$

where the permittivity is given by (3.2). If $N^2 e^2 \ll 2\pi m$ then $\mu \sim e^2$ and the discontinuity reduces to

$$\Delta B \sim N\, n_e\, e^2/m \qquad (3.25)$$

which is small in comparison with $B_N \sim 2\pi n_e/N$. Indeed, we have $\Delta B/B_N \sim (\delta/\lambda)^2$. This indicates that there is no significant concentration of current in the boundary layer.

An expression for surface charge, analogous to (3.23), could be obtained by evaluating the integral

$$I_0 = \int_0^\delta dx_2\, (\Gamma_0 - n_e)$$

on the assumption that n_e is strictly constant. However, this does not seem to be a very reasonable assumption near the boundary. It would be more plausible to suppose that the neutralizing background, represented rather simplistically by n_e, would adjust itself so as to make the external electric field vanish. We shall not attempt to pursue this question.

To summarize, if N is not too large, more precisely if $e^2 N^2 \ll 2\pi m$, then the boundary layer is small in comparison with the penetration depth and it is permissible to use the translation invariant effective Lagrangian and assume that the field strengths are continuous across the boundary. On the other hand, if $e^2 N > 4\pi m$ then $\delta \sim \lambda$ and the exponential decay indicated in (3.6) would be significantly modified.

4. CONCLUSIONS

Two dimensional charged fermions are expected to aggregate in translation invariant states which sustain a uniform non–vanishing magnetic field. These states, comprising a whole number, N, of filled Landau levels are metastable if $N > 4\pi m/e^2$. If this number is not too large or, in other words, if the magnetic field is not too weak relative to the fermion density, then metastability is lost but edge effects will not be important. Weak perturbations around these states can be analyzed by means of an effective action formalism and, as far as long wavelength and low frequency modes of the electromagnetic field are concerned, the main effects are accounted for by electric and magnetic permittivities and by a parity violating Chern–Simons term whose strength is proportional to the number of filled Landau levels. Because of the Chern–Simons term, gradients in the magnetic field generate electric fields and vice versa. These are associated with a Hall current and the conductivity is proportional to the number of filled levels. In contrast, if the number of filled levels is sufficiently large, i.e. when the magnetic field is relatively weak, then metastable configurations are possible but the local form of the effective action is compromised near the boundary and the Chern–Simons description becomes unreliable.

It would be interesting to compute the lifetime of the metastable states and, more generally, the speed with which the system responds to changes in the external magnetic field. The system that we have considered is highly idealized: it is strictly 2–dimensional and does not include any remnant of 3–dimensional electrodynamics like B_1, B_2 or E_3. In a somewhat more realistic model it would be necessary to take account of these in addition to structures associated with the "neutralizing background", band structure, impurities, etc. Even within the limitations of the idealized model from which we started, the neglect of interactions is a gross simplication. It is quite conceivable that these interactions could destabilize the translation invariant states and favour the emergence of a lattice structure. We hope to come back to some of these questions in the future.

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FIGURE CAPTIONS

Fig.1 Energy density $U(B)$ at zero temperature. (The total energy density includes the electromagnetic contribution, $B^2/2e^2$, which is not represented in this figure.)

Fig.2 Magnetization density at zero temperature.

Fig.3 Locus of points $\partial\Omega/\partial A_0 = 0$ in the zero temperature limit. Straight lines radiating from the origin represent $A_0 = \varepsilon_N$. Extrema are located at the points $A_0 = 2\pi n_e/m$, $B = B_N = 2\pi n_e/N$.

Fig.4 Locus of $\partial\Omega/\partial A_0 = 0$ at low but finite temperature.

Fig.5 Trajectories of zeroes of the parabolic cylinder function $D_\nu(z)$.

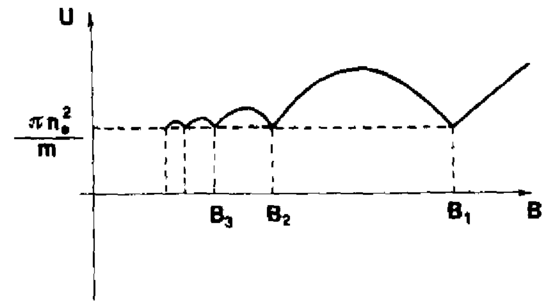

Fig.1

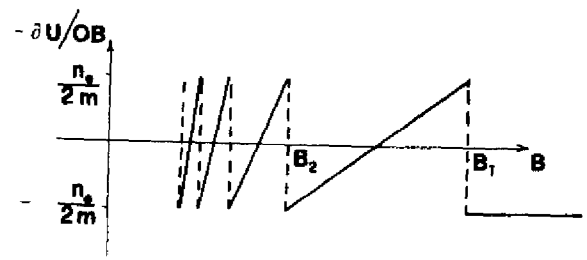

Fig.2

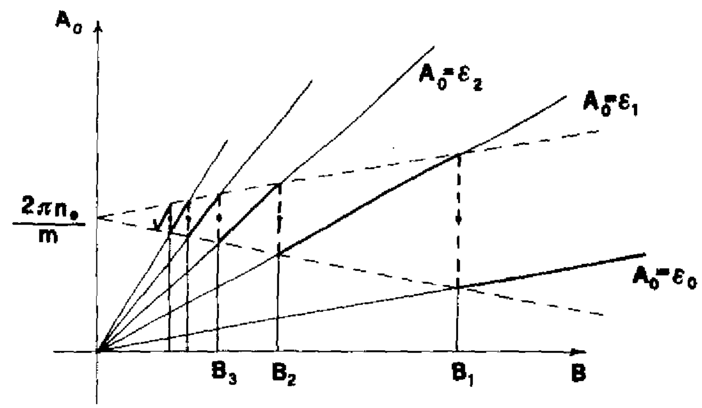

Fig.3

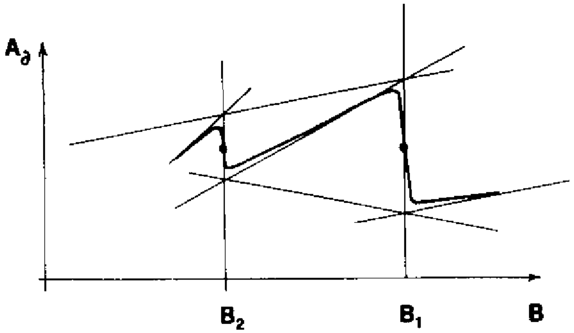

Fig.4

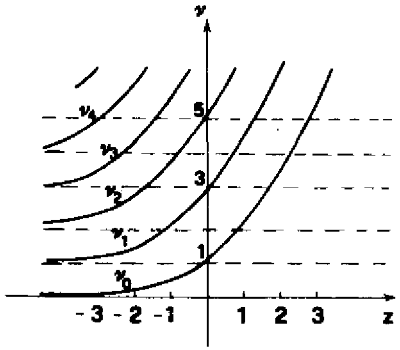

Fig.5